HIGHER SYMMETRIES OF THE SCHRÖDINGER EQUATION

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Complete sets of symmetry operators of arbitrary finite order are found for the Schrödinger equation with some types of potential, including the potential of a supersymmetric harmonic oscillator. Potentials that admit nontrivial higher symmetries are described.

Symmetry operators of higher orders attract increasing interest of investigators; see, for example, [1—7]. Study of such symmetry operators yields information about the hidden symmetry of the equations of mathematical physics, including Lie—Bäcklund symmetries [1] and supersymmetries [2]; they enable one to calculate explicitly the conservation laws and integrals of the motion that in principle cannot be found in the classical approach of Lie [3]. A very important application of symmetry operators of higher orders is the description of the coordinate systems in which an equation admits solutions in separated variables [4]. A review of results relating to the symmetry operators of the basic equations of quantum theory can be found in [3].

Investigations of the higher symmetries of the equations of mathematical physics are usually restricted to some definite class of symmetry operators, for example, first-order differential operators with matrix coefficients in the case of the Dirac equation [2,5]. Of course, there is a natural interest in the problem of describing symmetry operators of the highest possible order, ideally an arbitrary order. This interest is stimulated by the successful use of symmetry operators of higher order (exceeding the order of the equation) for the separation of variables [6,7].

In [8—11] complete sets of symmetry operators of arbitrary order $n<\infty$ were obtained for the d'Alembert, Klein—Gordon—Fock, Schrödinger, and Dirac equations describing free (noninteracting) particles. In this paper, we investigate the higher symmetries of the Schrödinger equation with various potentials.

Potentials admitting nontrivial Lie symmetries of the one-dimensional Schrödinger equation were obtained in [12—14]. Below, we find complete sets of symmetry operators of arbitrary order for the Schrödinger equation with all these potentials and also the potential of a supersymmetric oscillator. Potentials that admit higher symmetries are described; it is shown that the potentials corresponding to exactly solvable Schrödinger equations [15] admit symmetry operators of third order (see also [11]).

1. DETERMINING EQUATIONS

We write the investigated one-dimensional Schrödinger equation in the form

$$L\Psi\equiv(p_0-{}^1/_2(p^2+V(x)))\Psi=0, \tag{1.1}$$

where

$$p_0=i\frac{\partial}{\partial t},\qquad p=-i\frac{\partial}{\partial x}=-i\partial_x.$$

Investigation of the symmetry of Eq. (1.1) includes problems that can be nominally divided into two types:

1) the potential V is given the symmetry is to be found;

2) to determine potentials that admit a known (or some) symmetry.

In this section, we give general results relating to both types of problem.

Definition. A linear differential operator of order n,

$$Q^n=\sum_{i=0}^{n}(q_i\cdot p)_+, \tag{1.2}$$

where

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$$(q_i \cdot p)_i = [(q_i \cdot p)_{i-1}, p]_+, \quad (q_0 \cdot p)_0 = q_0, \quad [A, B]_+ = AB + BA, \quad q_i = q_i(t, x),$$

is called a symmetry operator (of order n) of Eq. (1.1) if

$$[L, Q^n] = 0. \tag{1.3}$$

Remark 1. The operator (1.2) does not depend on p_0, since on the set of solutions of Eq. (1.1) it is always possible to express p_0 in terms of p^2+V. This makes it possible, without loss of generality, to require vanishing of the commutator of L with the symmetry operator that carries solutions to solutions [3].

Remark 2. The representation of Q^n as a sum of i-fold anticommutators simplifies the subsequent calculations. Using the identity [11]

$$(q_i p)_i = (-1)^i \sum_{k=0}^{i} \frac{i! 2^{i-k}}{(i-k)!k!} (\partial_x^k q_i) \partial_x^{i-k},$$

we can always transfer the differentiation operators to the right.

