SUPERALGEBRAS OF SYMMETRY OPERATORS FOR COULOMB AND AHARONOV-BOHM-COULOMB SYSTEMS

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ABSTRACT. It is shown that the relativistic Hydrogen atom and the Aharonov-Bohm-Coulomb system are characterized by extended N=6 supersymmetry and admit a hidden symmetry algebra $gl(8, R)$.

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1 Introduction

Symmetries of differential equations present powerful tools for their study. In many cases symmetries make it possible to reduce the considered problem to a more simple one or even to find its exact solutions (refer to [1] for numerous examples).

In recent papers [2,3] a new invariance algebra A of the free Dirac equation was found. This algebra whose basis elements are involutions like parity transformations or special finite rotations, is isomorphic to the Lie algebra $gl(8,R)$. Moreover, as it was demonstrated in [2,3] rather extended subalgebras of A form invariance algebras of the Dirac equation for a charged particle interacting with an external electromagnetic field.

In this paper we show that relativistic Hydrogen atom and the relativistic Aharonov-Bohm-Coulomb (ABC) system (which is described by the Dirac equation including a superposition of the Coulomb and Aharonov-Bohm potentials) also admit symmetry algebras which are isomorphic to $gl(8,R)$. In addition we indicate that the above mentioned problems admit extended $N=6$ supersymmetry (SUSY).

These results can give a key for better understanding of ABC system which is in a focus of interest of a number of investigators (refer, e.g., to [4,5] and references cited therein). Moreover, we present a new origin of extended SUSY and so give one more argument that this symmetry is realised in realistic quantum mechanical systems.

2 Discrete symmetries of the Dirac equation

We start with the free Dirac equation

$$L\psi = 0, \qquad L = \gamma^\mu p_\mu - m, \tag{2.1}$$

where $p_\mu = i\frac{\partial}{\partial x^\mu}$, $\mu = 0,1,2,3$, γ^μ are the Dirac matrices.

It was demonstrated in paper [3] that equation (2.1) admits a 64 – dimensional algebra of involutive symmetries. This algebra can be constructed in the following way.

First, we remind that a linear operator Q is a *symmetry operator* (SO) of equation (2.1) if there exists such an operator α_Q that

$$[Q, L] = \alpha_Q L. \tag{2.2}$$

where $[.,.]$ is a commutator.

Well-known examples of SOs of the Dirac equation are parity and time reflection. Considering reflections of all independent variables $x=(x_0,x_1,x_2,x_3)$:

$$
\begin{array}{lll}
\theta_0 x = (-x_0, x_1, x_2, x_3)\,, & \theta_1 x = (x_0, -x_1, x_2, x_3)\,, & \theta_2 x = (x_0, x_1, -x_2, x_3)\,, \\
\theta_3 x = (x_0, x_1, x_2, -x_3)\,, & \theta x = (-x_0, -x_1, -x_2, -x_3)\,. &
\end{array}
\tag{2.3}
$$

we easily construct the corresponding SOs for (2.1):

$$
\Gamma_\mu = \gamma_4 \gamma_\mu \hat{\theta}_\mu, \quad \Gamma_4 = i\gamma_4 \hat{\theta} \tag{2.4}
$$

where $\gamma_4 = i\gamma_0\gamma_1\gamma_2\gamma_3$,
$hat\theta_\mu, \hat{\theta}$ are operators defined as follows:

$$
\hat{\theta}_\mu \psi(x) = \psi\left(\theta_\mu x\right), \quad \hat{\theta}\psi(x) = \psi(-x).
$$

Adding to (2.3) two more *antilinear* SOs

$$
\Gamma_5 = i\gamma_2 c, \quad \Gamma_6 = \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5 = \gamma_2 c \tag{2.5}
$$

(with c being the complex conjugation, $c\psi(x) = \psi^*(x)$), we come to a basis $(\Gamma_0, \quad \Gamma_1, \quad \cdots, \quad \Gamma_6)$ of the seven-dimension Clifford algebra. All linearly independent products of SOs (2.3), (2.4), i.e.,

$$
\Gamma_m, \quad \Gamma_m\Gamma_n, \quad \Gamma_k\Gamma_m\Gamma_n, \quad \hat{I}, \tag{2.6}
$$

$(k, m, n, = 0, 1, ...6$, $\hat{I}$ is the unit operator) form a basis of the 64-dimensional invariance algebra isomorphic to $gl(8, R)$ [3].

