IRREDUCIBLE REPRESENTATIONS OF THE EXTENDED POINCARÉ PARASUPERALGEBRA

J.Niederle,
Institute of Physics, Academy of Sciences of the Czech Republic,
Na Slovance 2, Praque 8, Czech Republic,

and

A.G.Nikitin,
Institute of Mathematics, Academy of Sciences of the Ukraine,
3 Tereshenkivska Str., Kiev–4, Ukraine

Abstract

We classify and explicitly construct all irreducible hermitian representations of the extended Poincaré parasuperalgebra. These representations which include the representations of the usual extended Poincaré superalgebra as a particular case can serve as a group–theoretical foundation of parasupersymmetric quantum field theory, i.e., as a general viewpoint to reformulate quantum field theory and quantum mechanics of identical particles on the general basis of paraquantisation and supersymmetry.

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1. Introduction

It is pretty well known that description of identical particles in classical and quantum physics is completely different. Whereas in classical physics the identical particles can be enumerated and individually tracked, in quantum physics, due to Heisenberg's uncertainty relations, the notion of particle trajectory loses sense and identical particles become completely indistinguishable. Due to this and superposition principle the functions describing states of more than one identical particle should be always either symmetric or antisymmetric w.r.t. exchange of identical particles. Moreover, the particles described by symmetric (antisymmetric) functions obey the Bose-Einstein (Fermi-Dirac) statistics . Due to the theorem about connection between spin and statistics, the particles obeying the Fermi-Dirac statistics are associated with half-integer spins, i.e., they are fermions, and the particles obeying the Bose-Einstein statistics – with integer spins, i.e., they are bosons.

The description of a system of fermions or bosons in the so–called field quantization schemes (known as "second quantization") are quite different too. The field operators for bosons (i.e., the corresponding creation and annihilation operators) satisfy commutation relations whereas those for fermions satisfy anticommutation relations. Thus bosons and fermions obey different rules and have at first sight completely different physical properties.

The question whether they exhaust all possibilities in nature or whether there exist particles which satisfy other statistics and quantization rules, has been raised many times together with the question how bosons and fermions are connected and how can be mutually transformed one to the other.

At present there are several approaches clarifying the Fermi-Bose similarities. The first is associated with the so–called *parastatistics*. It was Gentile [1] in 1940 who first mentioned that there might be other statistics and quantization rules than just those found by Fermi and Dirac or Bose and Einstein. This was based on the fact that physical observables are bilinear forms of creation and annihilation operators but not these operators alone.

In 1950 Wigner [2] stressed another important fact, namely that the usual canonical commutation or anticommutation relations of field operators are only sufficient to derive the equation of motion of a given physical system but not necessary. The necessary conditions are expressed in terms of double commutation (anticommutation) relations among the field operators - relations which are much weaker than the usual canonical ones.

In 1953 Green [3] put Wigner's quantization and Gentile's statistics together and introduced the so–called parastatistics.

Let us recall that creation and annihilation operators of para–fields satisfy the relations

$$\begin{aligned} [[a_k^+, a_j]_\pm, a_m]_- &= -2\delta_{km} a_j, \\ [[a_k, a_l]_\pm, a_m]_- &= 0, \end{aligned} \tag{1.1}$$

together with the condition of that there exists a no–particle state

$$a_k a_l^+ \Phi_0 = p\delta_{kl}\Phi_0. \tag{1.2}$$

Here a_k and a_l^+ denote annihilation and creation para–field operators respectively, Φ_0 is the vacuum state, the upper (lower) sign corresponds to para–bose (para–fermi) case and integer $p > 0$ is the order of a given parastatistics.

It is known [4] that the above parastatistics relations reduce for $p = 1$ to the usual relations for fermions and bosons, and that the limit $p \to \infty$ for para-bose (para-fermi) statistics yelds in some sense the Fermi (Bose) theories (for other properties of parastatistics, their generalizations, etc., see, e.g., [5]).

The second approach which clarifies the connections between fermions and bosons is associated with supersymmetry – a new kind of symmetry which transforms bosonic states into the fermionic ones and vice versa realizes a factorization of the Dirac operator, and so on. It was introduced 30 years ago first in quantum field theory [6] (refer to [7] for reviews).

Supersymmetric quantum field theory has been followed soon by supersymmetric quantum mechanics [8]. In 1988 this mechanics was generalized to a special parasupersymmetric one [9] which deals with bosons and parafermions. Another approach to parasupersymmetric quantum mechanics namely with positive-definite Hamiltonians was proposed in [10]. Since its birth parasupersymmetric quantum mechanics has been a topics of many papers (see, for example,[11] and the references cited therein) then parasupersymmetric quantum field theory has appeared and begun to be discussed [12]. In contradistinction to supersymmetric quantum field theory, in which field operators satisfy the usual Bose-Einstein or Fermi-Dirac statistics, the field operators in parasupersymmetric quantum field theory satisfy parastatistics.

In papers [13,14] irreducible representations (IRs) of the simplest $N = 1$ (i.e., including only one parasupercharge) Poincaré parasuperalgebra were described.

In this paper (using the generalised Wigner method of induced representations) irreducible representations of the extended Poincaré parasuperalgebra $p(1,3;N)$ (i.e. the Poincaré parasuperalgebra with an arbitrary number N of parasupercharges, which includes the internal symmetry algebra) are classified and explicitly constructed. These representations form a group-theoretical basis of parasupersymmetric quantum field theory with N parasupercharges, that is the general standpoint for reformulate quantum field theory and quantum mechanics of identical particles on the general basis of paraquantization and supersymmetry.

We shall see that some IRs of the extended Poincaré *parasuperalgebra* appear to be simultaneously also IRs of the extended Poincaré *superalgebra*, so that they bring a deeper insight into the usual supersymmetric quantum field theory.

2. The extended Poincaré parasuperalgebra

The extended Poincaré parasuperalgebra $p(1,3;N)$ includes ten generators $P_\nu,\ J_{\nu\sigma}$ of the Poincaré group satisfying the usual commutation relations

$$\begin{gathered}
[P_\mu, P_\nu] = 0, \qquad [P_\mu, J_{\nu\sigma}] = i(g_{\mu\nu}P_\sigma - g_{\mu\sigma}P_\nu), \\
[J_{\mu\nu}, J_{\rho\sigma}] = i(g_{\mu\sigma}J_{\nu\rho} + g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\rho}), \\
J_{\mu\nu} = -J_{\nu\mu}, \qquad \mu,\nu = 0,1,2,3, \qquad g_{\nu\nu} = (1,-1,-1,-1),
\end{gathered} \tag{2.1}$$

and $4N$ parasupercharges $Q_A^j,\ \bar{Q}_A^j$ ($A = 1,2, j = 1,2,...,N$) which satisfy the double commutation relations

