ON RELATIVISTIC EQUATIONS OF MOTION WITHOUT "REDUNDANT" COMPONENTS

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On the basis of a definite representation (2.1) for the generators of the proper Poincaré group all (to within unitary equivalence) operator functions H for which Eq. (1.1) is invariant under the complete Poincaré group (including space-time reflections) are described. For arbitrary spin a unitary operator is found that relates the representation (2.1) to the Foldy–Shirokov canonical representation. Explicit expressions are obtained for the operators of the coordinate, velocity, and spin in the representation (2.1) for an arbitrary spin s.

1. Introduction

Several recent investigations have been devoted to the problem of finding relativistically invariant equations that describe the free motion of a particle (and antiparticle) with arbitrary spin s whose wave functions have only 2(2s + 1) components. This problem can be reduced to that of describing all the operator functions H (the Hamiltonians of the particles with arbitrary spin s) depending on the momentum and spin operators of the particles for which the Schrödinger-type equation

$$i\frac{\partial\Psi(t,\mathbf{x})}{\partial t}=H\Psi(t,\mathbf{x}) \qquad (1.1)$$

is invariant under the complete Poincaré group $\tilde{\mathcal{P}}$ (1, 3) (including space-time reflections). In other words, H in (1.1) must be such that on the set of solutions $\{\Psi(t, \mathbf{x})\}$ of Eq. (1.1) an irreducible representation of $\tilde{\mathcal{P}}$ (1, 3) is realized.

This problem is solved in [1] and [2, 3] on the basis of a specific representation for the generators of the proper Poincaré group $\mathcal{P}$ (1, 3). Since this representation is related to the Foldy–Shirokov canonical representation by an isometric and not a unitary operator (except in the case s = 1/2) difficulties can arise in connection with the physical interpretation of the dynamical variables found in [1-5] and the introduction of an interaction into an equation of motion of the form (1.1).

To avoid this difficulty, we take a different representation which is related to the Foldy–Shirokov canonical representation by a unitary operator for all spins s.

2. Statement of the Problem

Our starting point is the following representation for the generators P_μ and $J_{\mu\nu}$ of $\mathcal{P}$ (1, 3):

$$P_0=H,\quad P_k\equiv p_k=-i\frac{\partial}{\partial x_k},\quad k=1,2,3,$$
$$J_{kl}=x_kp_l-x_lp_k+S_{kl},$$
$$J_{0k}=tp_k-\frac{1}{2}[x_k,H]_+,\quad [x_k,H]_+=x_kH+Hx_k, \qquad (2.1)$$

where H is an unknown operator function and S_{kl} are 2(2s + 1) × 2(2s + 1) matrices that realize the direct sum of two irreducible representations D(s) of the algebra SO(3). The operators P_μ and $J_{\mu\nu}$ are Hermitian with respect to the scalar product

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$$(\Psi_1, \Psi_2) = \int d^3x \, \Psi_1^\dagger(t, \mathbf{x}) \Psi_2(t, \mathbf{x}),$$

where † is the operation of Hermitian conjugation.

The representation (2.1) and the corresponding representation in [1] are the same only for s = 1/2. Accordingly, our results are quite different from those obtained in [1-3].

We define the operators of the space P and time $T^{(1)}$ and $T^{(2)}$ reflections in the usual manner

$$P\Psi(t, \mathbf{x}) = r\Psi(t, -\mathbf{x}), \quad P^2 \sim 1,$$
$$T^{(1)}\Psi(t, \mathbf{x}) = \tau^{(1)}\Psi^*(-t, \mathbf{x}), \quad (T^{(1)})^2 \sim 1,$$
$$T^{(2)}\Psi(t, \mathbf{x}) = \tau^{(2)}\Psi(-t, \mathbf{x}), \quad (T^{(2)})^2 \sim 1,$$

where, without loss of generality, the matrix r can be chosen in the form

$$r = I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad r = \sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where 1 is the (2s + 1) × (2s + 1) identity matrix. The matrices $\tau^{(1)}$ and $\tau^{(2)}$ can be taken, for example, in the form of the 2(2s + 1) × 2(2s + 1) Pauli matrices σ_1 and σ_2. Since it is not essential, we shall not specify the explicit form of $\tau^{(1)}$ and $\tau^{(2)}$. We shall not consider the operator of charge conjugation since it is equivalent (~) to the product of $T^{(1)}$ and $T^{(2)}$.

The operators P, $T^{(1)}$, $T^{(2)}$ and the generators P_μ, $J_{\mu\nu}$ satisfy relations

$$[P, H]_- = 0, \quad [P, P_k]_+ = 0, \quad [P, J_{kl}]_+ = 0, \quad [P, J_{0k}]_- = 0,$$
$$[T^{(1)}, H]_- = [T^{(1)}, J_{0k}]_- = 0, \quad [T^{(1)}, P_k]_+ = [T^{(1)}, J_{kl}]_+ = 0,$$
$$[T^{(2)}, H]_+ = [T^{(2)}, J_{0k}]_+ = 0, \quad [T^{(2)}, P_k]_- = [T^{(2)}, J_{kl}]_- = 0. \tag{2.2}$$

