A COMPLETE SET OF SYMMETRY OPERATORS FOR THE DIRAC EQUATION

A. G. Nikitin

UDC 517.9:519.46

A complete set of symmetry operators of arbitrary finite order admitted by the Dirac equation is found. The algebraic structure of this set is investigated and subsets of symmetry operators that form bases of Lie algebras and superalgebras are isolated.

1. Introduction. It is well known that for many equations of mathematical physics there exist integrals of motion and symmetry operators that in principle cannot be found in the framework of classical group analysis [1]. Indeed, in the classical infinitesimal approach of Lie, the investigation of the symmetries of a differential equation reduces to finding the generators of its invariance group, which are first-order differential operators in the dependent and independent variables [2]. This leaves outside the symmetry operators (SO) of higher orders, which belong to classes of differential operators of order $n > 1$.

SO of higher orders carry information on the hidden symmetry of the equation, among them, the symmetries of Lie-Bäcklund type [3] and the supersymmetries [4, 5]. One of the most important applications of such operators is the description of systems of coordinates in which the equation under study admits solutions in separable variables.

In the works [8, 9] a set of SO of arbitrary order n was obtained for a scalar wave equation (the Klein–Gordon–Fock (KGF) equation). This result opens new possibilities in the study of SO of wave equations for fields with spin – the equations of Dirac, Kemmer–Duffin–Petiau, and others.

The present work is devoted to the investigation of higher-order SO admitted by relativistic wave equations. Our main result is exhibiting in explicit form a complete set of SO of arbitrary finite order for the Dirac equation. We also investigate the algebraic properties of this set and we find new superalgebras of hidden symmetries of the Dirac equation.

2. Symmetry Operator of the KGF Equation. Let us write the KGF equation for a complex scaler function $\Psi(x)$, $x = (x_0, x_1, x_2, x_3)$, $\Psi \in L_2(R_4)$ in the form

$$L\Psi = 0, \tag{1}$$

where L is the linear differential operator given by

$$L = p_\mu p^\mu - \varkappa^2, \quad p_\mu = i\frac{\partial}{\partial x^\mu}, \quad \mu = 0, 1, 2, 3. \tag{2}$$

Let F_0 denote the solution set of Eq. (1) (the null-space of the operator (2)), i.e., $\Psi \in F_0$: $\Psi \in L_2(R_4)$, $L\Psi = 0$.

Definition. A linear differential operator of order n is called a SO (of order n) of the KGF equation if

$$[Q, L]\Psi = 0, \quad \Psi \in F_0. \tag{3}$$

Well known examples of SO of the KGF equation are the generators of the Poincaré group

$$\hat{P}_\mu = p_\mu, \quad \hat{J}_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu. \tag{4}$$

Institute of Mathematics, Academy of Sciences of the Ukraine, Kiev. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 43, No. 10, pp. 1388-1398, October, 1991. Original article submitted March 27, 1991.

0041-5995/91/4310-1287$12.50

A SO of arbitrary finite order $j \leq n$ can be represented in the form [8, 9]

$$Q^{(j)} = [[\ldots[F^{a_1 a_2 \ldots a_j}, p_{a_1}]_+, p_{a_2}]_+, \ldots]_+, p_{a_j}]_+, \tag{5}$$

where $[A,B]_+ = AB + BA$ and $F^{a_1 a_2 \ldots a_j}$ are symmetric tensors of rank j. Condition (3) for the operators (2), (5) reduces to the following equations for the coefficients of the SO:

$$p^{(a_{j+1}} F^{a_1 a_2 \ldots a_j)} = 0, \tag{6}$$

where symmetrization with respect to the indices closed in braces is understood.

In [8, 9] the general solution of Eqs. (6) was obtained and the explicit form of the corresponding SO was found. The number of linearly independent SO of order n is

$$N^{(n)} = \frac{1}{4!}(n+1)(n+2)(2n+3)(n^2+3n+4), \tag{7}$$

and the total number of SO of orders $j \leq n$ is given by the formula

$$\hat{N}^{(n)} = \frac{1}{72}(n+1)(n+2)^2(n+3)(n^2+4n+6). \tag{8}$$

Any SO of order n can be represented in the form [8]

$$Q^{(n)} = \sum_{c=0}^{n} \lambda^{a_1 a_2 \ldots a_c [a_{c+1} b_1] \ldots [a_n b_{n-c}]} \hat{P}_{a_1} \hat{P}_{a_2} \ldots \hat{P}_{a_c} \hat{J}_{a_{c+1} b_1} \ldots \hat{J}_{a_n b_{n-c}}. \tag{9}$$