We find equations for the coefficients q_i of the symmetry operators. Substituting (1.1) and (1.2) in (1.3), using the relations [11]

$$[p_0, (q_i \cdot p)_i] = i(\dot{q}_i \cdot p)_i, \quad \left[-\frac{1}{2} p^2, (q_i \cdot p)_i \right] = \frac{i}{2} (q_i' \cdot p)_{i+1},$$

$$[V, (q_{2k} \cdot p)_{2k}] = -i \sum_{m=0}^{k-1} (-1)^{m+k} \frac{2(2k)!}{(2k-2m-1)!(2m+1)!} (q_{2k} \partial_x^{2k-2m-1} V \cdot p)_{2m+1}, \quad k \geqslant 1,$$

$$[V, (q_{2k+1} \cdot p)_{2k+1}] = -i \sum_{m=0}^{k} (-1)^{m+k+1} \frac{2(2k+1)!}{(2k-2m+1)!(2m)!} (q_{2k+1} \partial_x^{2k-2m+1} V \cdot p)_{2m}, \quad k \geqslant 0$$

(where the dot and the prime denote the derivatives with respect to t and x), and equating the coefficients of the linearly independent terms of the form $(Ap)_i$, we arrive at the following system of equations for the coefficients q_i and the potential V:

$$q_n' = 0, \quad 2\dot{q}_{2m} + 2q_{2m-1}' + \sum_{k=m}^{\{(n-1)/2\}} (-1)^{m+k+1} \frac{2(2k+1)!}{(2k-2m+1)!(2m)!} q_{2k+1} \partial_x^{2k-2m+1} V = 0, \quad 2\dot{q}_{2l+1} + q_{2l}'$$
$$+ \sum_{k=l+1}^{\{n/2\}} (-1)^{k+l} \frac{2(2k)!}{(2k-2l-1)!(2l+1)!} q_{2k} \partial_x^{2k-2l-1} V = 0, \tag{1.4}$$

where

$$m = 0, 1, \ldots, \{n/2\}; \; l = 0, 1, \ldots, \{(n-1)/2\}; \; q_{-1} \equiv 0.$$

Equations (1.4) give necessary and sufficient conditions for the existence of a symmetry operator of arbitrary preassigned order n for Eq. (1.1). The general solution of Eq. (1.4) for V and q_i determines the explicit form of the potentials that admit a symmetry operator of order n and the explicit form of this operator.

2. COMPLETE SETS OF SYMMETRY OPERATORS OF THE ONE-DIMENSIONAL SCHRÖDINGER EQUATION

We consider problems of the type 1 for Eq. (1.1), in which the potential V is assumed known. We restrict ourselves to analysis of potentials of the form

$$V = V_1, \tag{2.1a}$$

$$V = V_2 x, \tag{2.1b}$$

$$V = V_3 x^2, \tag{2.1c}$$

$$V=V_4\frac{1}{x^2}, \tag{2.1d}$$

$$V=V_5x^2+V_6\frac{1}{x^2}, \tag{2.1e}$$

where V_1, ..., V_6 are arbitrary constants.

The expressions (2.1) give all inequivalent potentials that admit nontrivial Lie symmetries [12—14]. Here, we shall find complete sets of symmetry operators of arbitrary order n for Eq. (1.1) with the potentials (2.1).

In the case of the potential (2.1a), the problem reduces to the description of the symmetry operators of the free Schrödinger equation [11]. The corresponding equations (1.4) take the form

$$\dot{q}_0=0, \qquad q_n'=0, \qquad 2\dot{q}_k-q_{k-1}'=0. \tag{2.2}$$

where the dot and the prime denote the derivatives with respect to t and x, respectively. By successive differentiation of (2.2), we obtain

$$\partial_t^{k+1}q_k=0, \qquad \partial_x^{n-k+1}q_k=0, \tag{2.3}$$

whence

$$q_k=\sum_{p=0}^{n-k}\sum_{l=0}^{k}C_k^{p,l}x^pt^l, \tag{2.4}$$

where $C_k^{p,l}$ are arbitrary constant coefficients, the number of which is $(k+1)(n-k+1)$. From (2.2), we obtain a unique restriction on $C_k^{p,l}$:

$$2l(l+1)C_k^{p,l+1}+(p+1)C_{k-1}^{p+1,l}=0, \qquad k=1,2,\ldots,n. \tag{2.5}$$

Therefore, the total number of independent parameters in (2.4) is [11]

$$N^n=\sum_{k=0}^{n}(k+1)(n-k+1)-\sum_{k=1}^{n}k(n-k+1)={}^1/_2(n+1)(n+2). \tag{2.6}$$

