shown that the statistics $\kappa_{i,q}$ and $\chi_{i,q}$ introduced in this paper are indeed measures of closeness between samples belonging to the class of more refined differential measure of closeness than the coarse metric (22). In this connection, the test criterion constructed above is recommended for identification of distribution functions in mixture models

$$F_2(u) = (1-\alpha) F_1(u) + \alpha\Phi(u).$$

We now proceed to the problem of constructing the interval $\mathcal{I}_{i,q}$, which is the basis of the tests proposed in this paper. Evidently, such an interval must be selected on the region (a, b) of the line where the distinction between the probablity densities $f_1(u)$ and $f_2(u)$ (for example in the metric C or $\mathcal{L}_1$) is the largest. Since these densities are unknown, they should be replaced by the histograms $f_1^*(u)$ and $f_2^*(u)$ constructed with the aid of the well-known methods of histogram estimation based on learning samples $\bar{x}_1$ and $\bar{x}_2$, respectively. Usually it is not too difficult to determine, using the graphs of these histograms, the interval in which $f_1^*(u)$ and $f_2^*(u)$ differ to the largest extent: Having this interval (a, b), one could obtain order statistics $x_1^{(i)}$ and $x_1^{(i+q)}$ constructed by means of the sample $\bar{x}_1$ which contain the interval (a, b) or form an interval $\mathcal{I}_{i,q}$ that differs only slightly from the interval (a, b).

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GENERALIZED KILLING TENSORS OF ARBITRARY RANK AND ORDER

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We define Killing tensors and conformal Killing tensors of arbitrary rank and order which generalize in a natural way the notion of a Killing vector. We explicitly derive the corresponding tensors for a flat de Sitter space of dimension $p + q = m$, $m \le 4$, which permits us to calculate complete sets of symmetry operators of arbitrary order n for a scalar wave equation with m independent parameters.

1. Introduction. In recent years the classical group-theoretical approach [1] has been increasingly replaced with more modern methods for studying the symmetry of differential equations. In particular, more attention has been placed on the study of symmetry operators of higher orders which are a natural generalization of generators of Lie groups and which contain important information on the hidden symmetry of the equation. These operators are used to describe systems of coordinates in which the equation can be solved by a separation of variables [1-4], in the study of non-Noetherian conservation laws, etc. [5].

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It is known that the description of first-order symmetry operators (generators of Lie groups) is based on the explicit calculation of the Killing vector [6, 7] corresponding to the space of independent variables. Symmetry operators of higher orders have more complicated structures associated with them, which are called Killing tensors (or conformal Killing tensors) of rank j and order s (j, s = 1, 2, ...).

In this article we define these tensors as a complete set of linearly independent solutions of some overdetermined system of partial differential equations and compute them explicitly for all cases where the number of independent variables m is less than or equal to four.

The obtained results can be used to study the higher symmetries of a large class of equations of mathematical physics in m independent variables. As an example (which is of independent interest), in this article we describe a complete set of symmetry operators of arbitrary order n for the scalar wave equation in an m-dimensional space.

2. Symmetry Operators for a Wave Equation. To arrive at a natural definition of Killing tensors of arbitrary rank and order, we state the problem of determining symmetry operators of arbitrary order n = 1, 2, ... for a wave equation

$$L\varphi \equiv (g_{\mu\nu}\partial^{\mu}\partial^{\nu} - \varkappa^2)\,\varphi = 0, \quad \partial^{\mu} = \frac{\partial}{\partial x_{\mu}}, \tag{1}$$

where κ is a real parameter, $g_{\mu\nu}$ a metric tensor whose non-zero elements are equal to $g_{00} = -g_{11} = -g_{22} = \ldots = 1$, $\mu, \nu = 0, 1, \ldots, m-1$, $m \le 4$, and repeated indices mean summation.

For our purposes, it suffices to study only solutions of Eqs. (1) which are defined on some open set D of an m-dimensional manifold R_m consisting of points with coordinates $(x_0, x_1, \ldots, x_{m+1})$, and which are analytic with respect to the real parameters $x_0, \ldots, x_{m+1}$. The space of solutions of Eq. (1) for a fixed D is denoted by $\mathscr{F}_0$.

Let $\mathscr{F}$ be the vector space of all complex-valued functions defined on D which are real-analytic, and let L be a linear differential operator (1) defined on $\mathscr{F}$. Then $L\varphi \in \mathscr{F}$ if $\varphi \in \mathscr{F}$ and $\mathscr{F}_0$ is the nullspace (kernel) of the operator L.

Let $\mathfrak{M}_n$ be the set (class) of linear differential operators of order n defined on $\mathscr{F}$. Then the symmetry operator $Q \in \mathfrak{M}_n$ of Eq. (1) is defined as follows.