An irreducible representation of $\tilde{\mathcal{P}}$ (1, 3) (characterized, of course, by a mass m and spin s) must be realized on the set of solutions $\{\Psi(t, \mathbf{x})\}$ of Eq. (1.1). This means that

$$H^2 = p^2 + m^2. \tag{2.3}$$

The square of the Pauli–Lubanski vector is a multiple of the identity operator on $\{\Psi(t, \mathbf{x})\}$ if the matrices S_{kl} are taken in the form

$$S_{kl} = \begin{pmatrix} s_n & 0 \\ 0 & s_n \end{pmatrix} = S_n, \quad k, l, n \text{ is a cyclic permutation of } (1, 2, 3),$$

where s_n are (2s + 1) × (2s + 1) matrices that realize an irreducible representation of the algebra SO(3) and satisfy

$$[s_k, s_l]_- = i\varepsilon_{kln}s_n.$$

By hypothesis, (1.1) is invariant under $\tilde{\mathcal{P}}$ (1, 3); the operator H must therefore satisfy the commutation relations [4, 5]

$$[H, P_k]_- = [H, J_{kl}]_- = 0, \ [H, J_{0k}]_- = ip_k, \ [p_n, J_{0k}]_- = i\delta_{nk}H, \tag{2.4}$$
$$[J_{kl}, J_{0n}]_- = i\delta_{kn}J_{0l} - i\delta_{nl}J_{0k},$$

$$[J_{0k}, J_{0n}]_- = -iJ_{kn}, \tag{2.5}$$

$$[P, H]_- = [T^{(1)}, H]_- = 0, \quad [T^{(2)}, H]_+ = 0. \tag{2.6}$$

The problem of finding an operator function H for which Eq. (1.1) is invariant under $\tilde{\mathcal{P}}$ (1, 3) has now been reduced to the solution of the system of operator relations (2.4)-(2.6) subject to the condition (2.3).

3. Solution of the System (2.4)-(2.6)

To solve (2.4)-(2.6) we shall reduce the problem to the solution of functional equations by decomposing the operators in (2.4)-(2.6) with respect to a complete system of orthogonal projection operators.

1. Consider the system of projection operators

$$\Lambda_{s_3}=\prod_{\substack{s_3'=-s\\ s_3'\neq s_3}}^{s}\frac{S_p-s_3'}{s_3-s_3'},\quad -s\leqslant s_3\leqslant s, \tag{3.1}$$

where $S_p = S_k p_k/p$, $p = |\mathbf{p}|$. Using the theory of projection operators [2] one can readily show that the system (3.1) is indeed a set of operators of orthogonal projection onto subspaces that are the proper subspaces of the operator S_p with eigenvalues s_3 (the particle helicity), i.e.,

$$\Lambda_{s_3}\Lambda_{s_3'}=\delta_{s_3 s_3'}\Lambda_{s_3'},\quad \sum_{s_3=-s}\Lambda_{s_3}=I,$$

$$S_p^n=\sum_{s_3=-s}(s_3)^n\Lambda_{s_3},\quad n=0,1,\ldots,2s.$$

Instead of the system of operators Λ_{s_3}, it sometimes convenient to use a different system of operators B_{s_3} and C_{s_3}:

$$B_{s_3}=\Lambda_{s_3}+\Lambda_{-s_3},\quad C_{s_3}=\Lambda_{s_3}-\Lambda_{-s_3},\quad 1/2\leqslant s_3\leqslant s,$$

$$B_0\equiv\Lambda_0,\quad \sum_{s_3\geqslant 0}B_{s_3}=I.$$

2. To satisfy the conditions (2.6), we take H in the form

$$H_s=\sum_{s_3=-s}^{s}(\sigma_1 g_{s_3}(p)+\sigma_3 f_{s_3}(p))\Lambda_{s_3}, \tag{3.2}$$

where the unknown functions g_{s_3} and f_{s_3} (which depend only on p) must have the following properties: if r = I, then

$$g_{-s_3}=g_{s_3},\quad f_{-s_3}=f_{s_3},\quad 0\leqslant s_3\leqslant s; \tag{3.3}$$

if $r=\sigma_3$, there are two cases:

$$g_{-s_3}=-g_{s_3},\quad f_{-s_3}=f_{s_3},\quad g_0=0,\quad f_0=\pm E,\quad 1/2\leqslant s_3\leqslant s, \tag{3.4}$$

$$g_{-s_3}=-g_{s_3},\quad f_{-s_3}=-f_{s_3},\quad g_0=f_0=0,\quad 1/2\leqslant s_3\leqslant s. \tag{3.5}$$

Note that (2.6) can also be satisfied by making the substitution

$$\sigma_3\to\sigma_2\quad\text{or}\quad\sigma_1\to\sigma_2\quad\text{or}\quad\sigma_1\rightleftarrows\sigma_3$$

in (3.2). We shall not consider any of the H_s obtained in this manner since they are all unitarily equivalent to (3.2).

An additional restriction is imposed on g_{s_3} and f_{s_3} by (2.3):

$$f_{s_3}^2+g_{s_3}^2=E^2=p^2+m^2,\quad -s\leqslant s_3\leqslant s. \tag{3.6}$$

Direct verification shows that the relations (2.4) with H_s in the form (3.2) are satisfied if (3.6) holds. It therefore remains to consider (2.5), which, in conjunction with (3.3)-(3.5), determines the final structure of H_s, i.e., the explicit form of g_{s_3} and f_{s_3} in (3.2).