$$\begin{gathered}
[Q_A^i, [Q_B^j, Q_C^k]] = [\bar{Q}_A^i, [\bar{Q}_B^j, \bar{Q}_C^k]] = 0, \\
[Q_A^i, [\bar{Q}_B^j, Q_C^k]] = 4\delta_{ij} Q_C^k (\sigma_\mu)_{AB} P^\mu, \\
[\bar{Q}_A^i, [Q_B^j, \bar{Q}_C^k]] = 4\delta_{ij} \bar{Q}_C^k (\sigma_\mu)_{BA} P^\mu
\end{gathered} \tag{2.2}$$

and the following commutation relations with generators of the Poincaré group

$$\begin{array}{l}
[J_{\mu\nu}, Q_A^j] = -\frac{1}{2i}(\sigma_{\mu\nu})_{AB} Q_B^j,\ [P_\mu, Q_A^j] = 0, \\
[J_{\mu\nu}, \bar{Q}_A^j] = -\frac{1}{2i}(\sigma_{\mu\nu})^*_{AB} \bar{Q}_B^j,\ [P_\mu, \bar{Q}_A^j] = 0.
\end{array} \tag{2.3}$$

Here σ_ν are the Pauli matrices, $\sigma_{\nu\sigma} = -\sigma_{\sigma\nu} = \sigma_\nu \sigma_\sigma$, $(.)_{AB}$ are the related matrix elements, and the asterisk denotes the complex conjugation.

It is easily to see that the algebra $p(1, 3; N)$ is a direct and natural generalization of the extended Poincaré superalgebra [15]. The last one also includes $10 + 4N$ elements which satisfy (2.1), (2.3), but instead of (2.2) the supercharges $Q^j_A,\ \bar{Q}^j_A$ fulfill the following anticommutation relations

$$\begin{gathered}[Q^i_A, Q^j_B]_+ = 0, \quad [\bar{Q}^i_A, \bar{Q}^j_B]_+ = 0, \\ [Q^i_A, \bar{Q}^j_B]_+ = 2\delta_{ij}(\sigma_\mu)_{AB} P^\mu,\end{gathered} \tag{2.4}$$

which, whenever satisfied, imply that relations (2.2) are valid as well. However, the converse is not true.

Thus, representations of the extended Poincaré *superalgebra* appear as a particular representations of a more general algebraic structure − the extended Poincaré *parasuperalgebra* (cf. relations of the usual Fermi-Dirac or Bose-Einstein statistics with parastatistics [5].

The Poincaré parasuperalgebra (2.1)–(2.3) for a particular case $N = 1$ was studied in paper [13].

In the next sections we present the classification and explicit construction of IRs of the N-extended Poincaré parasuperalgebra defined by relations (2.1)–(2.3).

3. Classification of IRs

The IRs of the algebra p(1,3;N) can be specified by eigenvalues of the appropriate Casimir operators.

First let us note that the Casimir operator of the Poincaré algebra $C_1 = P_\mu P^\mu$ commutes with all parasupercharges $Q^i_A,\ \bar{Q}^i_A$ so that it is a Casimir operator for the Poincaré parasuperalgebra too. The second less obvious however essential Casimir operator for the algebra $p(1, 3; N)$ can be obtained by extending the usual Pauli–Lubanski four–vector

$$W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma \tag{3.1}$$

to the following vector

$$B_\mu = W_\mu + X_\mu, \tag{3.2}$$

where X_μ is defined by the following bilinear combinations of parasuper-charges

$$\begin{array}{l} X_0 = \Sigma_{i=1}^N \frac{1}{8}\{[Q_1^i, \bar{Q}_1^i] + [Q_2^i, \bar{Q}_2^i]\}, \ X_1 = \Sigma_{i=1}^N \frac{1}{8}\{[Q_1^i, \bar{Q}_2^i] + [Q_2^i, \bar{Q}_1^i]\}, \\ X_2 = \Sigma_{j=1}^N \frac{i}{8}\{[Q_2^j, \bar{Q}_1^j] + [\bar{Q}_2^j, Q_1^j]\}, \ X_3 = \Sigma_{i=1}^N \frac{1}{8}\{[\bar{Q}_1^i, Q_1^i] + [Q_2^i, \bar{Q}_2^i]\}. \end{array} \tag{3.3}$$

As follows from (2.1) and (2.2) four–vector B_μ satisfies the relations

$$[B_\mu, P_\nu] = 0, \qquad [B_\mu, J_{\nu\sigma}] = i(g_{\mu\nu}B_\sigma - g_{\mu\sigma}B_\nu), \tag{3.4a}$$

$$\begin{array}{ll} [B_\mu, Q_A^i] = \frac{1}{2}P_\mu Q_A^i, & [B_\mu, \bar{Q}_A^i] = -\frac{1}{2}P_\mu \bar{Q}_A^i, \\ [B_\mu, B_\nu] = i\varepsilon_{\mu\nu\rho\sigma}P^\rho B^\sigma & \end{array} \tag{3.4b}$$

from which we conclude that the operator

$$C_2 = P_\mu P^\mu B_\nu B^\nu - (B_\mu P^\mu)^2 \tag{3.5}$$

is the second Casimir operator of algebra $p(1,3;N)$.

We shall investigate relations (2.2),(3.4b), which define an algebra of operators B_ν, Q_A^i and $\bar{Q}_A^i$ for any fixed set of eigenvalues p_ν of operators P_ν.

As in the case of the ordinary Poincaré algebra [16] we shall classify IRs of algebra $p(1,3;n)$ according to the eigenvalues of C_1. We distinguish three classes of IRs, namely :

I. The time-like four–momentum case for which

$$P_\mu P^\mu = M^2 > 0, \tag{3.6a}$$

II. The light–like four-momentum case for which

$$P_\mu P^\mu = 0 \tag{3.6b}$$

and

III. The space–like four-momenta case for which

$$P_\mu P^\mu = -\eta^2 < 0. \tag{3.6c}$$

They will be studied separately in Sections 4-7. The internal symmetries of algebra $p(1,3;n)$ and their representations will be introduced in Section 8, and physical relevance of the representations of $p(1,3;N)$ will be discussed in Summary.

