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EQUATIONS OF MOTION FOR PARTICLES OF ARBITRARY SPIN INVARIANT UNDER THE GALILEO GROUP

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Systems of differential equations of first and second order are derived that are invariant under the Galileo group and describe the motion of a particle with arbitrary spin. These equations admit a Lagrangian formulation and describe the dipole, spin–orbit, and Darwin couplings of the particle to an external electromagnetic field; traditionally, these have been regarded as purely relativistic effects. Examples are given of infinite-component equations that are invariant under the Galileo group. The problem of the motion of a nonrelativistic particle with spin $s = \frac{1}{2}$ in a homogeneous magnetic field is solved exactly.

Introduction

Relativistic equations of motion for particles with arbitrary spin stimulate great and sustained interest among physicists and mathematicians (see [1] and the literature quoted there). On the other hand, remarkably little literature has been devoted to equations invariant under the Galileo group. But as early as 1954, Bargmann [2] showed that by means of the central extension of the Galileo group the concept of particle spin can be introduced consistently in nonrelativistic quantum mechanics as well.

In [3, 4], Galileo invariant differential equations of first order describing the motion of a nonrelativistic particle of arbitrary spin were obtained. These equations describe the dipole interaction of a particle with an external field but do not take into account well-known physical effects such as the spin–orbit and Darwin couplings.

In the present paper, using the method developed in [1, 5, 6] to derive Poincaré invariant equations, we obtain Galileo invariant equations of motion for a particle with arbitrary spin s, these being capable of describing the above couplings. This is achieved by an extension of the Galileo group G to the group G', which includes the transformation of simultaneous reflection of the coordinates and the time. The obtained equations have the Schrödinger form

$$i\partial\Psi(t, \mathbf{x})/\partial t=H_s(\mathbf{p})\Psi(t, \mathbf{x}), \quad p_a=-i\partial/\partial x_a \tag{0.1}$$

(where $H_s(\mathbf{p})$ is a differential operator of second order, and Ψ is a $2(2s+1)$-component wave function), permit a Lagrangian formulation, and describe the dipole, spin–orbit, Darwin, and quadrupole couplings of a particle of spin s to an external electromagnetic field. This means, in particular, that the listed interactions, which are usually introduced as relativistic corrections, can be treated consistently in the framework of nonrelativistic quantum mechanics.

In the paper, we also obtain Galileo invariant differential equations of first order describing the

Institute of Mathematics, Ukrainian Academy of Sciences. Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol.44, No.1, pp.34–46, July, 1980. Original article submitted October 6, 1978.

 0040-5779/80/4401-0584$07.50

motion of a particle with arbitrary spin. After the minimal coupling $p_\mu \to p_\mu - eA_\mu$, these equations also describe the spin–orbit and Darwin couplings of the particle to a field. We also give an example of infinite-component equations invariant under the Galileo group.

1. Basic Definitions and Formulation of the Problem

The Galileo group G is the set of transformations of the coordinates x_a $(a=1, 2, 3)$ and the time t of the form

$$x_a \to x_a' = R_{ab}x_b + V_a t + b_a, \quad t \to t' = t + b_0, \tag{1.1}$$

where R_{ab} is an operator of a three-dimensional rotation, and V_a and b_μ are arbitrary real parameters.

The representation of the group G is uniquely determined by specifying the explicit form of the infinitesimal operators P_μ, J_a, and G_a corresponding to displacements, rotations, and Galilean boosts.

DEFINITION. We shall that Eq. (0.1) is invariant under the Galileo group if the Hamiltonian $H_s = P_0$ and the generators P_a, J_a, G_a satisfy the commutation relations

$$[P_a, P_b]=0, \quad [P_a, J_b]=i\varepsilon_{abc}P_c, \tag{1.2a}$$

$$[G_a, G_b]=0, \quad [G_a, J_b]=i\varepsilon_{abc}G_c, \tag{1.2b}$$

$$[P_a, G_b]=i\delta_{ab}M, \quad [M, P_\mu]=[M, J_a]=[M, G_a]=0, \tag{1.2c}$$

$$[H_s, P_a]=[H_s, J_a]=0, \tag{1.2d}$$

$$[H_s, G_a]=iP_a, \quad a, b=1, 2, 3, \quad \mu=0, 1, 2, 3. \tag{1.2e}$$

The relations (1.2) define the Lie algebra of the Galileo group. The algebra (1.2) has three invariant operators (Casimir operators):

$$2MC_1=2MP_0-P_aP_a, \quad C_2=M, \quad C_3=(MJ_a-\varepsilon_{abc}P_bG_c)(MJ_a-\varepsilon_{ade}P_dG_e). \tag{1.3}$$

The eigenvalues of the operators C_1, C_3, and C_2 are associated with the internal energy, spin, and mass of the particle described by the invariant equation (0.1).

We shall solve the problem of finding all possible (up to equivalence) Galileo invariant equations of the form (0.1) in two approaches, which are in general inequivalent. In approach I, the problem is formulated as follows: to find all Hamiltonians H_s^I such that the operators

$$P_0^I = H_s^I, \quad P_a^I = p_a = -i\partial/\partial x_a, \quad J_a = (x\times p)_a + S_a, \quad G_a = tp_a - mx_a + \lambda_a^I \tag{1.4}$$

satisfy the Lie algebra of the extended Galileo group (1.2). Here

$$S_c = \begin{pmatrix} s_c & 0 \\ 0 & s_c \end{pmatrix}, \quad (a, b, c) \text{ is cyclic perm. of } (1,2,3); \tag{1.5}$$

s_c are the generators of the irreducible representation D(s) of O(3), m is the parameter which specifies the particle mass, and λ_a^I are numerical matrices whose explicit form will be determined below.