With allowance for (2.1), the relations (2.5) take the form

$$\frac{1}{4}[[x_k,H_s]_-,[x_l,H_s]_-]_-=-iS_n,\quad k,\ l,\ n\ \text{is a cyclic permutation of }(1,2,3). \tag{3.7}$$

Multiply (3.7) by p_n, sum over n (n = 1, 2, 3), and use the structure of j_{kl}, $k\neq l$ [see (2.1)]:

$$S[x,H_s]_-H_s=3ipS_p=3ip\sum_{s_3=-s}s_3\Lambda_{s_3}. \tag{3.8}$$

To obtain equations for the coefficient functions substitute (3.2) into (3.8) and use (3.6) and the commutation relations (A. 1):

$$\sum_{s_3=-s}^{s}(g_{s_3}g_{s_3'}+f_{s_3}f_{s_3'})\left[-s_3'a_{s_3s_3'}+d_{s_3s_3'}(s(s+1)-(s_3')^2)\right]=2p^2s_3',$$

$$-s+1\leqslant s_3'\leqslant s-1, \tag{3.9}$$

$$\sum_{s_3=-s}^{s}(g_{s_3}g_s+f_{s_3}f_s)d_{s_3s}=2p^2, \tag{3.10}$$

$$\sum_{s_3=-s}^{s}(g_{s_3}g_{-s}+f_{s_3}f_{-s})d_{s_3,-s}=-2p^2. \tag{3.11}$$

The numerical values of $d_{s_3s_3'}$ (Appendix A) and also (3.6), (3.10), and (3.11) yield

$$f_{\pm s}f_{\pm(s-1)}+g_{\pm s}g_{\pm(s-1)}=m^2-p^2. \tag{3.12}$$

Now write down Eq. (3.9) for $s_3'=s-1$, $s-2$, $s-3$, etc. and use equations of the type (3.12) for $s_3=s$, $s-1$, $s-2$, etc.; the induction yields the recursion relation

$$f_{s_3}f_{s_3-1}+g_{s_3}g_{s_3-1}=m^2-p^2,\quad -s+1\leqslant s_3\leqslant s. \tag{3.13}$$

It follows from (3.13) and (3.6) that for each s_3

$$f_{s_3}=\frac{m^2-p^2}{E^2}f_{s_3-1}+\frac{2mp}{E^2}g_{s_3-1},\quad g_{s_3}=\frac{m^2-p^2}{E^2}g_{s_3-1}-\frac{2mp}{E^2}f_{s_3-1},\ -s+1\leqslant s_3\leqslant s; \tag{3.14}$$

$$f_{s_3}=\frac{m^2-p^2}{E^2}f_{s_3-1}-\frac{2mp}{E^2}g_{s_3-1},\quad g_{s_3}=\frac{m^2-p^2}{E^2}g_{s_3-1}+\frac{2mp}{E^2}f_{s_3-1},\ -s+1\leqslant s_3\leqslant s. \tag{3.15}$$

The recursion relations (3.14) and (3.15) in conjunction with the conditions (3.3)-(3.5) enable us to find all the coefficient functions f_{s_3} and g_{s_3} of H_s in (3.2) if we know at least one function in the set f_{s_3}, $-s\leq s_3\leq s$ (or g_{s_3}). It follows that the system (2.4)-(2.6) is satisfied if H_s has the form (3.2) and f_{s_3} and g_{s_3} satisfy the conditions (3.14), (3.15), (3.3)-(3.5).

At the same time, we obtain a description of all possible operator functions H_s for which Eq. (1.1) is invariant under the complete Poincaré group $\tilde{\mathcal{P}}$ (1, 3).

Remark 1. Equations (3.13)-(3.15) are also valid for $m=0$.

Remark 2. The class of operators H_s with functions f_{s_3} and g_{s_3} satisfying the conditions (3.5) describes particles with vanishing mass ($m=0$) and half-integral spin s since it is only in this case that the conditions (3.6) and (3.13) are satisfied. In this case (3.14) and (3.15) are identical and determine f_{s_3} and g_{s_3} to within an arbitrary function.

Remark 3. The class of operators H_s with functions f_{s_3} and g_{s_3} satisfying (3.3) describes particles with integral spin since (3.6) and (3.13) are compatible only for integral s. In this case (3.14) and (3.15) determine H_s to within an arbitrary function.

Remark 4. The class of operators H_s with functions f_{s_3} and g_{s_3} satisfying the conditions (3.4) describes particles with both integral and half-integral spin. In this case f_{s_3} and g_{s_3} are determined by (3.14) and (3.15) uniquely for both integral and half-integral spins since

$$g_0=0,\quad f_0=\pm E\quad \text{for integral } s, \tag{3.16}$$

$$g_{1/2}=\pm p,\quad f_{1/2}=\pm m\quad \text{for half-integral } s. \tag{3.17}$$

The relation (3.17) follows from the conditions (3.6) and (3.13).

The assertions of Remarks 2, 3, and 4 follow from an investigation of the compatibility of the conditions (3.3)-(3.6) and (3.13) for $s_3=0$, $1/2$.

It is helpful to write (3.2) as a recursion relation:

$$H_s=H_{s-1}+D(s), \tag{3.18}$$

where

$$D(s) = \sigma_1(g_s\Lambda_s + g_{-s}\Lambda_{-s}) + \sigma_3(f_s\Lambda_s + f_{-s}\Lambda_{-s}).$$

One can then find the Hamiltonian for spin s from the Hamiltonian for spin s − 1 (and conversely). Of course, H_{s-1} must be defined in the same space [of 2(2s + 1) dimensions with respect to the spin subscripts] as H_s, although it is actually defined in a space of 2(2s − 1) dimensions. In the 2(2s + 1)-dimensional space H_{s-1} has the same form as in the 2(2s − 1)-dimensional space except that the S_k are now 2(2s + 1) × 2(2s + 1) matrices. Equation (3.18) shows that the Hamiltonian for an arbitrarily high spin is completely determined by the Hamiltonian for the lowest spins s = 1/2, s = 1.