Definition 1. A linear differential operator Q of order n defined by

$$Q = \sum_{i=0}^{n} h_{a_1 a_2 \ldots a_i}\partial^{a_1}\partial^{a_2}\ldots\partial^{a_i}, \quad h_{a_1 a_2 \ldots a_i} \in \mathscr{F}, \tag{2}$$

is called a symmetry operator of Eq. (1) in class $\mathfrak{M}_n$ (or a symmetry operator of order n) if

$$[Q, L] = \alpha_Q L, \quad \alpha_Q \in \mathfrak{M}_{n-1}, \tag{3}$$

where [Q, L] = QL − LQ is the commutator of the operators L and Q.

In the case n = 1 the symmetry operators defined above can be regarded as generators of the invariance group of Eq. (1). Symmetry operators of order n > 1 do not generate a Lie group and instead define a generalized (non-Lie) symmetry. The problem of describing a complete set of symmetry operators of order n for Eq. (1) reduces to finding a general solution of operator equations (3).

3. Killing Tensors of Rank j and Order s. It is convenient to write all operators appearing in Eq. (3) as sums of j-multiple anticommutators

$$Q = \sum_{j=0}^{n} \hat{Q}_j, \quad \alpha_Q = \sum_{j=0}^{n-1} \hat{\alpha}_j, \quad \alpha_Q L = \frac{1}{4}\,[[\alpha_Q, \partial^{\mu}]_+, \partial_{\mu}]_+ + \frac{1}{2}\,[(\partial^{\mu}\alpha_Q), \partial_{\mu}]_+, \tag{4}$$

where

$$\hat{Q}_j = [[\ldots[F^{a_1 a_2 \ldots a_j}, \partial_{a_1}]_+, \partial_{a_2}]_+, \ldots, \partial_{a_j}]_+,$$
$$\hat{\alpha}_j = [[\ldots[\alpha^{a_1 a_2 \ldots a_j}, \partial_{a_1}]_+, \partial_{a_2}]_+, \ldots, \partial_{a_j}]_+, \tag{5}$$

$$[A, B]_+ = AB + BA,$$

$F^{a_1a_2\dots a_j}$ and $\alpha^{a_1a_2\dots a_j}$ are unknown functions which are symmetric tensors of rank j. We can always reduce operators (4) to their equivlent form (2) by opening up the anticommutators and transferring the commutator operators to the right.

Substituting (4) and (5) into (3) and equating coeffficients of equal powers of differentiation operators, we obtain the following system of equations for $x^2 \neq 0$:

$$\partial^{(a_{j+1}}F^{a_1a_2\dots a_j)} = 0; \tag{6}$$

$$\alpha^{a_1a_2\dots a_{j-1}} = 0, \tag{7}$$

where $F^{a_1a_2\dots a_j}$ and $\alpha^{a_1a_2\dots a_{j-1}}$ are symmetric tensors of rank j and the indices inside the round brackets denote symmetrization, i.e.,

$$\partial^{(a_{j+1}}F^{a_1a_2\dots a_j)} = \partial^{a_{j+1}}F^{a_1a_2\dots a_j} + \partial^{a_1}F^{a_{j+1}a_2\dots a_j} + \partial^{a_2}F^{a_1a_{j+1}a_3\dots a_j} + \dots + \partial^{a_j}F^{a_1a_2\dots a_{j-1}a_{j+1}}.$$

On the other hand, if $\kappa^2 = 0$ then the equations for the coefficients of the symmetry operator become

$$\partial^{(a_{j+1}}\tilde{F}^{a_1a_2\dots a_j)} - \frac{2}{m+2(j-1)}\,\partial_b\tilde{F}^{b(a_1a_2\dots a_{j-1}}g^{a_ja_{j+1})} = 0, \tag{8}$$

$$\tilde{\alpha}^{a_1a_2\dots a_{j-1}} = \frac{2}{m+2(j-1)}\,\partial_b\tilde{F}^{b(a_1a_2\dots a_{j-1}}g^{a_ja_{j+1})}, \tag{9}$$

where $\tilde{F}^{a_1a_2\dots a_j}$ and $\tilde{\alpha}^{a_1a_2\dots a_{j-1}}$ are symmetric traceless tensors and m is the number of independent variables.

Thus, the problem of describing symmetry operators of order n for wave equation (1) reduces to finding a general solution of either system (6) or (8). In the case j = 1 systems (6) and (8) coincide with Killing's equations [6, 7].

Sometimes the equations for the coefficients of symmetry operators of higher order for systems of partial differential equations are more general than (6), (8), and are as follows [5]:

$$\partial^{(a_{j+1}}\partial^{a_{j+2}}\dots\partial^{a_{j+s}}F^{a_1a_2\dots a_j)} = 0 \tag{10}$$

and

$$[\partial^{(a_{j+1}}\partial^{a_{j+2}}\dots\partial^{a_{j+s}}\tilde{F}^{a_1a_2\dots a_j)}]^{SL} = 0, \tag{11}$$

where the symbol $[\cdot]^{SL}$ denotes the traceless part of the tensor inside the square brackets, i.e.,

$$[G^{(a_1a_2\dots a_R)}]^{SL} = G^{(a_1a_2\dots a_R)} + \sum_{\alpha=1}^{\{R/2\}}(-1)^\alpha K_\alpha\Big(\prod_{i=1}^{\alpha} g^{(a_{2i-1}a_{2i}}\Big)G^{a_{2\alpha+1}\dots a_R)b_1b_2\dots b_{2\alpha}}g_{b_1b_2}g_{b_3b_4}\dots g_{b_{2\alpha-1}b_{2\alpha}}, \tag{12}$$

$$K_\alpha = \frac{n!}{(n-2\alpha)!\,2^{\alpha-1}}\prod_{i=1}^{\alpha}\frac{1}{2(n-i)+m-2}, \tag{13}$$

and {R/2} is the integer part of R/2.