The corresponding symmetry operators of order n (the number of which, obviously, is N) are determined by the relations (1.2), (2.4), and (2.5) (these last can be regarded as recursion relations). It is readily noted that all symmetry operators of Eq. (1.1), (2.1a) are polynomials of order n in the first-order symmetry operators $P=p$ and $G=tp-mx$.

For the potentials (2.1b) and (2.1c), Eqs. (1.4) reduce to the systems (2.7) and (2.8), respectively:

$$q_n'=0, \quad \dot{q}_0-2V_2q_1=0, \quad 2\dot{q}_n+\dot{q}_{n-1}=0, \quad 2\dot{q}_k+q_{k-1}-2(k+1)V_2q_{k+1}=0, \quad 0<k<n; \tag{2.7}$$

$$q_n'=0, \quad 2\dot{q}_n-q_{n-1}'=0, \quad \dot{q}_0-2V_3xq_1=0, \quad 2\dot{q}_k+q_{k-1}'-4(k+1)V_3xq_{k+1}=0, \quad 0<k<n. \tag{2.8}$$

Equations (2.7) can be solved in complete analogy with (2.2). We again have the differential consequences (2.3), and the representation (2.4) holds; however, instead of (2.5) we obtain from (2.7) the following conditions on $C_k^{p,l}$:

$$2m(l+1)C_k^{p,l+1}+(p+1)C_{k-1}^{p+1,l}-4(k+1)V_2C_k^{p,l}=0. \qquad k=1,\ldots,n. \tag{2.9}$$

Thus, Eq. (1.1) with the potential (2.1b) admits N^n symmetry operators of order n. The explicit form of these operators is given by (1.2), (2.4), and (2.9), and N^n is given by (2.6). All the symmetry operators are polynomials of degree n in the first-order symmetry operators $\hat{p}=p+Vt$, $\hat{G}=t\hat{p}-mx$.

To find the general solution of the system (2.8), we use the following differential consequences:

$$\partial_x^{n-k+1}q_k=0,$$

which enable us to represent q_k in the form

$$q_k=\sum_{i=0}^{n-k}a_{k,i}x^i, \tag{2.10}$$

where $a_{k,i}$ are arbitrary functions of t. Substituting (2.10) in (2.8) and equating the coefficients of identical powers of x, we arrive at a system of N^n ordinary differential equations of first order for the N^n unknowns $a_{k,i}$. Using the fact that the general solution of such a system depends on N^n arbitrary parameters [16], we immediately specify the explicit form of the corresponding linearly independent symmetry operators:

$$Q^n = \sum_{k=0}^{n} \sum_{\alpha=0}^{k} C_{k,\alpha} (p-i\omega x)^\alpha (p+i\omega x)^{k-\alpha} \exp[i(2\alpha-k)\omega t] \tag{2.11}$$

where $\omega=(V_3)^{1/2}$, and $C_{k,\alpha}$ are arbitrary constants, the number of which is N^n (2.6).

We see that all the symmetry operators of Eq. (1.1), (2.1c) of finite order n reduce to polynomials in the first-order symmetry operators $p_\pm=(p\pm i\omega x)\exp(\mp i\omega t)$. In the case $n=2$, our results reduce to those obtained earlier in [12].