4. IRs of class I

For the case time-like representations (3.6a) there exists an additional Casimir operator, namely, $C_3 = P_0/|P_0|$ whose eigenvalues are $\varepsilon = \pm 1$. First we shall consider the case $\varepsilon = +1$ and determine "a Wigner little parasuperalgebra " a_I associated with the time–like four–momentum taken in the form $P = (M, 0, 0, 0)$. For this particular choise of P we define the three-vector j_k by the indentities

$$B_k = W_k + X_k = -MS_k + X_k \equiv Mj_k, \qquad k = 1, 2, 3 \tag{4.1}$$

and find that relations (3.4b) take the form

$$[B_0, Q_A^i] = \frac{M}{2} Q_A^i, \qquad [B_0, \bar{Q}_A^i] = -\frac{M}{2} \bar{Q}_A^i, \tag{4.2}$$

$$[j_k, Q_A^i] = [j_k, \bar{Q}_A^i] = 0. \tag{4.3}$$

Moreover, vector j_k fulfills the commutation relations

$$[j_k, j_j] = i\varepsilon_{kjl} j_l, \tag{4.4}$$

and the corresponding relations (2.2) reduce to

$$\begin{gathered} [Q_A^i, [\bar{Q}_B^j, Q_C^k]] = 4\delta_{ij}\delta_{AB} M Q_C^k, \qquad [\bar{Q}_A^i, [Q_B^j, \bar{Q}_C^k]] = 4\delta_{ij}\delta_{AB} M \bar{Q}_C^k, \\ [Q_A^i, [Q_B^j, Q_C^k]] = [\bar{Q}_A^i, [\bar{Q}_B^j, \bar{Q}_C^k]] = 0. \end{gathered} \tag{4.5}$$

Let us remark two things: first, that according to (3.3), relations (4.2) turn to identities provided relations (4.5) are true, and secondly, that relations (4.5) are equivalent to relations (1.1) for parafermions.

Thus we see that the Wigner little parasuperalgebra a_I is just a direct sum of the Lie algebra whose basis elements are j_a and of the algebra of operators Q_A, $\bar{Q}_A$ characterized by double commutation relations (4.5).

Relations (4.4) define $so(3)$ – the Lie algebra of the rotation group $SO(3)$. IRs of this algebra are well known and are determined by integers or half–integers j (see, e.g., [17]).

Relations (4.5) determine the algebra of $2N$ creation and annihilation operators for parafermions. These operators form a representation of the algebra $so(4N + 1)$ [18].

To prove the isomorphism of algebra (4.5) and $so(4N+1)$ explicitly we express supercharges $Q_A^i, \bar{Q}_A^i$ and their commutators in terms of generators $S_{kl} = -S_{lk}$ $(k,l = 1,2,...,4N+1)$ of $so(4N+1)$:

$$\begin{array}{l} Q_A^i = -(-1)^A\sqrt{2M}(S_{4N+1\ 2N(A-1)+2i-1} - iS_{4N+1\ 2N(A-1)+2i}), \\ \bar{Q}_A^i = -(-1)^A\sqrt{2M}(S_{4N+1\ 2N(A-1)+2i-1} + iS_{4N+1\ 2N(A-1)+2i}), \end{array} \tag{4.6}$$

$$\begin{array}{c} [Q_A^i, \bar{Q}_B^k] = (-1)^{(A+B)}2M\left[-S_{2N(A-1)+2i-1\ 2N(B-1)+2k}\right. \\ -S_{2N(B-1)+2k-1\ 2N(A-1)+2i} + i\left(S_{2N(A-1)+2i-1\ 2N(B-1)+2k-1}\right. \\ \left.\left.+S_{2N(A-1)+2i\ 2N(B-1)+2k}\right)\right], \\ [Q_A^i, Q_B^k] = (-1)^{(A+B)}2M\left[S_{2N(A-1)+2i-1\ 2N(B-1)+2k}\right. \\ +S_{2N(A-1)+2i\ 2N(B-1)+2k-1} + i\left(S_{2N(A-1)+2i-1\ 2N(B-1)+2k-1}\right. \\ \left.\left.-S_{2N(A-1)+2i\ 2N(B-1)+2k}\right)\right], \\ [\bar{Q}_A^i, \bar{Q}_B^k] = (-1)^{(A+B)}2M\left[-S_{2N(A-1)+2i-1\ 2N(B-1)+2k}\right. \\ -S_{2N(A-1)+2i\ 2N(B-1)+2k-1} + i\left(S_{2N(A-1)+2i-1\ 2N(B-1)+2k-1}\right. \\ \left.\left.-S_{2N(A-1)+2i\ 2N(B-1)+2k}\right)\right]. \end{array} \tag{4.7}$$

Formulae (4.6),(4.7) are invertible, so that

$$\begin{array}{c} S_{4N+1\ 2N(A-1)+2i-1} = -(-1)^A\frac{1}{2\sqrt{2M}}\left(\bar{Q}_A^i + Q_A^i\right), \\ S_{4N+1\ 2N(A-1)+2i} = (-1)^A\frac{i}{2\sqrt{2M}}\left(\bar{Q}_A^i - Q_A^i\right), \\ S_{2N(A-1)+2i-1\ 2N(B-1)+2k} = -(-1)^{(A+B)}\frac{1}{8M}\left[\bar{Q}_A^i + Q_A^i,\ \bar{Q}_B^k - Q_B^k\right], \\ S_{2N(A-1)+2i-1\ 2N(B-1)+2k-1} = -(-1)^{(A+B)}\frac{i}{8M}\left[\bar{Q}_A^i - Q_A^i,\ \bar{Q}_B^k + Q_B^k\right], \\ S_{2N(A-1)+2i\ 2N(B-1)+2k} = (-1)^{(A+B)}\frac{i}{8M}\left[\bar{Q}_A^i - Q_A^i,\ \bar{Q}_B^k - Q_B^k\right]. \end{array} \tag{4.8}$$

Using (4.5), (4.8) we obtain the following commutation relations:

$$[S_{kl}, S_{mn}] = -i(g_{km}S_{ln} + g_{ln}S_{km} - g_{kn}S_{lm} - g_{lm}S_{kn}) \tag{4.9}$$

(where $g_{kl} = -\delta_{kl}$, δ_{kl} being the Kronecker symbol) which are specific for the Lie algebra $so(4N+1)$.

The IRs of algebra $so(4N+1)$ are determined by the set of numbers $(n_1, n_2, ..., n_{2N})$ which are either integer or half–integer and satisfy the inequalities $n_1 \geq n_2 ... \geq n_{2N} \geq 0$. For the explicit form of matrices S_{kl} see, e.g., [18].