Equations (1.4) define the general form of the generators of the Galileo group corresponding to the local transformations of a 2(2s + 1)-component wave function on the transition to a new coordinate system (1.1),

$$\Psi(t, \mathbf{x}) \to \Psi'(t', \mathbf{x}') = \exp[if(t, \mathbf{x})]D^s(R_{ab}, v_a)\Psi(t, \mathbf{x}), \tag{1.6}$$

where $D^s(R_{ab}, v_a)$ is a numerical matrix that depends on the transformation parameters (1.1), and $f(t, x)$ is a phase factor [2]:

$$f(t, \mathbf{x}) = mv_aR_{ab}x_b + {}^1/_2 mv_av_a. \tag{1.7}$$

We shall see below that the operators H_s^I can always be chosen to make Eq. (0.1) also invariant under the antiunitary transformation of reflection of the coordinates and the time:

$$\Psi(t, \mathbf{x}) \to r_1\Psi^*(-t, -\mathbf{x}), \quad r_1^2=1, \tag{1.8}$$

where r_1 is a matrix.

In the approach II, the problem reduces to determining all possible differential operators H_s^{II} such that the generators

$$P_0^{II} = H_s^{II}, \quad P_a^{II} = p_a = -i\partial/\partial x_a, \quad J_a = (\mathbf{x}\times\mathbf{p})_a + S_a, \quad G_a^{II} = tp_a - \sigma_3 mx_a + \lambda_a^{II} \tag{1.9}$$

satisfy the algebra (1.2). Here, σ_3 is one of the Pauli matrices

$$\sigma_0=\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},\quad \sigma_1=\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},\quad \sigma_2=i\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},\quad \sigma_3=\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

and I and 0 are (2s + 1)-row square unit and null matrices, respectively, and λ_a^{II} are certain operators (in the general case they depend on p_a) that we must also find. One can show that Eqs. (1.9) give the general form of the generators of G for which Eq. (0.1) is invariant under the unitary transformation $\Psi(t, \mathbf{x})\to r_2\Psi(-t, -\mathbf{x})$, $r_2=\sigma_2$.

We require that the generators (1.9) be Hermitian under the scalar product usually adopted in quantum mechanics:

$$(\Psi_1, \Psi_2)=\int d^3x\, \Psi_1^\dagger\Psi_2. \tag{1.10}$$

An important difference between the representations (1.4) and (1.9) is that the generators H_s^I and G_a^I are non-Hermitian under (1.10) but are Hermitian in the Hilbert space with scalar product

$$(\Psi_1, \Psi_2)=\int d^3x\, \Psi_1^\dagger\hat{M}\,\Psi_2, \tag{1.11}$$

where $\hat{M}$ is some positive-definite differential operator, or with respect to the indefinite metric when $\hat{M}$ in (1.11) is some numerical positive-indefinite matrix. The explicit form of $\hat{M}$ will be found below. Thus, the complication of the metric is the price that must be paid for the local transformations (1.6) of the wave function. The situation is the same for relativistic equations [1].

We require H_s^{II} to satisfy the condition

$$(H_s^{II})^2=(m+p^2/2m)^2. \tag{1.12}$$

This is equivalent to the requirement that the internal energy of the particle be equal to its mass.

Thus, the problem of finding Galileo invariant equations of the form (0.1) reduces to the solution of the system of relations (1.2) for the operators (1.4) and (1.9).

2. Explicit Form of the Hamiltonians H_s^I

We give the solution to problem I in the form of a theorem.

THEOREM 1. All possible (up to equivalence) Hamiltonians H_s^I satisfying together with the generators (1.4) the commutation relations (1.2) and (1.4) are given by the formulas

$$H_s^I=\sigma_3\eta m-2i\eta k\sigma_1\mathbf{S}\cdot\mathbf{p}+\frac{1}{2m}C_{ab}p_ap_b,\quad a, b=1, 2, 3, \tag{2.1a}$$

$$\tilde{H}_s^I=\sigma_1\tilde{\eta}m+\frac{p^2}{2m}-2\eta k(\sigma_2-i\sigma_3)\mathbf{S}\cdot\mathbf{p}, \tag{2.1b}$$

where $C_{ab}=\delta_{ab}-2\eta k^2(\sigma_3+i\sigma_2)(S_aS_b+S_bS_a)$, η, k, and $\tilde{\eta}$ are arbitrary parameters.

Proof. We determine first the explicit form of the matrices λ_a^I in (1.4). From (1.2b), we obtain for λ_a^I the equations

$$[\lambda_a^I, \lambda_b^I]=0,\quad [\lambda_a^I, S_b]=i\varepsilon_{abc}\lambda_c^I,\quad [S_a, S_b]=i\varepsilon_{abc}S_c. \tag{2.2}$$

From (1.5) and (2.2) we conclude that, without loss of generality, the matrices λ_a^I can be represented in the form

$$\lambda_a^I=k(\sigma_3+i\sigma_2)S_a, \tag{2.3}$$

where k is an arbitrary coefficient.

We find the general form of the Hamiltonian H_s^I in the representation in which $\lambda_a = 0$. The transition to such a representation is made by means of the operator [7]

$$V=\exp(i\lambda^I\cdot\mathbf{p}/m)=1+i\lambda^I\cdot\mathbf{p}/m. \tag{2.4}$$

Using (2.4), we obtain

$$(H_s^I)'=VH_s^IV^{-1},\quad (P_a^I)'=VP_a^IV^{-1}=p_a,\quad J_a'=VJ_aV^{-1}=J_a,\quad (G_a^I)'=VG_a^IV^{-1}=tp_a-mx_a. \tag{2.5}$$

From (2.5) and (1.2) we conclude that the general form of the operator $(H^I_s)'$ is given by the formula

$$(H_s^I)'=p^2/2m+A, \quad A=\sigma_\mu a^\mu m, \tag{2.6}$$

where a_μ are arbitrary coefficients and, without loss of generality, we can set $a_0=0$.