The integration of the system (1.4) with the potential (2.1d) requires somewhat more cumbersome calculations. We restrict ourselves to giving the explicit form of the corresponding symmetry operator of order $2n$:

$$Q^{2n} = \sum_{i=0}^{n} \lambda^{a_1 a_2 \ldots a_i} Q_{a_1} Q_{a_2} \cdot \ldots \cdot Q_{a_i}, \tag{2.12}$$

where $\lambda^{a_1 \cdots a_i}$ are arbitrary symmetric tensors, $a_k=1, 2, 3$,

$$Q_1 = \frac{p^2}{2m} - \frac{V_4}{x^2}, \qquad Q_2=2tQ_1-xp+\frac{i}{2}, \qquad Q_3=t^2Q_1-tQ_2-{}^1/_2 mx^2.$$

The number of linearly independent operators (2.12) is N^n (2.6). Symmetry operators of odd order do not exist for Eq. (1.1), (2.1d).

The relations (2.6) and (2.12) determine a complete set of symmetry operators also for Eq. (1.1), (2.1e).

3. SYMMETRY OPERATORS OF THE SUPERSYMMETRIC OSCILLATOR

The Schrödinger equation for the supersymmetric oscillator has the form [17]

$$L\Psi \equiv \left[i\frac{\partial}{\partial t} - {}^1/_2(p^2+\omega^2 x^2+\sigma_3\omega) \right] \Psi = 0, \tag{3.1}$$

where Ψ is a two-component wave function, ω is an arbitrary real parameter, and σ_3 is one of the Pauli matrices:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Equation (3.1) possesses a specific symmetry in the class of first-order differential operators with matrix coefficients; it is determined by the superalgebra $sqm(2)$ [17]. This algebra is formed by the symmetry operators

$$Q_1 = \frac{1}{\sqrt{2}}[\sigma_1 p+\sigma_2\omega x], \qquad Q_2 = \frac{1}{\sqrt{2}}(\sigma_2 p-\sigma_1\omega x), \qquad Q_3={}^1/_2(p^2+\omega^2x^2+\sigma_3\omega),$$

which satisfy the relations

$$[Q_1, Q_2]_+ = 0, \quad Q_1^2=Q_2^2=Q_3, \quad [Q_1, Q_3] = [Q_2, Q_3] = 0. \tag{3.2}$$

The invariance with respect to the algebra (3.3) is the main property of the equations of supersymmetric quantum mechanics [17].

In [18] a complete set of second-order symmetry operators was obtained for Eq. (3.1). We find all inequivalent symmetry operators of arbitrary order, the description of which reduces, essentially, to investigation of the symmetry of Eq. (1.1), (2.1d). Indeed, subjecting Ψ and $\hat{L}$ in (3.1) to the transformation

$$\Psi \to \Psi' = \exp\left(\frac{i}{2}\omega t\sigma_3\right)\Psi, \qquad L \to L' = \exp\left(-\frac{i}{2}\omega t\sigma_3\right) L \exp\left(\frac{i}{2}\omega t\sigma_3\right), \tag{3.3}$$

we arrive at the equation $L'\Psi'=0$, where

$$L' = i\frac{\partial}{\partial t} - \frac{1}{2}(p^2+\omega^2x^2)$$

is the operator representing the direct sum of the two operators (1.1) and (2.1d). The corresponding symmetry operators can obviously be represented in the form $Q'^a=\sigma^\mu Q^n_\mu$, where Q^n_μ are the symmetry operators of Eq. (1.1), (2.1d) determined by the relation (2.11) (in which $C_{k,\alpha}\to C^\mu_{k,\alpha}$). Returning by means of the transformation that is the inverse of (3.3) to the original equation (3.1), we obtain a complete set of symmetry operators of this equation in the form

$$Q^n = \sigma'^n Q_n^{-}. \tag{3.4}$$

where

$$\sigma_0' = \sigma_0, \quad \sigma_3' = \sigma_3, \quad \sigma_1' = \sigma_1 \cos\frac{\omega t}{2} + \sigma_2 \sin\frac{\omega t}{2}, \quad \sigma_2' = \sigma_2 \cos\frac{\omega t}{2} - \sigma_1 \sin\frac{\omega t}{2}.$$

The number of linearly independent symmetry operators of order n is $4N^n$, where N^n is given in (2.6).