Thus we have proved that for $P_\nu P^\nu > 0$ the algebra a_I is equivalent to a direct sum of the algebras $so(3)$ and $so(4N+1)$

$$a_I = so(3) \oplus so(4N+1). \tag{4.10}$$

As follows from the above discussion the IRs of algebra $p(1,3;n)$, which belong to class I with positive energy, are labelled by the following sets of numbers $(M, j, \varepsilon = 1, n_1, n_2, ..., n_{2N})$. The explicit expression of the corresponding Pauli–Lubanski vector and of the parasupercharges can be found from (3.2), (4.1), (4.6) by means of the Lorentz transformation (specified in the Appendix) and are of the form:

$$W_0 = p_a S_a, \qquad W_a = \varepsilon M S_a + \frac{p_a S_b p_b}{(E+M)}, \tag{4.11}$$

$$\begin{gathered}
Q_1^i = \tfrac{1}{\sqrt{E+M}}[(S_{4N+1\ 2i-1} - iS_{4N+1\ 2i})(E+M+\varepsilon p_3) \\
-\varepsilon(S_{4N+1\ 2N+2i-1} - iS_{4N+1\ 2N+2i})(p_1 - ip_2)], \\
Q_2^i = \tfrac{1}{\sqrt{E+M}}[\varepsilon(S_{4N+1\ 2i-1} - iS_{4N+1\ 2i})(p_1 + ip_2) \\
-(S_{4N+1\ 2N+2i-1} - iS_{4N+1\ 2N+2i})(E+M-\varepsilon p_3)], \\
\bar{Q}_A = Q_A^+,
\end{gathered} \tag{4.12}$$

where

$$\begin{gathered}
\varepsilon = 1, \qquad E = \sqrt{M^2 + p^2}, \qquad p^2 = p_1^2 + p_2^2 + p_3^2, \\
S_a = \frac{1}{2}\Sigma_{i=0}^{n-1}\left(\frac{1}{2}\varepsilon_{a\ b+4i\ c+4i} S_{b+4i\ c+4i} + S_{4(i+1)\ a+4i}\right)
\end{gathered} \tag{4.13}$$

and ε_{abc} is absolutely antisymmetric tensor of rank 3. The explicit expressions for the generators of the Poincaré group, corresponding to the Pauli–Lubanski vector (4.11) are given by

$$\begin{gathered}
P_0 = \varepsilon E, \qquad P_a = p_a, \\
J_{ab} = x_a p_b - x_b p_a + \varepsilon_{abc} S_c, \\
J_{0a} = x_0 p_a - \tfrac{i\varepsilon}{2}[\tfrac{\partial}{\partial p_a}, E]_+ - \varepsilon \tfrac{\varepsilon_{abc} p_b S_c}{E+M},
\end{gathered} \tag{4.14}$$

where $x_a = i\frac{\partial}{\partial p_a}$, and x_0 is a parameter which can be set zero without loss of generality.

Let us remark that operators (4.12)–(4.14) are hermitian w.r.t. the following scalar product

$$(\psi_1, \psi_2) = \int_{\infty}^{\infty} \psi_1^{+} \psi_2 d^3 p, \tag{4.15}$$

where $\psi_\alpha = \psi_\alpha(\mathbf{p})$ are m–component wave functions, forming a basis of the m– dimensional IR $D(n_1, n_2, ..., n_{2N})$ of the algebra $so(4N+1)$.

In contrast to the case of Poincaré superalgebra [7] for which the energy sign operator has positive eigenvalues only, the algebra $p(1,3;N)$ admits representation with both signs of the Casimir operator C_3.

For the case $\varepsilon = -1$, i.e., for negative energy, the rest frame four-momentum is of the form $P = (-M, 0, 0, 0)$ with $M > 0$. In this frame relations (2.2) are of the form

$$\begin{gathered} [Q^i_A, [\bar{Q}^j_B, Q^k_C]] = -4\delta_{ij}\delta_{AB} M Q^k_C, \;\; [\bar{Q}^i_A, [Q^j_B, \bar{Q}^k_C]] = -4\delta_{ij}\delta_{AB} M \bar{Q}^k_C, \\ [Q^i_A, [Q^j_B, Q^k_C]] = [\bar{Q}^i_A, [\bar{Q}^j_B, \bar{Q}^k_C]] = 0 \end{gathered} \tag{4.16}$$

(cf. (4.5)).

Similarly to (4.6)–(4.8) it is possible to show that algebra (4.16) is isomorphic to algebra $so(1,4N)$ whose representations are discussed in [19]. The corresponding basis elements of the algebra $p(1,3;N)$ have again the form (4.12)-(4.14) where, however, $\varepsilon = -1$ and S_{kl} are basis elements of an IR of the algebra $so(1,4N)$ which satisfy commutation relations (4.9) (with nonzero components of $g_{\mu\nu}$ being $g_{\nu\nu} = -1, \nu \neq 4N+1$ and $g_{4N+1\ 4N+1} = 1$). A description of IRs of the algebra $so(1,4N)$ in all details can be found in the book [19].

Thus, in this section we have enumerated all IRs of class I of the extended Poincaré parasuperalgebra $p(1,2;N)$ and found the explicit expressions of the basis elements.

5. IRs of class II

To this class of IRs we have again the additional Casimir operator $C_3 = P_0/|P_0|$ with the eigenvalues $\varepsilon = \pm 1$. As previously we shall consider first the case $\varepsilon = +1$.

To determine the corresponding little Wigner parasuperalgebra a_{II} we choose the light–like four–momentum P in the form $P = (M, 0, 0, M)$. Then algebra (2.2) reduces to the form

$$\begin{gathered}[Q_2^i,[\bar{Q}_2^k,Q_2^j]] = 8M\delta_{ik}Q_2^j, \qquad [\bar{Q}_2^i,[Q_2^k,\bar{Q}_2^j]] = 8M\delta_{ik}\bar{Q}_2^j, \\ [Q_2^i,[Q_2^k,Q_2^j]] = [\bar{Q}_2^i,[\bar{Q}_2^k,\bar{Q}_2^j]] = 0,\end{gathered} \tag{5.1}$$

$$\begin{gathered}[Q_2^i,[\bar{Q}_2^k,Q_1^j]] = 8M\delta_{ik}Q_1^j, \qquad [\bar{Q}_2^i,[Q_2^k,\bar{Q}_1^j]] = 8M\delta_{ik}\delta_{AB}\bar{Q}_1^j, \\ [Q_1^i,[\bar{Q}_1^k,Q_A^j]] = [\bar{Q}_1^i,[Q_1^k,Q_A^j]] = 0, \\ [Q_A^i,[Q_B^k,Q_C^i]] = [\bar{Q}_A^i,[\bar{Q}_B^k,\bar{Q}_C^j]] = 0.\end{gathered} \tag{5.2}$$

Now expressing Q_A^j in terms of S_{kl} $(k, l = 1, 2, ...2N + 1)$, namely

$$\begin{aligned} Q_2^j &= 2\sqrt{M}\left(S_{2N+1\ 2j} + iS_{2N+1\ 2j-1}\right), \\ \bar{Q}_2^j &= 2\sqrt{M}\left(S_{2N+1\ 2j} - iS_{2N+1\ 2j-1}\right), \\ \left[\bar{Q}_2^k, Q_2^j\right] &= 4M\left(iS_{2k\ 2j} + iS_{2k-1\ 2j-1} + S_{2k-1\ 2j} - S_{2k\ 2j-1}\right). \end{aligned} \tag{5.3}$$

we find that operators S_{kl} satisfy relations (4.9) with $g_{kl} = -\delta_{kl}$, i.e., they form a basis of algebra $so(2N + 1)$. Since relations (5.3) are invertible,

$$\begin{aligned} S_{2N+1\ 2j} &= \tfrac{1}{4\sqrt{M}}\left(Q_2^j + \bar{Q}_2^j\right), \\ S_{2N+1\ 2j-1} &= \tfrac{i}{4\sqrt{M}}\left(-Q_2^j + \bar{Q}_2^j\right), \\ S_{2j\ 2k} &= -\tfrac{i}{16M}\left[Q_2^j + \bar{Q}_2^j, Q_2^k + \bar{Q}_2^k\right], \\ S_{2j-1\ 2k-1} &= \tfrac{i}{16M}\left[Q_2^j - \bar{Q}_2^j, Q_2^k - \bar{Q}_2^k\right], \\ S_{2j\ 2k-1} &= -\tfrac{1}{16M}\left[Q_2^j + \bar{Q}_2^j, Q_2^k - \bar{Q}_2^k\right], \end{aligned} \tag{5.4}$$

algebra (5.1) reduces to the algebra $so(2N + 1)$ whose IRs are labelled by the set of N numbers which are all integer or half integer and satisfy the inequalities $n_1 \geq n_2 \geq ... \geq n_N \geq 0$.