In the case s = 1, Eqs. (10) and (11) become Eqs. (6) and (8), respectively.

Definition 2. A symmetric tensor $F^{a_1a_2\dots a_j}$ satisfying system (10) is called a Killing tensor of rank j and order s. A symmetric traceless tensor $\tilde{F}^{a_1a_2\dots a_j}$ satisfying Eqs. (11) is called a conformal Killing tensor of rank j and order s.

In the case s = 1 the above definitions are equivalent to those state in [8].

4. An Explicit Form of Killing Tensors of Arbitrary Rank j and Order s = 1. We look for a general solution of system (6). This system is overdetermined, since it includes C_{j+m}^{j+1} equations for C_{j+m-1}^{j} unknowns (C_b^a is the number of combinations containing a elements from a set of b elements).

We study a set of differential conditions obtained from system (6) by a successive differentiation of each term with respect to x_{b_1}, x_{b_2}, ...:

$$\partial^{b_1}\partial^{b_2}\ldots\partial^{b_k}\partial^{(a_{j+1}}F^{a_1a_2\ldots a_j)} \equiv F^{(a_1a_2\ldots a_j,a_{j+1})b_1b_2\ldots b_k} = 0 \tag{14}$$

(the indices after the comma denote derivatives with respect to the corresponding argument).

Equation (14) defines a system of linear homogeneous algebraic equations in unknowns $F^{a_1a_2\ldots a_j,a_{j+1}b_1b_2\ldots b_k}$, and the numbers of equations (N_y^k) and unknowns (N_H^k) are

$$N_y^k = C_{j+m}^{j+1}C_{k+m-1}^{k}, \quad N_H^k = C_{j+m-1}^{j}C_{m+k}^{k+1}. \tag{15}$$

Clearly, $N_y^k < N_n^k$, $k < j$; $N_y^k = N_H^k$, $k = j$.

<u>LEMMA 1</u>. System of linear algebraic equations (14) is non-degenerate.

This lemma is proven in [9].

The results proved earlier imply that for k = j system (14) has only trivial solutions, so therefore $F^{a_1a_2\ldots a_j}$ are polynomials of order j. According to (15) these polynomials include $\hat{N}_j^m$ arbitrary parameters, where

$$\hat{N}_j^m = \sum_{k=0}^{j}(N_H^k - N_y^k) = \frac{1}{m}C_{j+m-1}^{m-1}C_{j+m}^{m-1}. \tag{16}$$

<u>LEMMA 2</u>. A general solution of Eqs. (6) can be written as

$$F^{a_1a_2\ldots a_j} = \sum_{c=0}^{j}\lambda^{a_1a_2\ldots a_c[a_{c+1}b_1]\ldots[a_jb_{j-c}]}x_{b_1}x_{b_2}\ldots x_{b_{j-c}}, \tag{17}$$

where $\lambda^{a_1\ldots[a_jb_{j-c}]}$ are arbitrary parameters which are tensors that are symmetric with respect to permutations of indices a_μ and a_ν, μ, ν = 1, 2, ..., c, are anti-symmetric with respect to permutations of indices a_{c+i} and b_i, $1 \le i \le j - c$, and cyclic permutations of every triplet of indices is equal to zero.

To prove the above lemma it suffices to check that Eqs. (6) hold for functions (17) and compute the number of independent parameters $\lambda^{a_1a_2\ldots a_c[a_{c+1}b_1]\ldots[a_jb_{j-c}]}$ which is equal to that defined in Eq. (16).

Factoring tensors $\lambda^{a_1\ldots a_c[a_{c+1}b_1]\ldots[a_jb_{j-c}]}$ into irreducible ones (i.e., ones that in addition to having the properties listed in Lemma 2 also have zero traces over any pair of indices), we obtain the following representation of solutions of system (6):

$$F^{a_1a_2\ldots a_j} = g^{(a_{j-1}a_j}F^{a_1a_2\ldots a_{j-2})} + f^{a_1a_2\ldots a_j}, \tag{18}$$

where $F^{a_1a_2\ldots a_{j-2}}$ is a general solution of Eqs. (6) for $j \to j - 2$, and $f^{a_1\ldots a_j}$ is a solution of Eqs. (6) that depends on $\hat{N}_j^m - \hat{N}_{j-2}^m$ arbitrary parameters, which we explicitly list below.

1. m = 1, in which case $f^{a_1a_2\ldots a_j}$ reduces to a scalar that does not depend on the only available variable.