We can show that by means of transformations that do not change the general form of λ^I_a (2.3) the matrix A (2.6) can be reduced to one of the forms

$$A=\sigma_3\eta m \text{ or } A=\sigma_1\tilde{\eta}m. \tag{2.7}$$

Substituting (2.7) in (2.6) and using the inverse of the transformation (2.5), we arrive at Eqs. (2.1). This proves the theorem.

Equations (2.1) define nonrelativistic Hamiltonians for particles with arbitrary spin. In the case $s={}^1/_2, k=-i, \eta=1$, Eq. (0.1), (2.1a) can be written in the compact form

$$(\gamma_\mu p^\mu-m)\Psi=(1+\gamma_4-\gamma_0)\frac{n^2}{2m}\Psi, \tag{2.8}$$

where $\gamma_0=\sigma_3$, $\gamma_a=-2i\sigma_2 S_a$, $\gamma_4=i\gamma_0\gamma_1\gamma_2\gamma_3$ are Dirac matrices.

Note that all the Hamiltonians (2.1) belong to the class of second-order differential operators, which was not required a priori. In the framework of the Poincaré group, the Hamiltonians of particles with arbitrary spin are, as a rule, integro-differential operators [1, 5].

The parameters k, η, and $\tilde{\eta}$ can always be chosen to make Eqs. (0.1) and (2.1) invariant under the antiunitary operation (1.8) of reflection of the coordinates and the time. A necessary and sufficient condition of such invariance is the simultaneous fulfillment of the relations

$$\eta^*=\pm\eta, \quad k^*=\pm k \text{ or } \tilde{\eta}^*=\tilde{\eta}, \quad k^*=k, \tag{2.9}$$

where $r_1=\sigma_1\Delta$, if $\eta^*=-\eta$, $k^*=-k$ or $\tilde{\eta}^*=\eta$, $k^*=k$, $r_1=\Delta$, if $\eta^*=\eta$, $k^*=k$, $\Delta=\begin{pmatrix}\Delta' & 0\\ 0 & \Delta'\end{pmatrix}$, where Δ' are matrices defined up to the phase by the relations [8]

$$\Delta' s_a=-s_a^*\Delta', \quad (\Delta')^2=(-1)^{2s}.$$

Thus, under the restrictions on the parameters η, $\tilde{\eta}$, and k imposed by Eqs. (2.9), Eqs. (0.1) and (2.1) are invariant under the extended Galileo group including the transformations (1.8).

The Hamiltonians (2.1) and the operators (1.4) and (2.3) are non-Hermitian in the scalar product (1.10). However, these operators are Hermitian in the metric (1.11), where $\hat{M}$ is a positive-definite operator:

$$\hat{M}=(V^{-1})^+V^{-1}=1+[i(k-k^*)\sigma_3-(k+k^*)\sigma_2]\mathbf{S}\cdot\mathbf{p}/m+2(k^*k)(1+\sigma_1)(\mathbf{S}\cdot\mathbf{p})^2/m^2. \tag{2.10}$$

In addition, if η, k, and $\tilde{\eta}$ satisfy the conditions (2.9), the Hamiltonians (2.1) are Hermitian in an indefinite metric of the form (1.11), when

$$\hat{M}=\xi=\begin{cases}\sigma_3, & \text{if } \eta^*=\eta,\ k^*=k,\ \tilde{\eta}^*=-\tilde{\eta},\\ \sigma_2, & \text{if } \eta^*=-\eta,\ k^*=-k,\ \tilde{\eta}^*=-\tilde{\eta}.\end{cases} \tag{2.11}$$

If (2.11) is satisfied, Eqs. (0.1) and (2.1) can be obtained by means of a variational principle. The corresponding Lagrangians have the form

$$L(t,\mathbf{x})=\frac{i}{2}\left(\bar{\Psi}\frac{\partial\Psi}{\partial t}-\frac{\partial\bar{\Psi}}{\partial t}\Psi\right)-\eta m\,\bar{\Psi}\sigma_3\Psi-\eta k\left(\bar{\Psi}\sigma_1 S_a\frac{\partial\Psi}{\partial x_a}-\frac{\partial\bar{\Psi}}{\partial x_a}\sigma_1 S_a\Psi\right)-\frac{1}{2m}\frac{\partial\bar{\Psi}}{\partial x_c}C_{ab}\frac{\partial\Psi}{\partial x_b}, \tag{2.12a}$$

when H^I_s is given by Eq. (2.1a), and

$$L(t,\mathbf{x})=\frac{i}{2}\left\{\bar{\Psi}\frac{\partial\Psi}{\partial t}-\frac{\partial\bar{\Psi}}{\partial t}\Psi-2i\tilde{\eta}m\bar{\Psi}\sigma_1\Psi+2\tilde{\eta}k\left[\bar{\Psi}(\sigma_2-i\sigma_3)S_a\frac{\partial\Psi}{\partial x_a}-\frac{\partial\bar{\Psi}}{\partial x_a}(\sigma_2-i\sigma_3)S_a\Psi\right]\right\}-\frac{1}{2m}\frac{\partial\bar{\Psi}}{\partial x_a}\frac{\partial\Psi}{\partial x_b}, \tag{2.12b}$$

if the Hamiltonian has the form (2.1b). Here $\bar{\Psi}=\Psi^\dagger\xi$.