We notice that operators (2.5) include reflections Γ_μ, Γ_4 and pure rotations $\Gamma_\mu\Gamma_\nu$ as well.

It is evident that the Dirac equation with non-trivial potentials

$$
L\psi = \left(\gamma^\mu \pi_\mu - m\right)\psi = 0, \quad \pi_\mu = p_\mu - eA_\mu \tag{2.7}
$$

cannot admit all SOs (2.6) (for instance, neither Γ_5 nor Γ_6 are admissible). In the next sections we present examples of quantum mechanical systems which include interactions with external fields and admit symmetry algebras isomorphic to (2.6).

3 Symmetries of the Coulomb system

For the case when the external field is generated by a point charge, the vector-potential can be reduced to the form

$$A_1 = A_2 = A_3 = 0, \quad A_0 = \frac{\alpha}{e|x|}, \quad |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}. \tag{3.1}$$

Symmetries of the Dirac equation (2.6) with the Coulomb potential (3.1) are well-studied (refer, e.q., to [6,7]). It is generally accepted that the maximal dynamical symmetry of system (2.6), (3.1) is described by algebra $so(2.4)$; in addition, equation (2.6), (3.1) admits hidden supersymmetry generated by two supercharges [7].

New non-unitary symmetry of this equation w.r.t. algebra $su(2)$ was found recently [8].

Here we show that the Coulomb system (2.6), (3.1) admits more extended symmetry algebras generated by discrete symmetries.

It is known since 1950 that the Dirac Hamiltonian

$$H = \gamma_0 \gamma_a P_a + \gamma_0 m + \frac{\alpha}{|x|} \tag{3.2}$$

commutes with the Johnson-Lippman operator [9] which we write in the form

$$Q = m\alpha \frac{\vec{\sigma} \cdot \vec{x}}{|x|} + iD \left(\vec{\sigma} \cdot \vec{p} + i\gamma_4 \frac{\alpha}{|x|} \right) \tag{3.3}$$

where

$$D = \gamma_0 \left(\vec{\sigma} \cdot \vec{J} - \frac{1}{2} \right), \quad \vec{\sigma} = \frac{i}{2} \vec{\gamma} \times \vec{\gamma}, \quad \vec{J} = \vec{x} \times \vec{p} + \frac{1}{2} \vec{\sigma}. \tag{3.4}$$

Operators Q, D and H together with Q^2 and D^2 form a five dimensional symmetry superalgebra of the Coulomb system, characterized by the following commutation and anticommutation relations

$$\begin{aligned} &[D, H] = [Q, H] = [D^2, H] = [Q^2, H] = [D, Q^2] = [Q, D^2] = 0, \\ &\{D, Q\} = 0. \end{aligned} \tag{3.5}$$

We notice that the squared operators D and Q can be represented as follows

$$D^2 = J^2 + \frac{1}{4}, \quad Q^2 = D^2 \left(H^2 + m^2 \right) - \alpha^2 m^2. \tag{3.6}$$

Denoting eigenvalues of the commuting operators D^2, H^2 and H by κ^2, q^2 and E correspondingly, we obtain from (3.6) the following algebraic relation

$$q^2 = \kappa^2 E^2 + \left(\alpha^2 - \kappa^2\right) m^2, \quad \kappa = 1, 2, \cdots, \tag{3.7}$$

or

$$E^2 = \frac{q^2}{\kappa^2} + \left(1 - \frac{\alpha^2}{\kappa^2}\right). \tag{3.8}$$

In other words, the eigenvalues of the Hamiltonian of the relativistic Hydrogen atom can be found starting with eigenvalues of SOs Q (3.3) and D (3.4).

Let us show that equation (2.6), (3.1) admits a more extended symmetry superalgebra generated by the following six supercharges

$$\begin{aligned} Q_1 &= (1 + i\Gamma_5\Gamma_1\Gamma_2)\, Q, \quad \bar{Q}_1 = (1 - i\Gamma_5\Gamma_1\Gamma_2)\, Q, \\ Q_2 &= i\,(\Gamma_1 + \Gamma_5)\,\Gamma_2\Gamma_3 Q, \quad \bar{Q}_2 = i\,(\Gamma_1 - \Gamma_5)\,\Gamma_2\Gamma_3 Q, \\ Q_3 &= \Gamma_5\,(1 + i\Gamma_1\Gamma_3)\, Q, \quad \bar{Q}_3 = \Gamma_5\,(1 - i\Gamma_1\Gamma_3)\, Q. \end{aligned} \tag{3.9}$$