2. m = 2, in which case the number of linearly independent solutions $f^{a_1a_2\ldots a_j}$ is equal to $N_j^2 - N_{j-2}^2 = 2j + 1$. The solutions are numerated by integers c, $0 \le c \le j$, and in cases c = 0 and c > 0 they have one and two, respectively, arbitrary parameters that define the independent components of a symmetric traceless tensor $\lambda^{a_1a_2\ldots a_{j-c}}$ of rank j − c. We have the following explicit expression for $f^{a_1a_2\ldots a_j}$

$$f^{a_1a_2\ldots a_j} = \varepsilon_c\hat{f}_c^{a_1a_2\ldots a_j} + (1 - \varepsilon_c)\hat{f}_c^{(a_1a_2\ldots a_{j-1}}\varepsilon^{a_j)b}x_b,$$

where ε^{a_jb} is a unitary anti-symmetric tensor, $\varepsilon_c = 1/2(1 - (-1)^c)$,

$$\hat{f}_c^{a_1a_2\ldots a_j} = \lambda^{(a_1a_2\ldots a_{j-c}}\sum_{\mu=0}^{\{c/2\}}\Big(\prod_{i=j-c+1}^{j-c+2\mu}x^{a_i}\Big)^*\Big(\prod_{k=(j-c)/2+\mu+1}^{\min(\{j/2\},\{(j+1)/2\}-l)}g^{a_{2k}a_{2k+l}}\Big)^*(-1)^\mu C_{\{c/2\}}^{\mu}(x^2)^{\{c/2\}-\mu}, \tag{19}$$

$$x^2 = g^{ab}x_a x_b, \quad l = (-1)^{j+c+1}, \Big(\prod_{\lambda=A}^{B}\varphi_\lambda\Big)^* = \begin{cases} \prod_{\lambda=A}^{B}\varphi_\lambda, & B \geqslant A, \\ 1, & B < A, \end{cases} \tag{19}$$

and indices $(a_1, a_2 \dots a_j)$ on the right-hand side of (19) denote symmetrization.

3. m = 3, in which case the number of independent solutions is equal to $\hat{N}_j^3 - \hat{N}_{j-2}^3 = 1/3(j + 1)(2j^2 + 4j + 3)$. The solutions are numerated by pairs of integers $c = (c_1, c_2)$ satisfying conditions

$$0 \leqslant c_1 \leqslant 2\{j/2\}, \quad \varepsilon_{c_1} \leqslant c_2 \leqslant j - 2\{(c_1 + 1)/2\} \tag{20}$$

and for every c include a set of $2j - 2c_1 + 1$ arbitrary parameters that define the independent components of a symmetric traceless tensor $\lambda_c^{a_1 a_2 \dots a_{j-c_1}}$ of rank $j - c_1$. The corresponding solutions can be explicitly written as follows:

$$f_c^{a_1 a_2 \dots a_j} = \varepsilon_{c_2}\hat{f}_{c_1 c_2}^{a_1 a_2 \dots a_j} + (1 - \varepsilon_{c_2})\hat{f}_{c_1 c_2}^{b(a_1 a_2 \dots a_{j-1}}\varepsilon_b^{a_j)c}x_c,$$

where $\varepsilon_b^{a_jc}$ is a unitary anti-symmetric tensor and

$$\hat{f}_{c_1 c_2}^{a_1 a_2 \dots a_j} = \sum_\mu K_\mu \lambda_c^{B_\mu, A_\mu}\Big(\prod_{i=A_\mu+1}^{A_\mu+L_\mu} x^{a_i}\Big)^* (x^2)^{F_\mu}\Big(\prod_{k=\{(A_\mu+L_\mu)/2\}+1}^{\min(\{j/2\},\{(j+1)/2\}-l)} g^{a_{2k}a_{2k+l}}\Big)^*. \tag{21}$$

Here

$$\begin{gathered} \mu = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5), \quad x^2 = g^{ab}x_a x_b, \\ K_\mu = (-1)^{\mu_1+\mu_4+\mu_5}2^{\mu_5}\frac{r!}{\mu_2!\,\mu_3!\,\mu_4!}C^{\mu_1}_{\{c_1/2\}}, \quad r = \{c_2/2\}, \\ A_\mu = j - c_1 - B_\mu, \quad B_\mu = 2\mu_2 + \mu_4 + \mu_5, \quad l = (-1)^{c_2+j+1}, \\ L_\mu = c_1 + \mu_3 - 2\mu_1 - \mu_5, \quad F_\mu = \mu_1 + \mu_4, \\ \lambda^{B_\mu, A_\mu} = \lambda^{b_1 b_2 \dots b_{B_\mu} a_1 a_2 \dots a_{A_\mu}} x_{b_1} x_{b_2} \dots x_{b_{B_\mu}}. \end{gathered} \tag{22}$$

The summation in (21) is carried out over all possible nonnegative integer values of μ such that

$$0 \leqslant \mu_1 \leqslant \{c_1/2\}, \quad \mu_2 + \mu_3 + \mu_4 = \{c_2/2\}, \quad 0 \leqslant \mu_5 \leqslant \varepsilon_{c_1}. \tag{23}$$