In order to describe the structure of the little Wigner parasuperalgebra a_{II} we notice two facts. First, the relations (5.2) have only trivial solutions for Q_1^j and $\bar{Q}_1^j$.

Secondly, in accordance with (3.4b), (5.4), (4.4) we obtain

$$B_3 = B_0,\ [B_0, B_1] = iMB_2,\ [B_0, B_2] = -iMB_1,\ [B_0, B_1] = 0. \tag{5.5}$$

Defining operators $T_0,\ T_1,\ T_2$ by the following formulae

$$B_0 = W_0 + X_0 \equiv M(T_0 - \tfrac{1}{2}(Nn_1 - \hat{S}_3)),\ B_1 = W_1 \equiv T_1 \quad B_2 = W_2 \equiv T_2, \tag{5.6a}$$

where n_1 is the main quantum number characterizing the IR of the algebra $so(2N+1)$,

$$\hat{S}_3 = S_{12} + S_{34} + \cdots + S_{2N-1\ 2N}, \tag{5.6b}$$

we obtain from (5.5) the following relations

$$[T_0, T_1] = iT_2, \qquad [T_0, T_2] = -iT_1, \qquad [T_1, T_2] = 0, \tag{5.7}$$

$$[T_0, S_{ab}] = [T_1, S_{ab}] = [T_2, S_{ab}] = 0. \tag{5.8}$$

We conclude from (5.8) that algebra a_{II} for representations of Class II is a direct sum of the algebras $so(2N+1)$ and $e(2)$, which are determined by relations (4.4) and (5.7) respectively.

Thus, we have found the explicit form of operators $W_\nu,\ Q^i_A,\ \bar{Q}^i_A$ in the reference frame $P = (M, 0, 0, M)$. To obtain explicit expressions for these operators (and the corresponding generators $P_\nu,\ J_{\nu\sigma}$) in an arbitrary reference frame it is sufficient to make the corresponding rotation. As a result we get

$$\begin{array}{l} Q^i_1 = \frac{\sqrt{2}(-p_1+ip_2)}{\sqrt{p+p_3}}(S_{2N+1\ 2i} - iS_{2N+1\ 2i-1}),\\ \bar{Q}^i_1 = \frac{\sqrt{2}(-p_1-ip_2)}{\sqrt{p+p_3}}(S_{2N+1\ 2i} + iS_{2N+1\ 2i-1}),\\ Q^i_2 = \sqrt{2(p+p_3)}(S_{2N+1\ 2i} - iS_{2N+1\ 2i-1}),\\ \bar{Q}^i_2 = \sqrt{2(p+p_3)}(S_{2N+1\ 2i} + iS_{2N+1\ 2i-1}), \end{array} \tag{5.9}$$

and

$$\begin{array}{c} P_0 = \varepsilon p, \qquad\qquad P_a = p_a,\\ J_{ab} = x_a p_b - x_b p_a + \varepsilon_{abc}\hat{T}_0\frac{p_c+\delta_{c3}p}{p+p_3},\\ J_{0a} = x_0 p_a - \frac{1}{2}\varepsilon[p, x_a]_+ + \frac{\varepsilon_{abc}T_b p_c}{p^2} - \frac{\varepsilon_{abc}p_b n_c(\varepsilon\hat{T}_0 p^2 - T_a p_a)}{p^2(p+p_3)}, \end{array} \tag{5.10}$$

where

$$p = \sqrt{p_1^2 + p_2^2 + p_3^2},\ n_1 = n_2 = 0,\ n_3 = 1,\ T_3 = 0,\ \hat{T}_0 = T_0 - \frac{1}{2}(Nn_1 + \hat{S}_3),$$

$\hat{S}_3$ is defined in (5.6b), and T_0, T_1, T_2 are basis elements of an IRs of algebra $e(2)$ specified in(3.7).

In the important case $T_1^2+T_2^2=0$ (i.e., for representations with a discrete spin), formulae (5.10) are simplified and reduced to the form

$$\begin{gathered} P_0=\varepsilon p, \qquad P_a=p_a, \\ J_{ab}=x_ap_b-x_bp_a+\varepsilon_{abc}(2\lambda-Nn_1-\hat{S}_3)\tfrac{p_c+\delta_{c3}p}{p+p_3}, \\ J_{0a}=x_0p_a-\tfrac{1}{2}\varepsilon[p,x_a]_+\tfrac{\varepsilon_{abc}p_bn_c(2\lambda-Nn_1-\hat{S}_3)}{p^2(p+p_3)}, \end{gathered} \tag{5.11}$$

where both λ and $n_1>0$ are either arbitrary integers or half–integers.

For $\varepsilon=-1$ we can choose the light-like four-momentum in the form $P=(-M,0,0,-M)$. The relations corresponding to (2.2) can be obtained from (5.1), (5.2) by taking $M\to -M$. Then, using expression (5.3) again, we conclude, that S_{kl} satisfy relations (4.9) where the nonzero components of $g_{\mu\nu}$ are $g_{nn}=-1, n<2N+1$, and $g_{2N+1/2N+1}=1$. In other words, S_{kl} form just the algebra $so(1,2N)$, and the algebra a_{II} is a direct sum $e(2)\oplus so(1,2N)$.

Consequently the above IRs of the extended Poincaré parasuperalgebra for the light-like four-momenta are different qualitatively for $\varepsilon=+1$ and $\varepsilon=-1$. In the case $\varepsilon=+1$ these IRs are labelled by $N+2$quantum numbers $\varepsilon=1, r, n_1, n_2, ..., n_N$ (where $n_1, n_2, ..., n_N$ are Gelfand-Zetlin numbers for $so(2N+1)$, and r is an eigenvalue of the Casimir operator $T_1^2+T_2^2$ of algebra $e(2)$), or for $r=0$ by $(\varepsilon=1,\lambda,n_1,n_2,...,n_N)$ (where λ are integers or half integers). For $\varepsilon=-1$ the IRs are specified by eigenvalues of the Casimir operators of noncompact algebras $e(2)$ and $so(1,2N)$, which are described, e.g., in [15] and [19].