The symmetry of the supersymmetric Schrödinger equation with arbitrary potential was investigated in [19].

4. SYMMETRY OPERATORS OF THE THREE-DIMENSIONAL HARMONIC AND SUPERSYMMETRIC OSCILLATORS

The higher symmetries of the three-dimensional Schrödinger equation

$$L\Psi \equiv \left[i\frac{\partial}{\partial t} - \frac{1}{2}(\mathbf{p}^2 + V(\mathbf{x})\right] \Psi = 0 \tag{4.1}$$

can be investigated using the scheme employed in Sec. 2. The significant complication of the problem associated with the transition to partial differential equations with respect to the spatial variables can be overcome by using generalized Killing tensors [8,10].

We represent the symmetry operators of Eq. (4.1) of arbitrary order n in the form

$$Q^n = \sum_{k=0}^{n} [[\ldots[F^{a_1a_2\ldots a_k}, p_{a_1}]_+, p_{a_2}]_+, \ldots p_{a_k}]_+, \tag{4.2}$$

where $F^{a_1a_2\ldots a_k}$ is a symmetric tensor of rank k that depends on $\mathbf{x}$ and t.

Substituting L (4.1) and Q^n (4.2) in the invariance condition (1.3) and equating the coefficients of the linearly independent differentiation operators, we arrive at the following system of determining equations [cf. (1.4)]:

$$\begin{gathered}
\partial^{(a_{n+1}}F^{a_1a_2\ldots a_n)} = 0, \quad 2\dot{F}^{a_1a_2\ldots a_{2l+1}} + \partial^{(a_{2l+1}}F^{a_1a_2\ldots a_{2l})} + \sum_{k=m}^{\{(n-1)/2\}} (-1)^{m+k+1}\frac{2(2k+1)!}{(2k-2m+1)!(2m)!}U^{a_1a_2\ldots a_{2m}}, \\
2\dot{F}^{a_1a_2\ldots a_{2l+1}} + \partial^{(a_{2l+1}}F^{a_1a_2\ldots a_{2l})} + \sum_{k=l+1}^{\{n/2\}} (-1)^{k+l}\frac{2(2k)!}{(2k-2l-1)!(2l+1)!}W^{a_1a_2\ldots a_{2l+1}},
\end{gathered} \tag{4.3}$$

where

$$\partial^a = \frac{\partial}{\partial x_a}, \qquad m = 0, 1, \ldots, \{n/2\}, \qquad l = 0, 1, \ldots, \{(n-1)/2\},$$

$$U^{a_1a_2\ldots a_{2m}} = F^{a_1a_2\ldots a_{2m}b_1b_2\ldots b_{2k-2m+1}}\partial^{b_1}\partial^{b_2}\ldots\partial^{b_{2k-2m+1}}V, \quad W^{a_1a_2\ldots a_{2l+1}} = F^{a_1a_2\ldots a_{2l+1}b_1b_2\ldots b_{2k-2l-1}}\partial^{b_1}\partial^{b_2}\ldots\partial^{b_{2k-2l-1}}V$$

and symmetrization over the indices enclosed in the brackets is understood.

Equations (4.3) determine potentials V that admit nontrivial symmetries of order n and the coefficients $F^{a_1a_2\ldots a_k}$ of the corresponding symmetry operators. The general solution of these equations for $V \equiv 0$ was obtained in [10]. Here, we consider the case of the harmonic oscillator potential

$$V(\mathbf{x}) = \omega^2\mathbf{x}^2$$

and give without proof the number $\hat{N}^n$ of linearly independent symmetry operators of order n and the explicit form of these operators:

$$\begin{gathered}
N^n = \frac{1}{3!4!}(n+1)(n+2)^2(n+3)^2(n+4), \\
Q^n = \sum_{c=0}^{n}\sum_{k=0}^{c} \lambda^{a_1a_2\ldots a_cb_1b_2\ldots b_{n-c}} q_{a_1}^+ q_{a_2}^+ \ldots q_{a_k}^+ q_{a_{k+1}}^- \ldots q_{a_c}^- J_{b_1}J_{b_2}\ldots J_{b_{n-c}},
\end{gathered} \tag{4.4}$$