Indeed, using the relations

$$[\Gamma_5, H] = [\Gamma_5, D] = [\Gamma_a, H] = [\Gamma_a, D] = 0, \quad a, b = 1, 2, 3,$$

$$\{\Gamma_5, H\} = \{\Gamma_a, Q\} = \{\Gamma_a, \Gamma_5\} = \{\Gamma_5, i\} = 0, \quad \{\Gamma_a, \Gamma_b\} = 2\delta_{ab}$$

we easily verify that operators (3.9) commute with Hamiltonian (3.2) and satisfy the following anticommutation relations

$$\begin{aligned} &\left\{Q_a, \bar{Q}_b\right\} = 2\delta_{ab}\hat{H}, \\ &\{Q_a, Q_b\} = \left\{\bar{Q}_a, \bar{Q}_b\right\} = 0, \end{aligned} \tag{3.10}$$

where $\hat{H} = Q^2, a = 1, 2, 3$. In addition, it follows from (3.10) that

$$\left[\hat{H}, Q_a\right] = \left[\hat{H}, \bar{Q}_a\right] = 0. \tag{3.11}$$

Relations (3.10), (3.11) define a superalgebra which is the symmetry algebra of supersymmetric quantum mechanics [10] with six supercharges. Thus symmetry algebra (3.9) extends well-known $N = 2$ SUSY of the Hydrogen atom to $N = 6$ SUSY.

For any $q \neq 0$ we define

$$\hat{\Gamma}_0 = i\Gamma_1\Gamma_2\Gamma_3, \quad \hat{\Gamma}_a = \frac{1}{2q}\left(Q_a + \bar{Q}_a\right), \quad \hat{\Gamma}_{3+a} = \frac{1}{2iq}\left(Q_a - \bar{Q}_a\right). \qquad (3.12)$$

Operators (3.12) are SOs of the stationary Dirac equation

$$H\psi = E\psi \qquad (3.13)$$

where H is Hamiltonian (3.2) whose eigenvalues E have the form (3.8). These SOs form a basis of the seven-dimensional Clifford algebra, satisfying the following anticommutation relations

$$\{\Gamma_A, \Gamma_B\} = 2g_{AB}, \quad A, B, = 0, 1, 2, ..., 6$$

where nonzero elements of tensor g_{AB} are $g_{00} = g_{11} = g_{22} = g_{33} = -g_{44} = -g_{55} = -g_{66} = 1$. All linearly independent products of SOs (3.12) are given in (2.5) (where Γ_a have to be replaced by $\hat{\Gamma}_a$) and form a basis of algebra $gl(8, R)$.

Thus, the Coulomb system described by the Dirac equation (3.13) keeps the symmetry of free Dirac equation w.r.t. algebra $gl(8, R)$.

4 Symmetries of the Aharanov-Bohm-Coulomb system

Let us consider equation (2.6) for the case when the external field is generated by a superposition the Coulomb potential and the potential of solenoid directed along the third co-ordinate axis. The related four-vector A_μ can be chosen in the following form

$$A_1 = \frac{\xi}{e}\frac{x_2}{r^2}, \quad A_2 = -\frac{\xi}{e}\frac{x_1}{r^2}, \quad A_3 = 0, \quad A_0 = \frac{\alpha}{e}\frac{1}{|x|} \qquad (4.1)$$

where $r^2 = x_1^2 + x_2^2$.

Equation (2.6), (4.1) (and especially its reduced (1+2) – dimensional form with $\mu = 0, 1, 2$) has many intriguing physical applications [4,5]. Here we study symmetry aspects of this problem.

Using the well-known property of A_1, A_2 being locally a pure gauge, we easily find the analogue of the Lippman-Johnson constant of motion (3.3) for equation (2.6), (4.1):

$$Q = m\alpha\frac{\vec{\sigma}\cdot\vec{x}}{|x|} + i\hat{D}(\vec{\sigma}\cdot\vec{\pi} + i\gamma_4\frac{\alpha}{|x|}), \tag{4.2}$$

where

$$D = \gamma_0(\vec{\sigma}\cdot\vec{J} + \frac{1}{2} + \frac{\xi}{r^2}[\sigma_3|x|^2 - x_3(\vec{\sigma}\cdot\vec{x})]), \tag{4.3}$$

Operators (3.2), (3.3) commute with the corresponding Dirac Hamiltonian

$$H = \gamma_0\gamma_a\pi_a + \gamma_0 m + \frac{\alpha}{|x|}$$

and so are SOs of equation (2.6), (4.1).