4. m = 4, in which case the number of independent solutions is equal to $\hat{N}_j^4 - \hat{N}_{j-2}^4 = 1/4!(j + 1)(j + 2)(2j + 3)(j^2 + 3j + 4)$. The solutions are numerated by triples of integers $c = (c_1, c_2, c_3)$, satisfying conditions (20) and a condition $0 \leq c_3 \leq j - 2\{(c_1 + 1)/2\} - 2c_2$, and for every c they include a set of N_c arbitrary parameters (that define the independent components of an irreducible tensor $\lambda^{b_1 b_2 \dots a_1 a_2 \dots \{a_{A_\mu+D_c} d_{D_c}\}}$) where

$$N_c = \begin{cases} (j - c_1 - c_2 + 1)^2, & c_3 = 0; \\ 2(j - c_1 - c_2 + 1)(j - c_1 - c_2 + 2c_3 + 1), & c_3 \neq 0. \end{cases}$$

The corresponding solutions can be explicitly written as

$$f_c^{a_1 a_2 \dots a_j} = \sum_\mu K_\mu \lambda^{B_\mu, A_\mu, c_3}\Big(\prod_{i=A_\mu+c_3+1}^{A_\mu+L_\mu+c_3} x^{a_i}\Big)^*\Big(\prod_{k=\{(A_\mu+L_\mu+c_3)/2\}+1}^{\{j/2\}} g^{a_{2k}a_{2k+l}}\Big)^*(x^2)^{F_\mu}, \tag{24}$$

where μ, x^2, K_μ, B_μ, L_μ, F_μ are defined in (22) and (23),

$$l = (-1)^{j+1}, \; A_\mu = j - c_1 - b_\mu - c_3,$$

$$\lambda^{B_\mu, A_\mu, c_s} = x_{b_1} x_{b_2} \dots x_{b_{B_\mu}} x_{d_1} x_{d_2} \dots x_{d_{c_s}} \lambda^{b_1 b_2 \dots b_{B_\mu}(a_1 a_2 \dots a_{A_\mu} [a_{A_\mu+1} d_1][a_{A_\mu+2} d_2] \dots [a_{A_\mu+c_s} d_{c_s}]},$$

and the indices $a_1 a_2 \dots a_j$ on the right-hand side of (24) denote symmetrization.

We state our results derived so far as the following theorem.

THEOREM. Equations (6) have $\hat{N}_j^m$ linearly independent solutions that are polynomials of order j. These solutions are given explicitly by Eqs. (18)-(24), and $\hat{N}_j^m$ by Eq. (16).

Thus, we have explicitly computed all linearly independent first-order Killing tensors of arbitrary rank that depend on m parameters (where $m \le 4$).

We now study first-order conformal Killing tensors of arbitrary rank, which by definition are solutions of Eqs. (8). Since theoretically the analysis of these equations is similar to the analysis of system (6), we cite without proof an explicit expression for their general solution in the case $m \le 4$.

The case m = 1 is trivial, since the corresponding solution is an arbitrary constant.

In the case m = 2 relations (8) become Cauchy–Riemann equations. The corresponding solutions are defined up to arbitrary analytic functions $\varphi(x_0, x_1)$ and $\tilde{\varphi}(x_0, x_1)$ as follows:

$$j = 0, \quad \tilde{F} = \varphi_0; \qquad j > 0, \quad \tilde{F}^{11\dots1} = (\varphi_j + \varphi_j^*) + i(\tilde{\varphi}_j + \tilde{\varphi}_j^*),$$
$$\tilde{F}^{11\dots12} = \tilde{\varphi}_j - \tilde{\varphi}_j^* + i(\varphi_j - \varphi_j^*). \tag{25}$$

The remaining components $F^{a_1 a_2 \dots a_j}$ can be expressed in terms of (25) by using the properties of tracelessness and symmetry.

In the case m = 3 the number of independent solutions is equal to $1/3(j + 1)(2j + 1)\cdot(2j + 3)$. The solutions are numerated by pairs of integers $c = (c_1, c_2)$, where $0 \le c_1 \le j$, $0 \le c_2 \le 2c_1$, and they contain $2c_1 + 1$ arbitrary parameters that define the independent components of a symmetric traceless tensor $\lambda^{a_1 \dots a_j}$. The corresponding solutions can be written explicitly as

$$\tilde{F}_c^{a_1 a_2 \dots a_j} = [\varepsilon_{c_2} f_{c_1 c_2}^{a_1 a_2 \dots a_j} + (1 - \varepsilon_{c_2}) f_{c_1 c_2 - 1}^{b(a_1 a_2 \dots a_{j-1} a_j)} \varepsilon_b^c x_c]^{SL}, \tag{26}$$

where

$$f_{c_1 c_2}^{a_1 a_2 \dots a_j} = \sum_{k=0}^{\{c_2/2\}} (-2)^k C_k^{\{c_2/2\} - k} \lambda^{b_1 b_2 \dots b_k (a_1 a_2 \dots a_{c_1 - k}} x^{a_{c_1 - k + 1}} x^{a_{c_1 - k + 2}} \dots x^{a_j)} x_{b_1} x_{b_2} \dots x_{b_k} (x^2)^{(\{c_2/2\} - k)}. \tag{27}$$