Equation (3.18) may prove helpful in connection with the introduction of an interaction into Eq. (1.1) for s > 1/2.

4. Examples of the Operators H_s

We shall now apply our method to find the simplest operators H_s (m ≠ 0) whose coefficient functions satisfy (3.4). In addition, for m = 0, we shall find all possible [for the representation (2.1)] operators H_s that satisfy (2.4)-(2.6).

1. Since (3.14) and (3.15) are on an equal footing, they can be used in any order, different forms of H_s being obtained depending on the different order in which (3.14) and (3.15) follow each other. As a result, the number of possible H_s increases with increasing s.

Note that (3.14) and (3.15) are valid for $s_3 \geq 0$ and for $s_3 < 0$. For the actual calculations, to which we now turn, it is, however, expedient to use them only for $s_3 > 0$ and then find f_{s_3} and g_{s_3} for $s_3 < 0$ from (3.4).

We have seen (§ 3) that f_{s_3} and g_{s_3} can be found from (3.14) and (3.15) if any one of the functions f_{s_3} (or g_{s_3}), $-s \leq s_3 \leq s$ is known.

Let us consider half-integral spin; then [see (3.17)]

$$f_{1/2} = m, \quad g_{1/2} = p.$$

Generally speaking, (3.14) or (3.15) can be used to find $f_{3/2}$ or $g_{3/2}$. Opting for (3.14), we obtain $f_{3/2} = m$ and $g_{3/2} = -p$ and (3.4) yields $f_{-3/2} = m$ and $g_{-3/2} = p$.

For $f_{5/2}$ and $g_{5/2}$ there is the same freedom of choice between (3.14) and (3.15); taking the latter, we find $f_{5/2} = m$ and $g_{5/2} = p$ and (3.4) yields $f_{-5/2} = m$ and $g_{-5/2} = -p$.

Calculating the higher coefficient functions by regular alternation, i.e., using (3.14) for $s_3 = 7/2, 11/2,$ f_{s_3}, g_{s_3} and (3.15) for $s_3 = 9/2$, we obtain

$$H_s = \sigma_3 m + \sigma_1 p \sum_{s_3 > 1/2} (-1)^{s_3 - 1/2} C_{ss_3} \tag{4.1}$$

If the original functions are $f_{1/2} = m$, $g_{1/2} = -p$ [see (3.17)], we initiate a new alternation process by using (3.15) for $f_{3/2}$ and $g_{3/2}$ and (3.14) for $f_{5/2}$ and $g_{5/2}$ etc. [alternating (3.14) and (3.15) for $s_3 = 9/2, 11/2,$ etc.]. The upshot is

$$H_s = \sigma_3 m - \sigma_1 p \sum_{s_3 > 1/2}^{s} (-1)^{s_3 - 1/2} C_{ss_3} \tag{4.2}$$

If $f_{1/2} = -m$ and $g_{1/2} = \pm p$ [see (3.17)], similar calculations yield

$$H_s = -\sigma_3 m \pm \sigma_1 p \sum_{s_3 > 1/2}^{s} (-1)^{s_3 - 1/2} C_{ss_3} \tag{4.3}$$

A similar procedure yields f_{s_3} and g_{s_3} for integral spins. If $f_0 = E$ and $g_0 = 0$ [see (3.16)],

$$H_s = \sigma_3 \left(E - \frac{2p^2}{E} \sum_{n=0}^{N} B_{2n+1} \right) + \sigma_1 \frac{2mp}{E} \sum_{n=0}^{N} C_{2n+1}, \tag{4.4}$$

where B_{2n+1} and C_{2n+1} are the operators defined in § 3 and

$$N=\begin{cases}\dfrac{s-1}{2}, & \text{if } s \text{ is odd},\\[2mm] \dfrac{s}{2}-1, & \text{if } s \text{ is even}.\end{cases} \tag{4.5}$$

If $f_0 = -E$ and $g_0 = 0$, exactly the same procedure yields an operator H_s that differs only in sign from (4.4).

Remark 1. If $s = 1/2$ in (4.1), H_s is identical with the Dirac Hamiltonian and if $s = 1$ in (4.4) we obtain the Jordan–Mukunda [7] Hamiltonian by a completely different derivation.

Remark 2. A definite alternation of (3.14) and (3.15) was used to obtain (4.1)-(4.4). A different order of these formulas would have yielded more complicated expressions for H_s not amenable to compact expression for arbitrary s. For example, taking the same $f_{1/2}$ and $g_{1/2}$ [see (3.17)] and using (3.15) to calculate f_{s_3} and g_{s_3}, without the use of (3.14), we obtain

$$H'_{3/2}=\pm\left\{\sigma_3 m\left(1-\frac{4p^2}{E^2}B_{3/2}\right)\pm\sigma_1 p\left(C_{1/2}+\frac{3m^2-p^2}{E^2}C_{3/2}\right)\right\}. \tag{4.6}$$

If $s = 2$ and $f_0 = E$, calculation of f_2 and g_2 by (3.15) yields

$$H_2=\sigma_3\left(E-\frac{2p^2}{E}B_1-\frac{8m^2p^2}{E^3}B_2\right)+\sigma_1\frac{2mp}{E}\left(C_1+2\frac{m^2-p^2}{E^2}C_2\right). \tag{4.7}$$

If $f_0 = -E$, we obtain an H_s that differs only in sign from (4.7).