6. IRs of class III

To obtain the corresponding little Wigner parasuperalgebra a_{III} we choose the space–like four–momentum in the form $P=(0,0,0,\eta)$. The corresponding double commutation relations (2.2) reduce to the form

$$\begin{gathered} [Q_A^i,[\bar{Q}_B^j,Q_C^k]]=(-1)^A\delta_{ij}\delta_{AB}4\eta Q_C^k,\ [\bar{Q}_A^i,[Q_B^j,\bar{Q}_C^k]]=(-1)^A\delta_{ij}\delta_{AB}4\eta\bar{Q}_C^k, \\ [Q_A^i,[Q_B^J,Q_C^k]]=[\bar{Q}_A^i,[\bar{Q}_B^j,\bar{Q}_C^k]]=0. \end{gathered} \tag{6.1}$$

Introducing operators $\tilde{J}_{01}, \tilde{J}_{02}$ and $\tilde{J}_{12}$ in accordance with the following relations

$$\begin{gathered} \eta\tilde{J}_{01}\equiv B_2=J_{01}\eta+X_2,\ \eta\tilde{J}_{12}\equiv B_1=-J_{02}\eta+X_1, \\ \eta\tilde{J}_{02}\equiv B_0=-J_{12}\eta+X_0, \end{gathered} \tag{6.2}$$

and taking into account that $B_3 = X_3$, we find from (3.4b), that

$$[\tilde{J}_{\alpha\beta}, Q_A] = [\tilde{J}_{\alpha\beta}, \bar{Q}_A] = 0, \qquad \alpha, \beta = 0, 1, 2, \tag{6.3}$$

$$[\tilde{J}_{\alpha\beta}, \tilde{J}_{\rho\sigma}] = i(g_{\alpha\sigma}\tilde{J}_{\beta\rho} + g_{\beta\rho}\tilde{J}_{\alpha\sigma} - g_{\alpha\rho}\tilde{J}_{\beta\sigma} - g_{\beta\sigma}\tilde{J}_{\alpha\rho}), \tag{6.4}$$

where $g_{00} = -g_{11} = -g_{22} = 1, \qquad g_{\alpha\beta} = 0, \qquad \alpha \neq \beta.$

In accordance with (6.1)–(6.4) the algebra a_{III} corresponding to space–like momenta is equivalent to the direct sum of algebra $so(1,2)$ (defined by relations (6.4)) and the algebra, defined by the double commutation relations (6.1). Expressing the parasupercharges in terms of S_{kl} via (4.6), (4.7) with M replaced by η we conclude that the corresponding matrices S_{kl} satisfy relations (4.9), where the nonzero components of $g_{\mu\nu}$ are $g_{nn} = 1, n < 2N+1$ and $g_{nn} = -1, n \geq 2N$. In other words, relations (6.1) specify the algebra $so(2N, 2N+1)$.

Thus we have shown that the Wigner little parasuperalgebra a_{III} for representations of class III is isomorphic to a direct sum of two noncompact algebras, namely, of $so(1,2)$ and $so(2N, 2N+1)$.

Taking the generators $P_\mu, J_{\mu\nu}, Q^i_A$ and $\bar{Q}^i_A$ in the form expressed in (6.2), (4.6), (3.1) and performing Lorentz transformation and rotation corresponding to their transition to an arbitrary reference frame (see (A.5), (A.6)), we find that the basis elements of the extended Poincaré parasuperalgebra can be written as

$$\begin{gathered}
P_\mu = p_\mu, \quad J_{ab} = x_a p_b - x_b p_a + \tilde{S}_{ab}, \\
J_{0a} = x_0 p_a - \tfrac{1}{2}[x_a, p_0]_+ + \tilde{S}_{0a}, \\
J_{a3} = x_a p_3 - x_3 p_a - \tfrac{\tilde{S}_{ab}p_b - \tilde{S}_{a0}p_0}{p_3+\eta}, \\
J_{03} = x_0 p_3 - \tfrac{1}{2}[x_3, p_0]_+ - \tfrac{\tilde{S}_{0a}p_a}{p_3+\eta}, \\
Q^i_1 = \tfrac{1}{\sqrt{\eta+p_3}}[(S_{4N+1\ 2i-1} - iS_{4N+1\ 2i})(\eta + p_3 - p_0) \\
+(S_{4N+1\ 2N+2i-1} - iS_{4N+1\ 2N+2i})(p_1 - ip_2)], \\
Q^i_2 = \tfrac{1}{\sqrt{\eta+p_3}}[(S_{4N+1\ 2i-1} - iS_{4N+1\ 2i})(p_1 + ip_2) \\
-(S_{4N+1\ 2N+2i-1} - iS_{4N+1\ 2N+2i})(\eta + p_3 + p_0)], \\
\bar{Q}_A = Q^+_A,
\end{gathered} \tag{6.7}$$

where

$$\begin{gathered}
p_0^2 = \mathbf{p}^2 - \eta^2 \quad \tilde{S}_{12} = \tilde{J}_{12} + S_3 \\
\tilde{S}_{01} = \tilde{J}_{01} + S_1, \quad \tilde{S}_{02} = \tilde{J}_{02} + S_2.
\end{gathered}$$

Here $\tilde{J}_{\alpha\beta}$ are basis elements of the algebra $so(1,2)$ introduced in (6.4) and S_a are defined in (4.13) with S_{kl} being elements of the algebra $so(2N+1, 2N)$.

7. Special representations

In this section we shall show how to construct special representations of the algebra $p(1,3;N)$ namely representations in which the Poincaré generators P_μ, $J_{\mu\nu}$ have the form

$$P_\mu = i\frac{\partial}{x_\mu}, \qquad J_{\mu\nu} = x_\mu\frac{\partial}{x_\nu} - x_\nu\frac{\partial}{x_\mu} + S_{\mu\nu} \tag{7.1}$$

with $S_{\nu\sigma}$ being numerical matrices. These representations (in which the spin part $S_{\mu\nu}$ of any generator $J_{\mu\nu}$ commutes with the orbital part $x_\mu\frac{\partial}{x_\nu} - x_\nu\frac{\partial}{x_\mu}$) are frequently used in various physical applications.