The Lagrangians (2.12) are scalars under the transformations (1.1) and (1.6), where

$$D^s(R_{ab}, V_a)=(1+i\lambda^I\cdot v)D^s(R_{ab}), \tag{2.13}$$

and $D^s(R_{ab})$ are matrices that realize the direct sum $D(s)\oplus D(s)$ of two irreducible representations of SO(3).

3. Explicit Form of the Hamiltonians H_s^{II}

We solve problem II, i.e., we find differential operators that satisfy together with (1.9) the relations (1.2) and (1.12).

THEOREM 2. All possible (up to equivalence transformations) differential operators H_s^{II} that are Hermitian in the metric (1.10) and satisfy the conditions (1.2), (1.9), and (1.12) are given by the formulas

$$H_s^{II}=\sigma_3\left[m+\frac{p^2}{2m}-\frac{(S_aS_b+S_bS_a)p_ap_b}{2mS^2}\sin^2\theta_s\right]+\sigma_2\sqrt{2}\sin\theta_s\frac{\mathbf{S}\cdot\mathbf{p}}{S}+\sigma_1\left[a_s\frac{p^2}{2m}+\frac{b_s}{4ms^2}(S_aS_b+S_bS_a)p_ap_b\right], \tag{3.1}$$

where

$$a_{1/2}=\sin 2\theta_{1/2},\quad b_{1/2}=0,\quad a_1=1,\quad b_1=\sin 2\theta_1,\quad a_{3/2}=b_{3/2}-{}^5/_4\sin 2\theta_{3/2}=-{}^1/_8\sin 2\theta_{3/2}-{}^3/_4\sin\theta_{3/2}(1-{}^1/_9\sin^2\theta_{3/2})^{1/2},$$

$$a_s=b_s=\theta_s=0,\quad s>{}^3/_2,$$

and $\theta_{1/2}$, θ_1, $\theta_{3/2}$ are arbitrary real parameters.

Proof. We show first that the operators H_s^{II} can include derivatives of not higher than second order. Indeed, suppose $H_s^{II}=\sum\limits_{i=0}^{N}H_i$, where H_i contains derivatives of only i-th order; then from (1.12), we obtain

$$H_NH_N=H_N^+H_N=0 \text{ or } H_N=0,\quad \text{if}\quad N>2. \tag{3.2}$$

We represent the required differential operators H_s^{II} in the form of an expansion with respect to the spin matrices and 2(2s + 1)-row Pauli matrices (1.9):

$$H_s^{II}=\sum_{\mu=0}^{3}\left[a_\mu^s m+b_\mu^s\frac{p^2}{2m}+c_\mu^s S\cdot p+d_\mu^s\frac{(S\cdot p)^2}{2m}\right]\sigma_\mu, \tag{3.3}$$

where a_μ^s, b_μ^s, c_μ^s, d_μ^s are arbitrary real coefficients. Using the orthogonal projection operators [1, 5]

$$\Lambda_r=\prod_{r'\neq r}\frac{(\mathbf{S}\cdot\mathbf{p})p^{-1}-r'}{r-r'},\quad r,r'=-s,-s+1,\ldots,s,\quad \Lambda_r\cdot\Lambda_{r'}=\delta_{rr'},\quad \sum_r\Lambda_r=1,\quad \sum_r r^l\Lambda_r=\left(\frac{\mathbf{S}\cdot\mathbf{p}}{p}\right)^l,$$

we can rewrite H_s^{II} in the form

$$H_s^{II}=\sum_{\mu=0}^{3}\sum_{r=-s}^{s}\left[a_\mu^s m+(b_\mu^s+r^2d_\mu^s)\frac{p^2}{2m}+rpc_\mu^s\right]\sigma_\mu\Lambda_r. \tag{3.4}$$

The operators (3.4) obviously satisfy the conditions (1.2d) and (1.10). We require that (1.12) hold. Substituting (3.4) in (1.12), using the orthogonality of the operators Λ_r, and equating the independent terms, we find that a_μ^s, b_μ^s, c_μ^s, d_μ^s must satisfy one of the system of equations

$$\sum_{i=1}^{3}(a_i^s)^2=0,\quad \sum_{i=1}^{3}[r^2(c_i^s)^2+a_i^s(b_i^s+r^2d_i^s)]=1,\quad \sum_{i=1}^{3}rc_i^s(b_i^s+r^2d_i^s)=0,\quad \sum_{i=1}^{3}rc_i^sa_i^s=0,\quad \sum_{i=1}^{3}(b_i^s+r^2d_i^s)^2=1,$$
$$a_0^s=b_0^s=d_0^s=c_0^s=0 \tag{3.5}$$

or

$$a_0^s=b_0^s=1,\quad d_0^s=c_0^s=a_i^s=b_i^s=c_i^s=d_i^s=0,\quad i=1,2,3. \tag{3.6}$$

The general solution of Eqs. (3.5) and (3.4) (up to equivalence transformations realized by numerical matrices) is also given by Eqs. (3.1). One can show that the solution (3.6) is incompatible with (1.2a), (1.2b), and (1.2e).

To complete the proof of the theorem, it is now sufficient to indicate the explicit form of the operators λ_a^{II} for which the operators (1.8) satisfy Eqs. (1.2b) and (1.2e). It is easy to show that λ_a^{II} can be chosen in the form

$$\lambda_a^{II}=[U,\sigma_3x_a]U^+, \tag{3.7}$$

where

$$U=(E+\sigma_3H_s^{II})\,/\,\sqrt{2E(E+{}^1/_2H_s^{II}\sigma_3+{}^1/_2\sigma_3H_s^{II})},\quad E=m+p^2/2m, \tag{3.8}$$

is the operator that diagonalizes the Hamiltonian (3.1) and the generators (1.8):

$$U^\dagger H_s^{II}U=\sigma_3E,\quad U^\dagger G_aU=tp_a-\sigma_3mx_a. \tag{3.9}$$

The theorem is proved.