Thus, the operators H_s defined by (4.1)-(4.7) satisfy (2.4)-(2.6) and Eq. (1.1) with such H_s describes a particle (and antiparticle) with integral and half-integral spin.

Remark 3. The explicit form of the H_s for given s depends not only on the given initial functions (of the type $f_{1/2}$, $g_{1/2}$, f_0, and g_0) but also on the order in which (3.14) and (3.15) are used. The number of operators H_s compatible with (2.4)-(2.6) increases with increasing spin in accordance with the greater number of different orders in which (3.14) and (3.15) can be used.

Remark 4. Although the Hamiltonians for given s have different explicit structures, they are all unitarily equivalent in the case of a free theory. At the same time, it must be emphasized that they are physically inequivalent in the sense that the introduction of an interaction into Eq. (1.1) in accordance with, say, the rule $p_k \to p_k - eA_k$ leads to different results for the different H_s. We shall discuss this question in a following paper.

2. If the particle mass vanishes ($m = 0$), (3.14) and (3.15) take the identical form

$$f_{s_3}=-f_{s_3-1},\quad g_{s_3}=-g_{s_3-1},\quad -s+1\leqslant s_3\leqslant s. \tag{4.8}$$

The use of (3.16) and (4.8) for $m = 0$ and integral spins yields

$$H_s=\pm\sigma_1 p\sum_{s_3=-s}^{s}(-1)^{s_3}\Lambda_{s_3}=\pm\sigma_1 p\sum_{s_3\geqslant 0}^{s}(-1)^{s_3}B_{s_3}. \tag{4.9}$$

For half-integral spins (3.17) for $m = 0$ and (4.8) yield

$$H_s=\pm\sigma_1 p\sum_{s_3=-s}^{s}(-1)^{s_3-1/2}\Lambda_{s_3}=\pm\sigma_1 p\sum_{s_3\geqslant 1/2}^{s}(-1)^{s_3-1/2}C_{s_3}. \tag{4.10}$$

For $s = 1/2$, this operator is identical with the Cini–Touschek Hamiltonian in the ultrarelativistic limit.

Equations (4.8) are also valid if f_{s_3} and g_{s_3} satisfy (3.3) and (3.5) (for $m = 0$).

Applying (3.3) to (4.8) and recalling Remark 2 in § 3, we obtain

$$H_s=\sum_{s_3\geqslant 0}^{s}(-1)^{s_3}(\sigma_3 f_s+\sigma_1 g_s)B_{s_3},\qquad g_s^2+f_s^2=p^2. \tag{4.11}$$

Applying (3.5) to (4.8), and recalling Remark 3 of § 3, we obtain

$$H_s = \sum_{s_3=1/2}^{s} (-1)^{s_3-1/2}(\sigma_3 f_{1/2} + \sigma_1 g_{1/2}) C_{s_3}, \quad f_{1/2}^2 + g_{1/2}^2 = p^2. \tag{4.12}$$

Note that the H_s in (4.11) and (4.12) are defined to within an arbitrary function f_0 (or g_0) for integral s and $f_{1/2}$ (or $g_{1/2}$) for half-integral s, since it is only required that

$$f_{s_3}^2 + g_{s_3}^2 = p^2, \quad s_3 = 0, \ 1/2.$$

Now the wave function of a particle (antiparticle) with vanishing mass should have only two components corresponding to the spin projections $s_3 = s$ and $s_3 = -s$. Since a function that satisfies Eq. (1.1) with the operators H_s of the form (4.9)-(4.12) has $2(2s+1)$ components, we must impose additional relativistically invariant conditions to single out just two physically realizable components. These conditions have the form

$$\left\{1 - \frac{1}{2}\left(B_s \pm \frac{H_s}{p} C_s\right)\right\}\Psi(t, x) = 0, \quad B_{s_3}\Psi(t, x) = 0, \quad 0 \leqslant s_3 \leqslant s - 1, \tag{4.13}$$

or

$$\left\{1 - \frac{1}{2}(B_s \pm C_s)\right\}\Psi(t, x) = 0, \quad B_{s_3}\Psi(t, x) = 0, \quad 0 \leqslant s_3 \leqslant s - 1, \tag{4.14}$$

or

$$\left\{1 - \frac{1}{2}\left(1 \pm \frac{H_s}{p}\right) B_s\right\}\Psi(t, x) = 0, \quad B_{s_3}\Psi(t, x) = 0, \quad 0 \leqslant s_3 \leqslant s - 1. \tag{4.15}$$

Equation (3.1), and also the commutation relations (2.2), show that: 1) the conditions (4.13) are $T^{(1)}$ and $CP^{(k)}$ invariant ($k = 1, 2, 3$), but C and $T^{(2)}$ noninvariant; 2) the conditions (4.14) are C, $T^{(1)}$, and $T^{(2)}$ invariant, but $P^{(k)}$ noninvariant; 3) the conditions (4.15) are $T^{(1)}$ and $P^{(k)}$ invariant, but C and $T^{(2)}$ noninvariant.