In the case m = 4 the number of independent solutions of Eqs. (8) is equal to $1/12(j + 1)^2(j + 2)^2(2j + 3)$. The solutions are numerated by triples of integers $c = (c_1, c_2, c_3)$ such that

$$0 \le c_1 \le j, \quad -c_1 \le c_2 \le c_1, \quad 0 \le c_3 \le \{(c_1 - |c_2|)/2\}, \tag{28}$$

and for every c contain N_c arbitrary variables, where

$$N_c = \begin{cases} (|c_2| + 1)^2, & c_1 = |c_2|, \\ 2(|c_2| + 2c_3 + 1)(2c_1 - |c_2| - 2c_3 + 1), & c_1 \ne |c_2|. \end{cases}$$

These parameters define the independent components of an irreducible tensor $\lambda^{b_1 \dots [a_{c_1 - m + i + 2c_3} d_{c_1 - |c_2|}]}$ of rank $R_1 + 2R_2$, where $R_1 = |c_2| + 2c_3$, $R_2 = c_1 - |c_2| - 2c_3$.

The corresponding solutions have the following explicit form:

$$\widetilde{F}_c^{a_1 a_2 \ldots a_j} = \Big[\sum_{i=0}^{k+2c_3} (-1)\, C^i_{k+2c_3} x^{2i} \lambda^{b_1 b_2 \ldots b_{k+2c_3-i}(a_1 a_2 \ldots a_{|c_2|-k+i}[a_{|c_2|-k+i+1} d_1] \ldots [a_{|c_1|-k-2c_3+i} d_{c_1-|c_2|}]} \times$$
$$\times x^{a_{c_1-k+i+1}} x^{a_{c_1-k+i+2}} \ldots x^{a_j)} x_{b_1} x_{b_2} \ldots x_{k+2c_3+i} x_{d_1} x_{d_2} \ldots x_{d_{c_1-|c_2|}} \Big]^{SL}, \tag{29}$$

where $k = -c_2$, $c_2 < 0$ and $k = 0$, $c_2 \geq 0$. The indices $a_1 \ldots a_j$ on the right-hand side of (29) denote symmetrization.

5. Explicit Expression for Killing Tensors of Arbitrary Order. To completely describe Killing tensors of arbitrary order it is necessary to find a general solution of system (10). In analogy to the results derived for Eq. (6) in the previous section (for details see [9]), we conclude that such a solution is a polynomial whose order is no larger than $\hat{N}^{sj}$ and which contains $N_m{}^{sj}$ arbitrary parameters, where

$$\hat{N}^{sj} = j + s - 1, \quad N_m^{sj} = \frac{s}{m} C^{m-1}_{j+m-1} C^{m-1}_{j+s+m-1}.$$

The corresponding solutions can be explicitly written as [9]

$$F_{(s)}^{a_1 a_2 \ldots a_j} = \sum_{\alpha=1}^{s} F^{a_1 a_2 \ldots a_{j+\alpha-1}} x^{a_{j+1}} x^{a_{j+2}} \ldots x^{a_{j+\alpha-1}} + g^{(a_{j-1} a_j} F_{(s)}^{a_1 a_2 \ldots a_{j-2})} + \varepsilon_j \hat{f}^{a_1 a_2 \ldots a_j}, \tag{30}$$

where $F^{a_1 a_2 \ldots a_{j+\alpha-1}}$ are first-order Killing tensors of rank $j + \alpha - 1$ which are explicitly described in the above theorem, $F_{(s)}^{a_1 a_2 \ldots a_{j-2}}$ is a Killing tensor of rank $j - 2$ and order s,

$$\hat{f}^{a_1 a_2 \ldots a_j} = \sum_{\mu=0}^{\{j/2\}-1} (-1)^\mu C^\mu_{\{j/2\}-1} x^{(a_1} x^{a_2} \ldots x^{a_{2\mu}} g^{a_{2\mu+1} a_{2\mu+2}} g^{a_{2\mu+3} a_{2\mu+4}} \ldots g^{a_{j-2} a_{j-1}} \lambda^{[a_j) c]} x_c,$$

and $\lambda^{[a_j c]}$ is an arbitrary anti-symmetric tensor of rank two.

Equation (30) gives recursive relations for explicitly calculating the Killing tensor of order s and rank j in terms of known tensors of rank $j + \alpha - 1$ and order one and rank $j - 2$ and order s. This calculation can be easily carried out by using explicit expressions for first-order Killing tensors given in the previous section.