Additional (involutive) symmetries of this equation can be found in the form

$$\begin{array}{l} R_{12} = i\gamma_1\gamma_2\hat{\theta}_1\hat{\theta}_2, \quad R_{31} = \exp(i\varphi)i\gamma_3\gamma_1\hat{\theta}_3\hat{\theta}_1, \\ R_{23} = \exp(i\varphi)i\gamma_2\gamma_3\hat{\theta}_2\hat{\theta}_3, \quad R = i\gamma_4\gamma_0\hat{\theta}, \quad \hat{C} = \exp(i\varphi)i\gamma_2 c \end{array} \tag{4.4}$$

where $\hat{\theta}$, $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_3$ and c are operators defined in Section 2, $\varphi = 2\arctan\frac{x_1}{x_2}$.

Operators (4.4) commute with operator L of (2.6) and satisfy the following (anti)commutation relations

$$\begin{array}{l} [\hat{C}, H] = [\hat{C}, D] = [R_{ab}, H] = [R_{ab}, D] = [R_{ab}, \hat{C}] = [R, D] = [R, H] = 0, \\ \{\hat{C}, Q\} = \{\hat{C}, i\} = 0 \end{array} \tag{4.5}$$

Using (4.5) we can construct the analogues of the SOs (3.9) for the ABC system

$$\begin{array}{lll} Q_1 = (1 + \hat{C}R_{12})Q, & Q_2 = (R + \hat{C}R_{12})Q, & Q_3 = \hat{C}(1 + R_{13})Q, \\ Q_1 = (1 - \hat{C}R_{12})Q, & \bar{Q}_2 = (R - \hat{C}R_{12})Q, & \bar{Q}_3 = \hat{C}(1 - R_{13})Q \end{array} \tag{4.6}$$

Operators (4.6) satisfy relations (3.10) and so generate $N = 6$ SUSY for the ABC system. In analogy with (3.12) it is possible to construct a basis of algebra $ql(8, R)$ starting with SOs (4.6). In other words, this algebra is a symmetry algebra for the ABC system.

5 Discussion

We had shown that both the Coulomb and ABC systems admit rather extended symmetry which is much wider than the well known $so(2,4)$ symmetry of the relativistic Hydrogen atom. Here we discuss three aspects of this observation.

1. One of goals of the present paper is to show that the $gl(8,R)$ symmetry of the free Dirac equation indicated in papers [2,3] is kept also for the cases of the Coulomb and ABC systems. Using the technique proposed in [2,3] this symmetry can be used to decouple the related Dirac equation and to construct the corresponding complete sets of solutions.

2. It was demonstrated recently [1, 11] that Dirac and Schrődinger-Pauli equations for a charged particle interacting with the *magnetic* field admit extended SUSY provided the corresponding vector-potential has well defined properties w.r.t. parity transformations. This result presents a strong indication that extended SUSY is not a purely theoretical construction only, but is a rather common symmetry of a wide class of quantum mechanical systems.

In the present paper we show that extended SUSY is admitted by quantum mechanical systems which include *electric* field also. In other words, we indicate new origins of the extended SUSY in Nature.

3. A natural question arises, what kind of SUSY degeneration of energy spectra corresponds to the above mentioned extended symmetries. Starting with exact solutions $\psi_{E\kappa m}$ of equation (3.13) which are eigenfunctions of the commuting operators H, D and J_3 (refer, e.g., to book [1]) we recognize that operators (3.9) transform solutions with a fixed E into solutions with the same E, but change signs of κ and m and change relative phases of wave functions with different m. Such a degeneracy of the Hamiltonian eigenstates for the Hydrogen atom takes place indeed. We demonstrate that it admits SUSY interpretation.

We notice that the degeneracy w.r.t. the sign of κ became observable if we take into account the hyperfine splitting of the Hydrogen spectra. The degeneracy w.r.t. the sign of m and relative phases is not observable, but it can be in principle indicated in Coulomb systems perturbed by a magnetic field.

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