Thus, Eq. (1.1) with an operator H_s of the form (4.9)-(4.12) and one of the additional conditional conditions (4.13)-(4.15) is invariant under the proper Poincaré group $\mathcal{P}(1, 3)$ but only partially invariant under $P^{(k)}$, $T^{(i)}$, and C transformations. In (4.13)-(4.15), one must take one + or − sign.

For $s = 1/2$, Eq. (1.1) with the additional condition (4.13) is equivalent to Maxwell's equations in vacuum.

5. Transition to the Canonical Representation

In the Foldy–Shirokov canonical representation, the generators P_μ and $J_{\mu\nu}$ of $\mathcal{P}(1, 3)$ have the form

$$P_0^c = H^c = \sigma_3 E, \quad P_k^c = p_k, \quad k = 1, 2, 3,$$
$$J_{kl}^c = x_k p_l - x_l p_k + S_{kl},$$
$$J_{0k}^c = t p_k - \frac{1}{2}[x_k, H^c]_+ - \sigma_3 \frac{S_{kr} p_r}{E + m}. \tag{5.1}$$

In this representation, an equation of the type (1.1) that is invariant under the complete Poincaré group $\tilde{\mathcal{P}}(1, 3)$ has the form

$$i\frac{\partial \Phi(t, x)}{\partial t} = H^c \Phi(t, x), \tag{5.2}$$

where $\Phi(t, x)$ is a $2(2s+1)$-component wave function. Since an irreducible representation of $\tilde{\mathcal{P}}(1, 3)$ is realized on the set of solutions $\{\Phi(t, x)\}$ of Eq. (5.2), the wave functions Ψ and Φ are clearly related:

$$\Phi(t, x) = U\Psi(t, x),$$

where U is a unitary operator that will be determined below.

It is now clear that the problem we have solved in § 3 is equivalent to the problem of finding (describing) all unitary operators U for which the algebra (5.1) goes over into the algebra (2.1). Such operators are found in [6, 7] for $s = 1/2, 1$.

In this section, we shall describe the class of operators U for arbitrary spin and find expressions for the operators of the coordinate X_k, $k = 1, 2, 3$, velocity $\dot{X}_k$, $k = 1, 2, 3$, spin Σ_{kl}, and sign of the energy $\hat{\varepsilon}$.

1. We shall seek U in the form

$$U_s = \sum_{s_3=-s}^{s} (a_{s_3} + i\sigma_3 b_{s_3})\Lambda_{s_3}, \tag{5.3}$$

where $a_{s_3}(p)$ and $b_{s_3}(p)$ are real functions of p. The unitarity condition $U_s U_s^\dagger = I$ implies

$$a_{s_3}^2 + b_{s_3}^2 = 1, \quad -s \leqslant s_3 \leqslant s. \tag{5.4}$$

The generators (5.1) are related to the generators (2.1) by the equations

$$J_{kl} = U_s^\dagger J_{kl}^c U_s = J_{kl}^c, \quad P_k = P_k^c, \tag{5.5}$$

$$H_s = U_s^\dagger H^c U_s,$$

$$J_{0k} = U_s^\dagger J_{0k}^c U_s \quad \text{or} \quad J_{0k}^c = U_s J_{0k} U_s^\dagger. \tag{5.6}$$

Substituting the explicit expressions for H^C and H_S [see (3.2) and (5.1)] into (5.5), we obtain

$$f_{s_3} = E(a_{s_3}^2 - b_{s_3}^2), \tag{5.7}$$

$$g_{s_3} = 2E a_{s_3} b_{s_3}, \quad -s \leqslant s_3 \leqslant s. \tag{5.8}$$

Using the explicit form of J_{0k} and J_{0k}^C [see (2.1) and (5.1)], and (5.6), we obtain

$$[[U_s, x_k]_- U_s^\dagger, H^c]_+ = 2\sigma_2 \frac{S_{kr}p_r}{E+m}. \tag{5.9}$$

On the other hand, with allowance for (5.3) and (5.4), we find

$$[[U_s, x_k]_- U_s^\dagger, H^c]_+ = 2\sigma_3 E \sum_{s_3, s_3'=-s}^{s} (a_{s_3'}a_{s_3} + b_{s_3'}b_{s_3})[\Lambda_{s_3}, x_k]_- \Lambda_{s_3'}. \tag{5.10}$$

Equations (5.9) and (5.10) with allowance for (A.1) and (A.4) yield

$$\sum_{s_3=-s}^{s} (a_{s_3}a_s + b_{s_3}b_s)d_{s_3 s} = \frac{E-m}{E}, \tag{5.11}$$

$$\sum_{s_3=-s}^{s} (a_{s_3}a_{-s} + b_{s_3}b_{-s})d_{s_3,-s} = \frac{m-E}{E}, \tag{5.12}$$

$$\sum_{s_3=-s}^{s} (a_{s_3}a_{s_3'} + b_{s_3}b_{s_3'})a_{s_3 s_3'} = \frac{m-E}{E}, \quad -s+1 \leqslant s_3' \leqslant s. \tag{5.13}$$

Using the numerical values of the coefficients $d_{s_3 s_3'}$ [see (A.3)] we reduce (5.11) and (5.12) to

$$a_{\pm s}a_{\pm(s-1)} + b_{\pm s}b_{\pm(s-1)} = m/E. \tag{5.14}$$

Writing down (5.13) for $s_3' = s-1, s-2, s-3$, etc, and using formulas of the type (5.14) for $s_3 = s, s-1, s-2$, etc., we can prove by induction that [see the proof of (3.13)]

$$a_{s_3}a_{s_3-1} + b_{s_3}b_{s_3-1} = m/E, \quad -s+1 \leqslant s_3 \leqslant s. \tag{5.15}$$