The explicit expressions for Killing vectors of order $s \leq 3$ in a three-dimensional space obtained from relations (30) are as follows:

$$F^a_{(1)} = \lambda^a + \varepsilon^{abc} \eta_b x_c;$$

$$F^a_{(2)} = F^a_{(1)} + \lambda^{ab} x_b + \lambda x^a + \xi^a x^2 - x^a \xi^b x_b + \varepsilon^{abc} \eta_{bd} x_c x^d;$$

$$F^a_{(3)} = F^a_{(2)} + \lambda^{abc} x_b x_c + x^a \eta^b x_b + \varepsilon^{abc} x_b \eta_c x^2 + \varepsilon^{abc} \eta_{bdf} x_c x^d x^f + \xi^{ab} x_b x^2 - x^a \xi^{bc} x_b x_c.$$

Here ε^{abc} is a unitary anti-symmetric tensor, and other Greek letters denote the arbitrary parameters that define the symmetric traceless tensors.

Conformal tensors of arbitrary rank j and order s are defined as general solutions of system (11). As in Sec. 4, we can show that these tensors are polynomials of x_a of order $2(j + s - 1)$. We cite without proof an explicit expression for these tensors in the case of arbitrary j and s and $m \leq 4$.

The number of linearly independent solutions in cases $m = 3, 4$ is equal to

$$m = 3, \quad N_3^{sj} = \frac{s}{6} (2j + 1)(2j + 2s + 1)(2j + s + 1);$$

$$m = 4, \quad N_4^{sj} = \frac{s}{12} (j + 1)^2 (j + s + 1)^2 (2j + s + 2),$$

and in the case m = 2 there are infinitely many of them. Solutions can be explicitly written as follows [9]:

$$\widetilde{F}_{(s)}^{a_1 a_2 \ldots a_j} = \sum_{i=0}^{s} \left(\widetilde{F}_i^{a_1 a_2 \ldots a_j} x^{2(i-1)} + \sum_{\alpha=0}^{s-i} f_{i-1\alpha}^{a_1 a_2 \ldots a_j} x^{2\alpha} \right), \tag{31}$$

where $\widetilde{F}_i^{a_1 a_2 \ldots a_j}$ are first-order conformal tensors defined by Eqs. (26) and (29) and $f_{i-1\alpha}^{a_1 a_2 \ldots a_j}$ are tensors of rank j which are explicitly described below.

In the case m = 2 we have $f_{i-1\alpha}^{a_1 a_2 \ldots a_j} = 0$. In the case m = 3 the independent functions $f_{i-1\alpha}^{a_1 a_2 \ldots a_j}$ are numerated by integers c, $0 \le c \le 2j$, and are determined up to an arbitrary symmetric traceless tensor $\lambda^{a_1 \ldots a_{2c+2j-1}}$. These functions can be explicitly written as

$$f_{i\alpha}^{a_1 a_2 \ldots a_j} = [\varepsilon_c \hat{f}_{i\alpha c}^{a_1 a_2 \ldots a_j} + (1-\varepsilon_c) \hat{f}^{b(a_1 a_2 \ldots a_{j-1}} \varepsilon_b^{a_j)c} x_c]^{SL}, \tag{32}$$

where

$$\hat{f}_{i\alpha c}^{a_1 a_2 \ldots a_j} = \sum_{n=0}^{\{c/2\}} (-2)^n C_n^{\{c/2\}-n} \lambda^{b_1 b_2 \ldots b_{\alpha+n}(a_1 a_2 \ldots a_{j-n}} x^{a_{j-n+1}} x^{a_{j-n+2}} \ldots x^{a_j)} x_{b_1} x_{b_2} \ldots x_{b_{\alpha+n}} (x^2)^{(\{c/2\}-n)}.$$

In the case m = 4 the function $f_{i\alpha}^{a_1 a_2 \ldots a_j}$ is described by a pair of supplementary integers $c = (c_1, c_2)$, $-j \le c_1 \le j$, $0 \le c_2 \le \{(j - |c_1|)/2\}$, and is determined up to an arbitrary irreducible tensor $\lambda^{b_1 \ldots [a_{j-n+i} d_{j-|c_1|}]}$ of rank $R_1 + 2R_2$, where $R_1 = |c_1| + 2c_2 + \alpha$, $R_2 = j - [c_1]$. These functions can be explicitly written as

$$f_{i\alpha}^{a_1 a_2 \ldots a_j} = \Big[\sum_{\alpha=0}^{n+2c_2} (-1)^\alpha C_{n+2c_2}^{\alpha} x^{2\alpha} \times$$

$$\times \lambda^{b_1 b_2 \ldots b_{n+2c_2-i+\alpha}(a_1 a_2 \ldots a_{|c_1|-n+i}[a_{|c_1|-n+i} d_1] \ldots [a_{j-n+i} d_{j-|c_1|}]} x^{a_{j-n+i+1}} x^{a_{j-n+i+2}} \ldots x^{a_j)} x_{b_1} x_{b_2} \ldots x_{b_{n+2c_2+\alpha-i}} x_{d_1} x_{d_2} \ldots x_{d_{j-|c_1|}}, \tag{33}$$

where $n = -c_1$, $c_1 < 0$ and $n = 0$, $c_1 \ge 0$, and the indices $a_1 \ldots a_j$ on the right-hand side of (33) denote symmetrization.