In the case $\theta_{1/2}=\pi/4$, Eq. (0.1), (3.1a) takes a particularly simple form (cf. Eq. (2.8)):

$$(\gamma_\mu p^\mu+m)\Psi=i\gamma_4\frac{p^2}{2m}\Psi. \tag{3.10}$$

Equation (3.10) differs from the relativistic Dirac equation only by the presence of the term on the right-hand side, which obviously destroys the invariance under the Poincaré group but preserves the invariance under the Galileo group.

4. Nonrelativistic Particle in an External Electromagnetic Field

To go over to the description of the motion of a charged particle in an external electromagnetic field, we make in Eq. (0.1) the usual substitution

$$p_\mu\to\pi_\mu=p_\mu-eA_\mu, \tag{4.1}$$

where A_μ is the four-vector potential of the external field. We then arrive at the equations

$$i\frac{\partial}{\partial t}\Psi(t,x)=H_s^\alpha(\pi,A_0)\Psi(t,x),\qquad \alpha=\mathrm{I},\mathrm{II}, \tag{4.2}$$

where $H_s^\alpha(\pi, A_0)$ is one of the Hamiltonians obtained from (2.1) and (3.1) by the substitution (4.1):

$$H_s^{\mathrm{I}}(\pi,A_0)=\sigma_3\eta m+\frac{\boldsymbol{\pi}^2}{2m}-2i\eta k\sigma_1\mathbf{S}\cdot\boldsymbol{\pi}+eA_0-(\sigma_3+i\sigma_2)\frac{\eta k^2}{m}[(\mathbf{S}\cdot\boldsymbol{\pi})^2+{}^1\!/_2 e\mathbf{S}\cdot\mathbf{H}], \tag{4.3a}$$

$$\tilde{H}_s^{\mathrm{I}}(\boldsymbol{\pi},A_0)=\sigma_3\tilde{\eta}m+\frac{\boldsymbol{\pi}^2}{2m}-2\eta k(\sigma_2-i\sigma_3)\mathbf{S}\cdot\boldsymbol{\pi}+eA_0, \tag{4.3b}$$

$$H_s^{\mathrm{II}}(\boldsymbol{\pi},A_0)=\sigma_3\left[m+\frac{\boldsymbol{\pi}^2}{2m}-\frac{(\mathbf{S}\cdot\boldsymbol{\pi})^2}{ms^2}\sin^2\theta_s-e\frac{S\cdot H}{2ms^2}\sin^2\theta_s\right]+\sigma_1\left[a_s\frac{\pi^2}{2m}+b_s\frac{(S\cdot\pi)^2}{2ms^2}+eb_s\frac{\mathbf{S}\cdot\mathbf{H}}{4ms^2}\right]+\sigma_2\sqrt{2}\frac{\mathbf{S}\cdot\boldsymbol{\pi}}{s}\sin\theta_s+eA_0. \tag{4.3c}$$

In (4.3) $H_a=-i\varepsilon_{abc}\pi_b\pi_c$ is the magnetic field intensity.

Equations (4.2) and (4.3) are obviously invariant under gauge transformations. In addition, as they were before the introduction of the interaction, Eqs. (4.3) with the Hamiltonians (4.3a) and (4.3b) are invariant under transformations in the Galileo group (1.6), (2.13), if the vector potential transforms in accordance with the law [3]

$$A_b\to A_b'=R_{bc}A_c,\quad A_0\to A_0'=A_0+v_aA_a. \tag{4.4}$$

It is convenient to analyze Eqs. (4.2) in the representation in which the operators (4.3) are quasi-diagonal (i.e., commute with one of the σ matrices). As in the case of the Dirac equation, the Hamiltonians (4.3) can be diagonalized only approximately. Below, we implement such a diagonalization and represent the Hamiltonian of a particle with arbitrary spin in the form of a series in powers of $1/m$, which is convenient for calculations in perturbation theory.

Diagonalization of the Hamiltonians (4.3) up to terms of order $1/m^2$ is implemented by means of the operators

$$V^\alpha=\exp\left(iC_s^\alpha+\sigma_3\frac{1}{2\eta^\alpha m}\frac{\partial B_s^\alpha}{\partial t}\right)\exp(iB_s^\alpha)\exp(iA_s^\alpha),\quad \alpha=\mathrm{I},\mathrm{II},\quad \tilde{V}^{\mathrm{I}}=\exp(i\tilde{C}_s^\alpha)\exp(i\tilde{B}_s^\alpha)\exp(i\tilde{A}_s^\alpha), \tag{4.5}$$

where

$$A_s^{\mathrm{I}}=-i\sigma_2k\frac{\mathbf{S}\cdot\boldsymbol{\pi}}{m},\quad A_s^{\mathrm{II}}=-\sigma_1\frac{\sqrt{2}\sin\theta_s}{2ms}\mathbf{S}\cdot\boldsymbol{\pi},\quad \eta^{\mathrm{I}}=\eta,\quad \eta^{\mathrm{II}}=1,\quad B_s^{\mathrm{I}}=\sigma_1\frac{k}{2m^2}\left\{\frac{1}{2\eta}[\mathbf{S}\cdot\boldsymbol{\pi},\boldsymbol{\pi}^2]+ik[2(\mathbf{S}\cdot\boldsymbol{\pi})^2+e\mathbf{S}\cdot\mathbf{H}]+\frac{e}{\eta}\mathbf{S}\cdot\mathbf{E}\right\},$$