where

$$q_a^{\pm}=(p_{a_j}\pm i\omega x_{a_j})\exp(\mp i\omega t),\qquad J_b=\varepsilon_{bcd}q_c^{+}q_d^{-},$$

and $\lambda^{a_1\ldots a_c b_1\ldots b_{n-c}}$ are arbitrary constant tensors that are symmetric with respect to the substitutions $a_i \leftrightarrow a_j$, $b_k \leftrightarrow b_l$ and have zero trace with respect to any pair of indices (a_j, b_i).

The symmetry of Eq. (4.1) with the potential of the supersymmetric oscillator

$$V(\mathbf{x})=\omega^2\mathbf{x}^2+\omega\sigma_3$$

can be made in complete analogy with Sec. 3. The general expression for the corresponding symmetry operator of order n is given by (3.4), where Q_μ^n are the operators (4.5) $(\lambda^{a_1\ldots a_c b_1\ldots b_{n-c}}\to\lambda_\mu^{a_1\ldots a_c b_1\ldots b_{n-c}})$, and the number of linearly independent symmetry operators is $4N^n$.

5. POTENTIALS THAT ADMIT NONTRIVIAL SYMMETRIES

We now consider the problem of the second type and describe the class of potentials for which Eq. (1.1) admits nontrivial symmetries. In principle, all such potentials are described by Eqs. (4.1) if both V and q_i are regarded as unknown.

We consider successively the cases $n=1$, 2 (which correspond to Lie symmetries) and $n=3$ (simplest non-Lie symmetry).

Setting $n=1$ in (1.4), we arrive at the system

$$q_1'=0,\quad 2\dot{q}_1+q_0'=0,\quad \dot{q}_0-V'q_1=0. \tag{5.1}$$

By definition $q_1\neq 0$, and therefore the following differential consequences of the system (5.1) hold:

$$q_0''=0,\quad V'''=0,$$

and from them we obtain the general form of a potential V admitting a first-order symmetry operator:

$$V=V_1+V_2x+V_3x^2,$$

where V_1, V_2, and V_3 are arbitrary constants. The corresponding symmetry operators are given in Sec. 3.

For $n=2$, the system (1.4) takes the form

$$q_2'=0,\quad 2\dot{q}_2+q_1'=0,\quad \dot{q}_0-V'q_1=0,\quad 2\dot{q}_1+q_0'-2q_2V'=0, \tag{5.2}$$

from which we obtain by analogy with the above

$$V=V_0+V_1x+V_2x^2+\frac{V_{-2}}{(C_0+C_1x)^2}. \tag{5.3}$$

We see that Eqs. (1.4) permit an elementary calculation of the general form of a potential admitting a nontrivial Lie symmetry [second-order symmetry operators reduce on the set of solutions of Eq. (1.1) to first-order differential operators which are generators of a Lie group].

The case $n=3$ already corresponds to a non-Lie symmetry. The corresponding equations (1.4) take the form

$$q_3'=0,\quad 2\dot{q}_3+q_2'=0,\quad 2\dot{q}_2+q_1'-6q_3V'=0, \tag{5.4a}$$

$$2\dot{q}_1+q_0'-4q_2V'=0, \tag{5.4b}$$

$$\dot{q}_0-q_1V'+q_3V'''=0. \tag{5.4c}$$

It is easy to show that the general solution of Eqs. (5.4a) has the form

$$q_3=a,\quad q_2=b-2\dot{a}x,\quad q_1=2\ddot{a}x^2-2\dot{b}x+6aV+c, \tag{5.5}$$

where a, b, c are arbitrary functions of t. Differentiating (5.4b) with respect to t and (5.4c) with respect to x, and eliminating $\dot{q}_0'$, we arrive, using (5.5), at the equation

$$(aV''-3aV^2-cV)''-2\ddot{a}[(V'x^2)'+4(xV)'+2V]+2\dot{b}[(V'x)'+2V']=4\ddddot{a}x^2-4\dddot{b}x+2\ddot{c}. \tag{5.6}$$