The compatibility of (5.4) and (5.15) also yields the recursion relations

$$a_{s_3} = \frac{m}{E}a_{s_3-1} + \frac{p}{E}b_{s_3-1}, \quad b_{s_3} = \frac{m}{E}b_{s_3-1} - \frac{p}{E}a_{s_3-1}, \quad -s+1 \leqslant s_3 \leqslant s;$$

$$a_{s_3} = \frac{m}{E}a_{s_3-1} - \frac{p}{E}b_{s_3-1}, \quad b_{s_3} = \frac{m}{E}b_{s_3-1} + \frac{p}{E}a_{s_3-1}, \quad -s+1 \leqslant s_3 \leqslant s. \tag{5.16}$$

Equations (5.4), (5.16), and (5.17) determine all possible functions a_{s_3} and b_{s_3} if we know any one of the functions in the set a_{s_3}, $-s \leq s_3 \leq s$ (or b_{s_3}). This function a_{s_3} (or b_{s_3}) must be chosen, for example, for $s_3 = 0, 1/2$ such that (3.3)-(3.6), (5.7), (5.8), (5.16), and (5.17) hold.

Thus, Eqs. (5.4), (5.7), (5.8), (5.16), and (5.17) in conjunction with the conditions (3.3)-(3.5) taken for any one $|s_3|$ solve our problem, i.e., these formulas describe all unitary operators U_s [see (5.3)] that transform the algebra (5.1) into (2.1).

For example, taking the original functions a_{s_3} and b_{s_3} in the form

$$a_{1/2} = a_{-1/2} = \frac{E+m}{\sqrt{2E(E+m)}}, \quad b_{1/2} = -b_{-1/2} = \frac{p}{\sqrt{2E(E+m)}} \tag{5.17}$$

for half-integral s and in the form $a_0 = 1$ and $b_0 = 0$ for integral s, we obtain the following operators from (5.16) and (5.17):

$$U_s = \frac{E + \sigma_3 H_s}{\sqrt{2E(E+m)}} \quad \text{for half-integral } s, \tag{5.18}$$

$$U_s = 1 + \frac{m-E}{E} \sum_{n=0}^{N} B_{2n+1} + i\sigma_2 \frac{p}{E} \sum_{n=0}^{N} C_{2n+1} \quad \text{for integral } s, \tag{5.19}$$

where the number N is defined in (4.5).

The operators (5.18) and (5.19) transform H^c into the operators (4.1) and (4.4), respectively. For s = 1/2 the operator (5.18) is identical with the Foldy–Wouthuysen operator.

For completeness we may mention that if H_s is given in the representation (2.1) (and, hence, all the f_{s_3} and g_{s_3} are given) the coefficient functions a_{s_3} and b_{s_3} can be expressed in terms of f_{s_3} and g_{s_3} by means of (5.8) and (5.20), i.e.,

$$a_{s_3} = \pm \sqrt{\frac{E + f_{s_3}}{2E}}, \quad b_{s_3} = \sqrt{\frac{E - f_{s_3}}{2E}}, \quad -s \leqslant s_3 \leqslant s. \tag{5.20}$$

Equations (5.20) are solutions of the system (5.4) and (5.7).

If a unitary operator U_s [see (5.3)] with the coefficient functions (5.20) satisfying (5.7) and (5.8) is to transform the algebra (2.1) into (5.1) it is also necessary that a_{s_3} and b_{s_3} satisfy (5.15) [and, hence, (5.16) and (5.17)].

2. The operators of the coordinate X_k, velocity $\dot{X}_k$, spin Σ_{kl}, and sign of the energy $\hat{\varepsilon}$ in the representation (2.1) have the form

$$X_k = x_k + \frac{S_{kr}p_r}{E(E+m)} + \sigma_1 \Big\{ -i \frac{S_{kr}p_r}{p^2} \sum_{s_3'=-s+1}^{s-1} \Lambda_{s_3'} \sum_{s_3=-s}^{s} (a_{s_3}b_{s_3'} - a_{s_3'}b_{s_3}) a_{s_3s_3'} + \sum_{s_3', s_3=-s}^{s} \Big(\frac{S_{nl}}{p} - \frac{p_k}{p^2} s_3' \Big) (a_{s_3}b_{s_3'}$$

$$- a_{s_3'}b_{s_3}) d_{s_3s_3'}\Lambda_{s_3'} \Big\} - \sigma_1 \frac{p_k}{p} \sum_{s_3=-s}^{s} \Big(a_{s_3} \frac{\partial b_{s_3}}{\partial p} - \frac{\partial a_{s_3}}{\partial p} b_{s_3} \Big) \Lambda_{s_3}, \quad k, n, l \text{ is a cyclic permutation of } (1, 2, 3);$$

$$\Sigma_{kn} = S_{kn} + \frac{pp_lS_p - S_{ln}p^2}{E(E+m)} + \sigma_2 \frac{S_{lr}p_r}{p} \sum_{s_3'=-s}^{s} \Lambda_{s_3'} \sum_{s_3=-s}^{s} d_{s_3s_3'}(a_{s_3}b_{s_3'} - b_{s_3}a_{s_3'}) - i\sigma_2 \sum_{s_3'=-s+1}^{s-1} \Lambda_{s_3'} \Big(\frac{p_l}{p} s_3' - S_{kn} \Big)$$

$$\times \sum_{s_3=-s}^{s} a_{s_3s_3'}(a_{s_3}b_{s_3'} - a_{s_3'}b_{s_3}), \quad k, n, l \text{ is a cyclic permutation of } (1, 2, 3);$$

$$\dot{X}_k = \frac{p_k H_s}{E^2}, \quad \hat{\varepsilon} = \frac{H_s}{E},$$

where $a_{s_3s_3'}$ and $d_{s_3s_3'}$ are given in (A.3) and a_{s_3} and b_{s_3} are determined by the method described above.