$$C_s^{\mathrm{I}}=\sigma_2\frac{k^2}{m^3}\left\{-\frac{2ik}{3}(\mathbf{S}\cdot\boldsymbol{\pi})^3+iek[\mathbf{S}\cdot\boldsymbol{\pi},\mathbf{S}\cdot\mathbf{H}]_++[(\mathbf{S}\cdot\boldsymbol{\pi})^2,eA_0]\right\},$$

$$B_s^{\mathrm{II}}=\sigma_2\frac{1}{4m^2}\left\{a_s\boldsymbol{\pi}^2+\frac{b_s}{2s^2}[2(\mathbf{S}\cdot\boldsymbol{\pi})^2+e\mathbf{S}\cdot\mathbf{H}]+\frac{e\sqrt{2}\sin\theta_s}{s}\mathbf{S}\cdot\mathbf{E}\right\},$$

$$C_s^{\mathrm{II}}=\sigma_1\frac{1}{8m^3}\left\{\frac{\sqrt{2}\sin\theta_s}{s}\left[\mathbf{S}\cdot\boldsymbol{\pi},\boldsymbol{\pi}^2-\frac{e\sin^2\theta_s}{s^2}\mathbf{S}\cdot\mathbf{H}\right]_+-\frac{4\sqrt{2}\sin^3\theta_s}{s^3}(\mathbf{S}\cdot\boldsymbol{\pi})^3-iea_s[\boldsymbol{\pi}^2,A_0]-\frac{ieb_s}{s^2}[(\mathbf{S}\cdot\boldsymbol{\pi})^2,A_0]\right\},$$

$$\tilde{A}_s^{\mathrm{I}} = -ik(\sigma_2 - i\sigma_3)\frac{\mathbf{S}\cdot\boldsymbol{\pi}}{m}, \quad B_s^{\mathrm{I}} = \frac{k}{2\eta m^2}(\sigma_2 - i\sigma_3) S\cdot E, \quad C_s^{\mathrm{I}} = -\frac{ik}{4\tilde{\eta}m^3}(\sigma_2 - i\sigma_3)[\pi^2, S\cdot\pi] - \frac{i}{2\tilde{\eta}m}\frac{\partial \tilde{B}_s^{\mathrm{I}}}{\partial t}.$$

By direct calculation, we obtain

$$[H_s^{\alpha}(\boldsymbol{\pi}, A_0)]' = V^{\alpha} H_s^{\alpha}(\boldsymbol{\pi}, A_0)(V^{\alpha})^{-1} = A^{\alpha} m + B^{\alpha}\left(\frac{\boldsymbol{\pi}^2}{2m} + eA_0\right) +$$

$$\sigma_3 e C^{\alpha}\frac{\mathbf{S}\cdot\mathbf{H}}{m} + \frac{e}{4m^2} D^{\alpha}\mathbf{S}\cdot(\boldsymbol{\pi}\times\mathbf{E} - \mathbf{E}\times\boldsymbol{\pi}) + \frac{e}{6m^2} F^{\alpha} s(s+1)\,\mathrm{div}\,\mathbf{E} +$$

$$\frac{1}{12m^2} G^{\alpha} Q_{ab}\frac{\partial E_a}{\partial x_b} + \frac{n^{\alpha}}{m^2}\mathbf{S}\cdot(\boldsymbol{\pi}\times\mathbf{H} - \mathbf{H}\times\boldsymbol{\pi}) + \frac{L^{\alpha} e}{m^2} Q_{ab}\frac{\partial H_a}{\partial x_b} + o\left(\frac{1}{m^3}\right), \tag{4.6}$$

$$[\tilde{H}_s^{\mathrm{I}}(\boldsymbol{\pi}, A_0)]' = \tilde{V}^{\mathrm{I}}\tilde{H}_s^{\mathrm{I}}(\boldsymbol{\pi}, A_0)(\tilde{V}^{\mathrm{I}})^{-1} = \sigma_3\tilde{\eta} m + \frac{\boldsymbol{\pi}^2}{2m} + eA_0 + o\left(\frac{1}{m^3}\right),$$

where

$$A^{\mathrm{I}} = \sigma_3\eta, \quad B^{\mathrm{I}} = 1, \quad C^{\mathrm{I}} = -\eta k^2, \quad -D^{\mathrm{I}} = F^{\mathrm{I}} = G^{\mathrm{I}} = k^2, \quad n^{\mathrm{I}} = -3L^{\mathrm{I}} = \eta k^3, \tag{4.7a}$$

$$A^{\mathrm{II}} = B^{\mathrm{II}} = \sigma_3, \quad -C^{\mathrm{II}} = D^{\mathrm{II}} = -F^{\mathrm{II}} = -G^{\mathrm{II}} = \frac{\sin^2\theta_s}{2s^2}, \quad n^{\mathrm{II}} = \frac{\sqrt{2}\sin\theta_s}{2s}\left(-a_s + \frac{b_s}{4s^2}\right), \quad L^{\mathrm{II}} = \frac{\sqrt{2}\, b_s\sin\theta_s}{24s^2}, \tag{4.7b}$$

$$Q_{ab} = (e/2)\{3[S_a, S_b]_+ - 2\delta_{ab} s(s+1)\}.$$

The operators (4.6) and (4.7) contain terms corresponding to dipole ($\sim\mathbf{S}\cdot\mathbf{H}$), spin–orbit ($\sim\mathbf{S}\cdot(\boldsymbol{\pi}\times\mathbf{E}-\mathbf{E}\times\boldsymbol{\pi})$), quadrupole ($\sim Q_{ab}\partial E_a/\partial x_b$), and Darwin ($\sim\mathrm{div}\,\mathbf{E}$) couplings of the particle to the field. The last two terms in (4.6) and (4.7) can be interpreted as the magnetic spin–orbit and magnetic quadrupole couplings. The approximate Hamiltonians obtained from relativistic equations [5, 6] have a similar structure. In the case $s=1/2$, $\eta=1$, $k^2=-1$, $\theta_s=\pi/4$, the first seven terms in (4.6) and (4.7) are identical with the Foldy–Wouthuysen Hamiltonian [9] obtained by diagonalizing the Dirac equation. Thus, in the $1/m^2$ approximation the nonrelativistic equations (4.2), (4.6), and (4.7) describe the motion of a particle with spin $s = \frac{1}{2}$ in an external electromagnetic field with the same accuracy as the relativistic Dirac equation.