The nonlinear equation (5.6) is fairly complicated, and we therefore restrict ourselves to an investigation of its particular solutions. We note first that all the potentials (5.3) satisfy (5.6) and, therefore, admit a nontrivial third-order symmetry operator. However, it turns out that the class of potentials admitting such a symmetry operator is much larger and includes, for example, the following solutions of Eq. (5.6) [11]:

$$V=\frac{2d^2}{\cos^2 dx},\qquad V=2d^2\operatorname{tg}^2 dx,\qquad V=2d(\operatorname{th}^2 dx-1),\qquad V=2d^2(\operatorname{cth}^2 dx-1),\qquad V=\frac{d^2(1\pm\operatorname{ch} dx)}{\operatorname{sh}^2 dx}, \tag{5.7}$$

where d is an arbitrary parameter.

Equations (1.1) with potentials (5.7) are exactly solvable [15]. It should be emphasized that these potentials do not admit a nontrivial Lie symmetry, but for the corresponding Schrödinger equations a third-order symmetry operator exists.

We give a number of other solutions of Eq. (5.6). Setting *a priori* $\ddot{a}=\dot{b}=\ddot{c}=0$, we can integrate this equation twice with respect to x and reduce it to the form

$$aV''-3aV^2-cV=k_1x+k_0. \tag{5.8}$$

An obvious solution of (5.8) is the function

$$V=\frac{1}{3}W-\frac{c}{6a},\qquad a\neq 0, \tag{5.9}$$

where W is the Weierstrass function, satisfying the equation $W''=W^2$, and at the same time $k_0=k_1=0$. We obtain other solutions of Eq. (5.8) using the handbook [20]:

a) for $c=k_0=0$, $k_1=2a\neq 0$, $V=2y$ we obtain the equation that determines the transcendental Painlevé function;

b) for $k_1=k_0=0$, $c=4a\neq 0$, $V=2y$ we obtain an equation whose solution leads to elliptic integrals. The solutions include, for example, the functions

$$y=\frac{1}{\sin^2(x+C_1)},$$

corresponding to the special case of the Pöschl—Teller potential [21].

CONCLUSIONS

We have shown that the problem of describing the complete set of symmetry operators of arbitrary finite order n for the one-dimensional Schrödinger equation reduces to finding the general solution of the system of linear equations (1.4) for the coefficients q_i of the operator (1.2). The integration of this system for a given interaction potential enables us to find all inequivalent symmetry operators of order n. Above, we have found these symmetry operators for all potentials that admit a nontrivial Lie symmetry and for the potential of the supersymmetric oscillator.

Much more complicated is the problem of describing the potentials that admit symmetry operators of a given order n. This problem also reduces to the solution of the system (1.4), in which both q_i and V are regarded as unknown. As a result, already in the case $n=3$ we arrive at a nonlinear equation for V for which we were only able to obtain particular solutions. However, they include the very important potentials (5.7) and (5.10), which correspond to exactly solvable Schrödinger equations [15,21]. In our view, the existence of a generalized (non-Lie) symmetry of exactly solvable equations that do not possess Lie symmetry is a fundamental fact, opening up new possibilities in the construction of exactly solvable models. Thus, it would be very interesting to investigate the possibility of constructing exact solutions of Eqs. (1.1) with the potential (5.9) and the other potentials listed above under a) and b) admitting third-order symmetry operators.

It should be noted that our approach makes it possible to calculate symmetries of infinite order too. The corresponding symmetry operators can be represented in the form (1.2) or (4.2), where $n\to\infty$, and the determining equations are specified by (1.4) or (4.3), where the first rows must be omitted and the summation replaced by infinite series, i.e., the upper limit of summation tends to infinity.

Our approach to the investigation of symmetry operators of higher orders of Eq. (1.1) is an alternative to the one employed in [22], which describes symmetry operators admitted by the Morse and Pöschl—Teller potentials.

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