For the operators U_s in (5.18) and (5.19), X_k and Σ_{kn} have the form

$$X_k = x_k + \frac{S_{kr}p_r}{E(E+m)} - \frac{S_{kr}p_rE - imp_k}{p^2E^2}(\sigma_3 H_s - m),$$

$$\Sigma_{kn} = \frac{m}{E} S_{kn} + \frac{p p_l S_p}{E(E+m)} + \frac{1}{E}\left(S_{kn} - \frac{p_l}{p} S_p\right)(\sigma_3 H_s - m),$$

where k, n, *l* is a cyclic permutation of (1, 2, 3) and s is half-integral;

$$X_k = x_k + \frac{S_{kr} p_r}{E(E+m)} + i\sigma_3 \frac{S_{kr} p_r}{pE} \sum_{s_3>0}^{s-1} (-1)^{s_3} C_{s_3} - \sigma_1 \frac{S_{nl}}{E} B_s - \sigma_2 \frac{m p_k}{E^2 p} \sum_{n=0}^{s} C_{2n+1} + i\sigma_1 (-1)^s \frac{S_{kr} p_r}{Ep} B_s,$$

$$\Sigma_{kn} = \frac{m}{E} S_{kn} + \frac{p_l p S_p}{E(E+m)} - \sigma_3 \frac{S_{lr} p_r}{E} B_s + \sigma_2 (-1)^s \frac{S_{lr} p_r}{E} B_s - i\sigma_2 \frac{p}{E}\left(S_{kn} - \frac{p_l}{p} S_p\right) \sum_{s_3>0}^{s-1} (-1)^{s_3} C_{s_3},$$

where k, n, *l* is a cyclic permutation of (1, 2, 3) and s is integral.

The operators X_k, $\dot{X}_k$, Σ_{kn}, and $\hat{\varepsilon}$ for s = 1/2 are identical with the operators obtained in [6].

APPENDIX

In this appendix we give (without proof) all the formulas used in the main text to derive our results:

$$[x_k, \Lambda_{s_3}]_- = \frac{S_{kr} p_r}{p^2} \sum_{s_3'=-s+1}^{s-1} a_{s_3 s_3'} \Lambda_{s_3'} + i \sum_{s_3'=-s}^{s} \left(\frac{S_{nl}}{p} - \frac{p_k}{p^2} s_3'\right) d_{s_3 s_3'} \Lambda_{s_3'}. \tag{A.1}$$

$$[S_{kn}, \Lambda_{s_3}]_- = \frac{i S_{lr} p_r}{p} \sum_{s_3'=-s}^{s} d_{s_3 s_3'} \Lambda_{s_3'} - \sum_{s_3'=-s+1}^{s-1} \left(S_{kn} - \frac{p_l}{p} s_3'\right) a_{s_3 s_3'} \Lambda_{s_3'}, \tag{A.2}$$

where $a_{s_3' s_3} \neq 0$, if $s_3' = s_3 - 1, s_3 + 1$,

$$\begin{gathered} a_{s_3 s_3} = -1,\ a_{s_3-1\, s_3} = a_{s_3+1\, s_3} = 1/2, \quad -s+1 \leq s_3 \leq s-1, \\ d_{s_3' s_3} \neq 0, \quad \text{if} \quad s_3' = s_3 - 1,\ s_3 + 1,\ -s+1 \leq s_3 \leq s-1, \\ d_{s_3+1\, s_3} = -d_{s_3-1\, s_3} = 1/2,\ -d_{s-1\, s} = -d_{-s+1\, -s} = d_{-s+1-s} = d_{ss} = 1. \\ \left(\frac{S_{kr} p_r}{p^2} \pm i \frac{S_{nl}}{p} \mp i \frac{p_k}{p^2} s_3\right) \Lambda_{s_3} = 0, \quad k, n, l \text{ is a cyclic permutation of } (1,2,3,), \quad s_3 = \pm s. \end{gathered} \tag{A.3}$$

$$[S_{nl}, \Lambda_{s_3}]_- = -[x_k p_l - x_l p_k, \Lambda_{s_3}]_-, \ [S_{kr} p_r, \Lambda_{s_3}]_- = -p^2 [x_k, \Lambda_{s_3}]_-, \tag{A.4}$$

$$S_{kr} p_r S_k = -S_k S_{kr} p_r = i p S_p = i p \sum_{s_3=-s}^{s} s_3 \Lambda_{s_3},$$

$$S_{kr} p_r S_{kr'} p_{r'} = p^2 \sum_{s_3=-s} [s(s+1) - s_3^2] \Lambda_{s_3}, \ S_k S_k = \mathbf{S}^2 = s(s+1).$$

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