Note that for some classes of external fields Eqs. (4.2) can be solved exactly. We give without proof the eigenvalues of the Hamiltonian (4.3b) for a particle with spin interacting with a constant homogeneous magnetic field [10]:

$$H_{1/2}^{\mathrm{II}}(\boldsymbol{\pi}, A_0)\Psi_{\varepsilon s_3 n p_3} = E_{\varepsilon s_3 n p_3}\Psi_{\varepsilon s_3 n p_3},$$

$$E_{\varepsilon s_3 n p_3} = \varepsilon\left\{m^2 + \xi^2 + p_3^2 + \frac{(\xi^2+p_3^2)^2}{4m^2} + \left(\frac{eH_3}{2m}\right)^2 + \varepsilon\frac{eH_3}{m}\left[m^2\cos^2 2\theta_{1/2} + \xi^2 + \frac{(\xi^2+p_3^2)^2}{4m^2}\right]^{1/2}\right\}^{1/2},$$

where $\xi^2 = (2n+1)eH_3$, $H_1 = H_2 = 0$, $n = 0, 1, 2, \ldots$, $\varepsilon = \pm 1$, $s_3 = \pm 1/2$.

5. First-Order Equations

We consider briefly the problem of describing Galileo invariant differential equations of the form

$$F\Psi = 0, \quad F = \beta_\mu p^\mu + \beta_5 m, \quad p_\mu = -i\partial/\partial x_\mu, \tag{5.1}$$

where β_μ, β_5 are numerical matrices.

By definition, Eq. (5.1) is invariant under the Galileo group if

$$[F, Q_A] = f_A F, \quad A = 1, 2, \ldots, 10, \tag{5.2}$$

where Q_A is an arbitrary generator of the group G: $\{Q_A\} = \{P_0, P_a, G_a, J_a\}$, and f_A are certain operators defined on the set of solutions of Eq. (5.1).

Setting $f_A \equiv 0$ and choosing the generators P_μ, J_a, G_a in the form (1.4), where S_a and λ_a are arbitrary matrices (which corresponds to local Galileo transformations (1.6) for the function Ψ), we obtain from (5.2) the following system of commutation relations for the matrices β_μ, β_5, λ_a, S_a:

$$[S_a, \beta_5] = [S_a, \beta_0] = 0, \quad [S_a, \beta_b] = i\varepsilon_{abc}\beta_c, \quad [\lambda_a, \beta_5] = i\beta_a, \quad [\lambda_a, \beta_b] = i\delta_{ab}\beta_0, \quad [\lambda_a, \beta_0] = 0, \tag{5.3}$$

where λ_a and S_a are matrices satisfying Eqs. (2.2).

Thus, the problem of describing Galileo invariant equations of the form (5.1) reduces in our formulation to finding matrices S_a, λ_a, β_5, β_a satisfying the conditions (2.2) and (5.3).

We give a special solution of the system (2.2), (5.3), which makes it possible to obtain equations of the form (5.1) for nonrelativistic particles of arbitrary spin. We denote by S_{kl}, $k, l=1, 2, 3, 4, 5, 6$, the generators of the irreducible representation of the group SO(6). Then the matrices

$$S_a={}^1/_2\varepsilon_{abc}S_{bc},\quad \lambda_a={}^1/_2(iS_{6a}+S_{5a}),\quad a=1,2,3,\quad \beta_a=2S_{4a},\quad \beta_0=iS_{46}+S_{45},\quad \beta_5=2(I+iS_{46}-S_{45}) \tag{5.4}$$

satisfy the commutation relations (2.2) and (5.3), i.e., Eqs. (5.4) solve the posed problem.

Setting in (5.1), (5.4) $S_{kl}=(i/4)[\gamma_k, \gamma_l]$, $S_{6k}={}^1/_2\gamma_k$, where γ_k are Hermitian four-row Dirac matrices, we obtain an equation equivalent to Levi-Leblond's equation [3] for a particle with spin $s = \frac{1}{2}$. Choosing other representations of the Lie algebra of the group SO(6), we obtain from (5.1) and (5.4) equations for particles with other values of the spin.

Equations (5.1) and (5.4), like the second-order equations considered earlier, make it possible to describe the spin–orbit interaction of the particle with an external field. For example, setting $S_{kl}=i[\hat{\beta}_k, \hat{\beta}_l]$, $S_{6k}=\hat{\beta}_k$, where $\hat{\beta}_k$ are the ten-row Kemmer–Duffin matrices (which can be chosen, for example, in the form given in the monograph [12]), and making in (5.1) the substitution $p_\mu\to\pi_\mu$, where $\pi_a=p_a$, $\pi_0=p_0-eA_0$, we obtain after simple but somewhat lengthy calculations an equation for the three-component wave function $\Psi^{(3)}$:

$$i\frac{\partial}{\partial t}\Psi^{(3)}=H\Psi^{(3)},\quad H=m+\frac{\pi^2}{2m}+eA_0+e\frac{\mathbf{S}\cdot\mathbf{E}}{4m}, \tag{5.5}$$

where S_a are spin matrices for $s = 1$. By means of the transformation $H \to H' = VHV^{-1}$, where $V = \exp(i\mathbf{S}\cdot\boldsymbol{\pi}/m)$, the Hamiltonian (5.5) can be reduced to a form analogous to (4.6),

$$H'=\frac{\boldsymbol{\pi}^2}{2m}+eA_0-\frac{1}{32m^2}\left[\mathbf{S}\cdot(\boldsymbol{\pi}\times\mathbf{E}-\mathbf{E}\times\boldsymbol{\pi})-\frac{1}{3}Q_{ab}\frac{\partial E_a}{\partial x_b}-\frac{4}{3}\operatorname{div}\mathbf{E}\right]+o\left(\frac{1}{m^3}\right). \tag{5.6}$$