Using (32), we obtain an explicit expression for conformal Killing vectors of order $s \le 3$ in the case m = 3 as follows:

$$\widetilde{F}_{(1)}^{a} = \lambda_{(1)}^{a} + \varepsilon_b^{ac} \eta_{(1)}^{b} x_c + \xi_{(1)}^{a} x^2 - 2x^a \xi_{(1)}^{b} x_b + \mu_{(1)} x^a,$$

$$\widetilde{F}_{(2)}^{a} = \widetilde{F}_{(1)}^{a} + \widetilde{F}_{(1)}^{\prime a} x^2 + \lambda_{(2)}^{ab} x_b + \varepsilon_b^{ac} \eta_{(2)}^{bd} x_c x_d + \xi_{(2)}^{ab} x_b x^2 - 2x^a \xi_{(2)}^{bc} x_b x_c,$$

$$\widetilde{F}_{(3)}^{a} = \widetilde{F}_{(2)}^{a} + \widetilde{F}_{(1)}^{\prime a} x^4 + x^2 (\lambda_{(3)}^{ab} x_b + \varepsilon_b^{ac} \eta_{(3)}^{bd} x_c x_d + \xi_{(3)}^{ab} x_b x^2 - 2x^a \xi_{(3)}^{bc} x_b x_c) + \lambda_{(3)}^{abc} x_b x_c + \varepsilon_b^{ac} \eta_{(3)}^{b\alpha k} x_c x_d x_k + \xi_{(3)}^{abc} x_b x_c x^2 - 2x^a \xi_{(3)}^{bcd} x_b x_c x_d.$$

6. Explicit Expression for Symmetry Operators of the Wave Equation. We have explicitly found all linearly independent solutions of Eqs. (6), (8), (10), and (11) that describe Killing tensors and conformal Killing tensors of arbitrary rank and order. These solutions allow us to describe symmetry operators for a large class of mathematical physics. In particular, for wave equation (1) such a description is accomplished by substituting solutions of Eqs. (6) and (8) into Eq. (4), which determines the general form of symmetry operators of arbitrary order. In the case $\chi^2 \neq 0$, j = 1 we use (4), (18)-(23) to obtain the following complete set of symmetry operators:

$$Q_a = P_a = i\partial_a, \quad Q_{ab} = J_{ab} = x_a P_b - x_b P_a. \tag{34}$$

Equations (34) describe generators of the Poincaré group. Using representation (17) for the general solution, we see that symmetry operator (4) for Eq. (1) is a polynomial of operators (34). In other words, all symmetry operators of arbitrary finite order for the wave equation belong to the enveloping algebra induced by the generators of the Poincaré group (34).

We shall count the number N(n, m) of linearly independent symmetry operators of order n. Analyzing Eq. (18), whose first term on the right-hand side corresponds to a complete set of symmetry operators of order j − 2, we conclude that N(n, m) is equal to the number of linearly independent solutions of system (6) for j = n and j = n − 1. According to (16) we have

$$N(n, m) = \hat{N}_m^n + \hat{N}_m^{n-1} = \frac{2n^2 + 2nm + m(m-1)}{m(m-1)} C_{n+m-2}^{m-2} C_{m+n-1}^{m-2}. \tag{35}$$

Equation (35) gives the number of linearly independent symmetry operators of order n for the wave equation with m variables. In particular, for m = 4 we have

$$N(n, 4) = \frac{1}{72}(n+1)(n+2)^2(n+3)(n^2+4n+b). \tag{36}$$

The corresponding symmetry operators can be explicitly obtained by substituting (18)-(23) into (4).

In the case $\kappa = 0$ the number of symmetry operators of order n is equal to

$$N(n, 4) = \frac{1}{12}\sum_{j=0}^{n}(j+1)^2(j+2)^2(2j+3);$$

$$N(n, 3) = \frac{1}{3}\sum_{j=0}^{n}(2j+1)(j+1)(2j+3), \quad N(n, 2) = \infty.$$

The symmetry operators in this case are explicitly given by Eqs. (4) and (29). It can be shown that these operators are polynomials of the generators of a conformal group $D = ix^\mu\partial_\mu + i$, $K_\mu = 2x_\mu D + x_\nu x^\nu i\partial_\mu$ and P_μ, $J_{\mu\nu}$ (34).

7. Conclusion. We have defined Killing tensors and conformal Killing tensors of arbitrary rank and order and derived them explicitly in the case of m independent variables, where m ≤ 4.

Equations (10) and (11), which define Killing tensors of arbitrary order, are a natural generalization of Killing's equations [6, 7]. They appear during a determination of symmetry operators of order s for systems of partial differential equations, in particular Maxwell's equations [5].

The obtained generalized Killing tensors can be used to study higher symmetries of the equations of mathematical physics. In this article we have used them to completely describe symmetry operators for the wave equation.

The notion of a (conformal) Killing tensor of arbitrary rank (and first order in our notation) was introduced in [8]. The general solution of Eqs. (6) for j = 2 appears in [10]. The authors of [9] cite solutions of Eqs. (6), (8), (10), and (11) along with some technical details and numerous examples. The use of symmetry operators of higher orders in describing coordinate systems in which equations can be solved by a separation of variables is discussed in [2-4, 11].

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