We take $S_{\nu\sigma}$ in the form

$$S_{ab} = \varepsilon_{abc}S_c, \qquad S_{0a} = iS_a, \tag{7.2}$$

where S_a denote matrices defined in (4.13). Then the corresponding parasupercharges can be expressed as

$$\begin{array}{c} Q_1^j = \sqrt{2M}(S_{4N+1\ 2j-1} - iS_{4N+1\ 2j}), \\ Q_2^j = -\sqrt{2M}[(S_{4N+1\ 2N+2j-1} - iS_{4N+1\ 2N+2j}) \\ \bar{Q}_1 = \sqrt{\frac{2}{M}}[(p_3 - p_0)(S_{4N+1\ 2j-1} + iS_{4N+1\ 2j}) \\ +(p_1 + ip_2)(S_{4N+1\ 2N+2j-1} + iS_{4N+1\ 2N+2j})], \\ \bar{Q}_2 = -\sqrt{\frac{2}{M}}[(p_0 + p_3)(S_{4N+1\ 4N+2j-1} + iS_{4N+1\ 4N+2j}) \\ +(p_1 - ip_2)(S_{4N+1\ 2j-1} + iS_{4n+1\ 2j})]. \end{array} \tag{7.3}$$

It is easy to verify that operators (7.1)-(7.3) satisfy relations (2.1), (2.2), (2.4) and consequently realise a representation of the extended Poincaré parasuperalgebra. Moreover, assuming that $P_\nu P^\nu = M^2 > 0$, $p_0 = E = (p^2 + M^2)^{1/2}$, it is possible to prove that this representation is irreducible. Indeed, the corresponding generators (7.2), (7.3) can be reduced to the form (4.18), (4.19) using the transformation

$$\begin{array}{ll} J_{\mu\nu} \to UJ_{\mu\nu}U^{-1}, & P_\mu \to UP_\mu U^{-1}, \\ Q_A \to UQ_AU^{-1}, & \bar{Q}_A \to U\bar{Q}_AU^{-1}, \end{array} \tag{7.4}$$

where

$$U = \exp\left(\frac{iS_{0a}p_a}{p}\text{artanh}\frac{p}{E}\right) \tag{7.5}$$

We notice that supercharges Q^j_A and $\bar{Q}^j_A$ are not conjugated w.r.t. the usual scalar product (4.15). However, they are conjugated w.r.t. the following scalar product

$$(\psi_1, \psi_2) = \int \psi_1 M \psi_2 d^3x$$

in which $M = U^+U = exp\left(\frac{2iS_{0a}p_a}{p}\text{artanh}\frac{p}{E}\right)$ is a positive definite metric operator.

8. Internal Symmetries

Now shall demonstrate that any IR of algebra $p(1,3;N)$ described in the previous sections, can be extended to a representation of the algebra including $p(1,3;N)$ and the internal symmetry algebra. In other words, the carier space of an IR of the extended Poincaré parasuperalgebra $p(1,2;N)$, appears to be also a carier space of a representation of the algebra $u(N)$ whose the N^2, generators $T_{ab}, \quad a,b = 1,2,...,N$, satisfy the following relations

$$[P_\mu, T_{ab}] = [J_{\mu\nu}, T_{ab}] = 0, \tag{8.1a}$$

$$\left[T_{ab}, Q^i_A\right] = f^{ik}_{ab} Q^k_A, \tag{8.1b}$$

$$[T_{ab}, T_{cd}] = \delta_{bc} T_{ab} - \delta_{ad} T_{bc}. \tag{8.1c}$$

The last relation just determines the algebra $u(N)$ in the Okubo basis [20].

Let us start with the case of light-like four-momenta considered in Section 5. The corresponding parasupercharges are represented in terms of matrices S_{kl} satisfying algebra $so(2N+1)$. The related generators of the internal symmetry algebra can be expressed in terms of basis elements of the algebra $so(2N+1)$ as follows:

$$\begin{array}{l} T_{nn} = \frac{1}{N}\left(1 - \hat{S}_3 + N S_{2n-1\ 2n}\right), \\ T_{ab} = \frac{1}{2}\left(S_{2a\ 2b-1} - S_{2a-1\ 2b} + iS_{2b-1\ 2a-1} - iS_{2a\ 2b}\right), \quad b > a, \\ T_{ba} = T^+_{ab}, \end{array} \tag{8.2}$$

where $\hat{S}_3$ is the matrix defined in (5.6b), $a, b, = 1, 2, ..., N$.

It is easy to verify that operators (8.2) satisfy relations (8.1), where

$$f^{ik}_{ab} = \left\{ \begin{array}{ll} \delta_{in}\delta_{ka} - \frac{1}{N}\delta_{ik}, & a = b = n, \\ 2\delta_{ia}\delta_{kb}, & b > a, \\ 2\delta_{ib}\delta_{ka}, & b < a. \end{array} \right. \tag{8.3}$$

For the case $P_\mu P^\mu > 0$ the parasupercharges can be expressed in terms of basis elements of algebra $so(4N+1)$ and the generators of the internal symmetry algebra can be chosen in the form

$$\begin{array}{c} T_{nn} = \frac{1}{N}\left(1 - \Lambda + NS_{2n-1\ 2n} + NS_{2N+2n-12N+2n},\right), \\ T_{ab} = \frac{1}{2}\left(S_{2a\ 2b-1} - S_{2a-1\ 2b} + S_{2N+2a\ 2N+2b-1} - S_{2N+2a-1\ 2N+2b}\right. \\ \left.+iS_{2b-1\ 2a-1} - iS_{2a\ 2b} + iS_{2N+2b-1\ 2N+2a-1} - iS_{2N+2a\ 2N+2b}\right), \quad a<b, \\ T_{ba} = T_{ab}^{+}, \end{array} \tag{8.4}$$

where

$$\Lambda = S_{12} + S_{34} + S_{56} + ... + S_{4N-1\ 4N}, \ \ a, b, = 1, 2, ..., N.$$

The commutation relations of T_{ab} with $P_\mu,\ J_{\mu\nu},\ Q_A^i$ are given in (8.1), (8.3).

For IRs of Class III generators of the internal symmetry group have again the form (8.4) where, however, S_{kl} are basis elements of the Lie algebra of the non-compact group $SO(2N, 4N+1)$.

Thus, as in the case of the Poincaré superalgebra (cf. [2,3]), algebra $p(1,3;N)$ can be extended by generators of internal symmetry group which is nothing else then the familiar group $U(N)$. These generators can be expressed in terms of linear combinations of basis elements of the orthogonal algebras which generate IRs of the extended Poincaré parasuperalgebra.

9. Discussion

In the previous sections we have classified and constructed IRs of the Poincaré parasuperalgebra with an arbitrary number of parasupercharges. Moreover, we present the explicit form of basis elements of algebra $p(1,3;N)$ in terms of matrices belonging to IRs of (pseudo)orthogonal algebras $so(4N+1), so(1,4N), ...$, etc. In other words we had taken an alternative to usual paragrasmanian variables [5].

Let us briefly discuss the spin content of the corresponding parasupermultiplets and possible physical interpretations of the obtained representations.

We start with IRs of class I. These representations are reducible w.r.t. the Poincaré algebra $p(1,3)$ which is a subalgebra of $p(1,3;N)$.