The operator (5.6), like (4.6) and (4.7), contains terms describing the Darwin, spin–orbit, and quadrupole couplings of the particle to the external electric field.

6. Concluding Remarks

1. We have obtained systems of differential equations of first and second order which are invariant under Galileo transformations and gauge transformations and describe dipole, quadrupole, spin–orbit, and Darwin couplings of particles of arbitrary spin to an external electromagnetic field. Thus, these interactions are not purely relativistic effects and can be treated consistently in the framework of nonrelativistic quantum mechanics (see also [10, 11]).

2. Equations (2.8) and (3.10) have a structure such that their left-hand side is identical to the relativistic Dirac equation, while the right-hand side contains terms which destroyed a symmetry under the Poincaré group and ensure invariance of the equation under the Galileo group. Such a method of destroying the Poincaré symmetry is one of the possible approaches for obtaining Galileo invariant equations of motion for particles with arbitrary spin. Thus, proceeding from the relativistic equations without redundant components given in [1, 6] we can, adding terms that destroy the Poincaré invariance but preserve the symmetry under the group E(3), obtain Eqs. (1.2) and (2.1a).

3. Equations of the form (0.1) and (5.1) do not, of course, exhaust all possible linear differential equations that are invariant under the Galileo group. For example, to describe the motion of a nonrelativistic particle with spin $s = 1$ one can use the Galileo invariant analog of the Proca equations

$$(2mp_0-p^2)\Psi_\nu=0,\quad \nu=0,1,2,3,\quad m\Psi_0-p_a\Psi_a=0,\quad a=1,2,3.$$

4. The non-Hermiticity of the generators (1.4) with respect to the usual scalar product (1.10) is due to the non-Hermiticity of the finite-dimensional representations of the algebra (2.2) (which is isomorphic to the Lie algebra of the Euclidean group E(3)). A similar situation obtains in relativistic theory, in which nonunitary representations of the homogeneous Lorentz group are always realized on the solutions of equations of motion that are finite dimensional with respect to the spin indices, the requirement of unitarity of such representations leading to infinite-component equations. It is therefore of interest to consider infinite-component equations invariant under the Galileo group. We give an example of such equations.

We denote by $\tilde{S}_{\mu\nu}$ ($\mu, \nu=0, 1, 2, 3, 4, 5$) the generators of the unitary infinite-dimensional representation of the group O(1, 5). Then an equation in the form of (5.1), (5.4), where $S_{kl}=\tilde{S}_{kl}$ ($k, l=1, 2, 3, 4, 5$), $S_{6k}=i\tilde{S}_{0k}$, is invariant under the Galileo group.

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QUASIPOTENTIAL MODELS OF A RELATIVISTIC OSCILLATOR

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Two exactly solvable one-dimensional models of a relativistic oscillator are investigated in the quasipotential approach in quantum field theory, and coherent states are constructed for them and the dynamical symmetry groups are found.

1. The harmonic oscillator, being one of the few exactly solvable problems in nonrelativistic quantum mechanics, has been widely used in different branches of theoretical physics such as statistical mechanics, superconductivity theory, nuclear physics, and so forth. The interest in the harmonic oscillator was sharpened after the appearance of the quark models, by means of which one can describe the basic properties of hadron structure. The further development of the quark models led to the need to construct relativistic wave equations of composite particles and, in particular, relativistic models of an harmonic oscillator [1-5].

A characteristic feature of the harmonic oscillator is the presence of a class of solutions in the form of coherent states. The use of coherent states makes it possible to employ a perspicuous classical language to describe quantum phenomena. Initially, coherent states were introduced for quantum systems with quadratic Hamiltonians, i.e., for systems that can be represented in the form of a finite or infinite set of harmonic oscillators. Coherent states of quadratic systems are defined as eigenstates of non-Hermitian boson annihilation operators [6] and are Gaussian wave packets that minimize the uncertainty product of the coordinate and momentum and preserve their form with the passage of time. A definition of coherent states for arbitrary quantum systems as eigenfunctions of the integrals of the motion was proposed in [7].

The representation of coherent states has also proved to be convenient in the investigation of hadron interaction at high energies. For example, in [8] a study was made of a high-energy model in which the excited states of the colliding hadrons have a coherent nature, and in [9] the method of coherent states was used to obtain a factorization of dual amplitudes of a semimultiperipheral type. The problems that arise in a systematic formulation of the quantum field theory in the representation of coherent states was studied in [10].

In the present paper, we study two exactly solvable one-dimensional models of a relativistic oscillator in the framework of the quasipotential approach in quantum field theory [11, 12]. We construct coherent states and find the dynamical symmetry groups for these models.

The one-dimensional quasipotential equation for the wave function in the p representation in the case of equal masses has the form

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Joint Institute for Nuclear Research, Dubna. Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol.44, No.1, pp.47-62, July, 1980. Original article submitted April 10, 1979.