Let us consider the case when the internal spin j equal to zero. Starting with (4.14) and calculating the corresponding Casimir operator $C = W_\nu W^\nu$ of the subalgebra $p(1,3)$ we obtain

$$W_\mu W^\mu = M^2 \mathbf{S^2} \tag{9.1}$$

where $\mathbf{S} = (S_1, S_2, S_3)$ and S_a are matrices defined in (4.13). These matrices form a subalgebra $so(3)$ of algebra $so(4N+1)$ and realize a reducible representation of this subalgebra. Reducing the IR $D(n_1, n_2, ..., n_{2N})$ to the corresponding representations of the spin algebra $so(3)$ we obtain the following set of eigenvalues of the Casimir operator of the Poincaré group

$$\begin{gathered} W_\mu W^\mu \psi = -M^2 s(s+1)\psi, \\ s = N\frac{n_1+n_2}{2}, N\frac{n_1+n_2-1}{2}, N\frac{n_1+n_2-2}{2}, \ldots, 0. \end{gathered} \tag{9.2}$$

where n_1 and n_2 is the first and second quantum number enumerating the corresponding IR of the algebra $so(4N+1)$.

For the case $j \neq 0$ (see (4.7), (4.14)) the possible spin values can be found as a result of summation of the two momenta, i.e., j and S of (4.13). Instead of (9.2) we get

$$\begin{gathered} s = N\frac{n_1+n_2}{2} + j, N\frac{n_1+n_2}{2} + j - 1, \ldots s_0, \\ s_0 = \begin{cases} 0, & \frac{N(n_1+n_2)}{2} \geq j, \\ j - \frac{n_1+n_2}{2}, & \frac{N(n_1+n_2)}{2} < j. \end{cases} \end{gathered} \tag{9.3}$$

Thus the IRs of algebra $p(1,3;N)$ can be viewed as into correspondence with parasupermultiplets of particles with spins given by formulae (9.2), (9.3).

Like in the case of Poincaré superalgebra [2], the parasupermultiplets contain bosons as well as fermions.

Let us consider now one example of IR of $p(1,3;N)$, namely, for $n_1 = n_2 = ... = n_{2N} = 1/2$.It appears that these representations are IRs of the Poincaré superalgebra since in this case the corresponding operators Q_A and $\bar{Q}_A$ satisfy the usual anticommutation relations (2.4) for supercharges.

Thus, we had obtained IRs of Poincaré superalgebra as a particular (and the simplest) case of representations of our more general algebra $p(1,3;N)$.

Now consider IRs of class II with discrete spins. The corresponding basis elements are defined in (5.9), (5.11).

The related IRs of the algebra $p(1,3;n)$ are reducible with respect to the subalgebra $p(1,3)$. This can be seen by calculating the additional Casimir operator C of the $p(1,3)$ in these representations. We obtain

$$C = \frac{J_{12}p_3 + J_{31}p_2 + J_{23}p_1}{p} = \lambda - \frac{1}{2}Nn_1 - \frac{1}{2}\hat{S}_3,$$

and its eigenvalues $\bar{\lambda}$ (associated with helicities of particles) in the form

$$\bar{\lambda} = \lambda, \lambda - \frac{1}{2}, \lambda - 1, \ldots, \lambda - Nn_1. \qquad (9.4)$$

Thus, the corresponding parasupermultiplets include both bosons and fermions, whose helicities are given in (9.4).

For $n_1 = n_2 = ... = n_N = 1/2$ these IRS of $p(1.3; N)$ reduce again to IRs of Poincaré superalgebra.

In conclusion we notice, that in the present paper we have studied the simplest "paraextension" of the Poincaré algebra, which includes parafermionic charges only. The other extensions of the Poincaré algebra, including central parasupercharges and also parafermionic and parabosonoc charges will be considered elsewere .

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Appendix

Lorentz transformations

Here we present the explicit Lorentz transformations of four-vectors and spinors which have been used in the previous sections.

The transformation $p'_\mu \to p_\mu = A_{\mu\nu} p'^\nu$, where p_μ are components of a time-like four-momenta $P = (p_0, p_1, p_2, p_3)$, and p'_μ are the related components in the c.m. frame, is performed by the matrix

$$A = \exp\left(-\frac{iS_{0a}p_a}{p}\text{artanh}\frac{p}{E}\right) = 1 - \frac{i}{M}S_{0a}p_a - \frac{1}{M(M+E)}\left(S_{0a}p_a\right)^2, \quad (A.1)$$

where $p = \sqrt{p_1^2 + p_2^2 + p_3^2}$, $E = \sqrt{p^2 + M^2}$, and matrices S_{0a} have the following non-zero elements:

$$(S_{01})_{12} = (S_{01})_{21} = (S_{02})_{13} = (S_{02})_{31} = (S_{03})_{14} = (S_{03})_{41} = i.$$

The corresponding transformation of the Weyl spinors is $Q_A \to B_{AB}Q_B$, where B_{AB} are elements of the following matrix

$$B = \exp\left(\frac{\sigma_a p_a}{2p}\text{artanh}\frac{p}{E}\right) = \frac{E + M + \sigma_a p_a}{\sqrt{2M(E+M)}}$$

with σ_a being the Pauli matrices.

The rotation transformation $(M, 0, 0, M) \to (p_0, p_1, p_2, p_3)$ for a light-like four-momenta is performed by the matrix

$$A = \exp\left(-\frac{iS_{3a}p_a}{\sqrt{p_1^2+p_2^2}}\arctan\frac{\sqrt{p_1^2+p_2^2}}{p_3}\right) = 1 - i\frac{S_{3a}p_a}{p} - \frac{(S_{3a}p_a)^2}{p(p+p_3)},$$

in which the nonzero elements of $S_{3a}(a = 1, 2)$ are given by

$$(S_{31})_{24} = -(S_{31})_{42} = (S_{32})_{34} = -(S_{32})_{43} = i.$$

The corresponding transformation matrix for spinors is

$$B = exp\left(\frac{i(\sigma_1 p_2 - \sigma_3 p_1)}{2\sqrt{p_1^2+p_2^2}}artan\frac{\sqrt{p_1^2+p_2^2}}{p_3}\right) = \frac{p + p_3 + \sigma_1 p_2 - i\sigma_2 p_1}{\sqrt{2p(p+p_3)}}.$$

Finally, the transformation of the space-like four-vector $(0, 0, 0, \eta) \to (p_0, p_1, p_2, p_3)$ is performed by the matrix

$$A = 1 + \frac{iS_{3\mu}p^\mu}{\eta} + \frac{(S_{3\mu}p^\mu)^2}{\eta(p_3+\eta)}. \qquad (A.5)$$

The related transformation matrix for spinors is given by

$$B = \frac{\eta + p_3 - \sigma_3 p_0 - i\sigma_2 p_1 + i\sigma_1 p_2}{\sqrt{\eta(\eta+p_3)}} \qquad (A.6)$$

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