Here $\hat{P}_a$, $\hat{J}_{ab}$ are the generators (4), and $\lambda^{a_1 a_2 \ldots a_c [a_{c+1} b_1] \ldots [a_n b_{n-c}]}$ are arbitrary parameters satisfying the following conditions:

1) Symmetry and tracelessness with respect to the indices a_1, a_2, ..., a_c;

2) Symmetry under permutations of the pairs of indices $[a_{c+i} b_i]$ and $[a_{c+j}, b_j]$; $i, j = 1, 2, \ldots, n-c$;

3) antisymmetry under the permutation of the indices a_{c+i} and b_i;

4) the contraction with respect to any triplet of indices with the completely antisymmetric tensor $\varepsilon_{\mu\nu\rho\sigma}$ is equal to zero.

The tensors $\lambda^{a_1 \cdots}$ with the properties 1)-4) will be referred to the basic tensors. These tensors are irreducible, since, generally speaking, they have a nonzero trace with respect to any pair of indices $(a_\ell, a_{\ell'})$ if $\ell > c$ and (or) $\ell' > c$. The general expression of a SO of arbitrary order n for the KGF equation which contains only irreducible parameters can be derived from (9) by decomposing the basis tensors into irreducible ones. Corresponding rather cumbersome formulas are given in [8, 9].

Thus, all SO of an arbitrary order n of the KGF equation are polynomials of order n in the generators of the Poincaré group (4), which can be expressed in the form (9).

3. General Form of a SO of Order n for the Dirac Equation. The Dirac equation, too, can be written in the form (1), where Ψ is a four-component bispinor and L is the following first-order linear differential operator with matrix coefficients:

$$L = \gamma^\mu p_\mu - m, \tag{10}$$

where γ_μ are numerical matrices of size 4 × 4 which obey the Clifford algebra rules

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}. \tag{11}$$

As the formal definition of an SO of order n admitted by the Dirac equation one can take relation (3), where F_0 denotes the null-space of the operator (10), $\Psi \in F_0$: $\Psi_\alpha \in L_2(R_4)$, $L\Psi = 0$ (Ψ_α, $\alpha = 1, 2, 3, 4$ are the components of the bispinor Ψ). Here one assumes that the coefficients of the SO are matrices of dimension 4 × 4 which, generally speaking, depend on x.

Well-known SO for the Dirac equation are the generators of the Poincaré group

$$P_\mu = p_\mu, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}, \tag{12}$$

where

$$S_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]. \tag{13}$$

References [4, 6, 10] give a description of a complete set of SO for the Dirac equations in classes of first- and second-order differential operators with matrix coefficients. The problem of describing such operators is connected with that of solving a very tedious system of defining equations for the coefficient functions [10].

Below we give a simple proof of the fact that all SO of arbitrary order n for the Dirac equation are polynomials in the generators (12), and we find the explicit form of all linearly independent SO. The basic idea of the proof is to exploit the fact that the solutions of the Dirac equation satisfy the KGF equation componentwise, and consequently the SO for the Dirac equation must also be SO for the equation (1), (2).

Let us subject the bispinor Ψ and the operator (10) to the invertible transformation

$$\Psi \to \Psi' = W_+\Psi, \quad L \to L' = W_-^{-1} L W_+^{-1}, \tag{14}$$

where

$$W_\pm = \exp\left(-\frac{1}{m} P_\pm \gamma^\mu p_\mu\right) \equiv 1 - \frac{P_\pm}{m}\gamma^\mu p_\mu,$$

$$W_\pm^{-1} = 1 + \frac{1}{m} P_\pm \gamma^\mu p_\mu, \quad P_\pm = \frac{1}{2}(1 \mp i\gamma_4), \quad \gamma_4 = i\gamma_0\gamma_1\gamma_2\gamma_3. \tag{15}$$

As a result, using relations (11), we arrive at the equivalent equation

$$L'\Psi' = 0, \quad L' = -P_+ m + P_-(p^\mu p_\mu - m^2), \tag{16}$$

which due to the orthogonality of the projectors P_+ and P_- splits into uncoupled subsystems

$$(p^\mu p_\mu - m^2)\Psi'_+ = 0, \tag{17}$$

$$\Psi'_- = 0, \quad \Psi'_\pm = P_\mp \Psi'. \tag{18}$$

The matrix $i\gamma_4$ can be taken diagonal with no loss of generality. Then Ψ' will have only two nonzero components.

With each SO Q of the Dirac equation one can associate, in a one-to-one manner, an SO Q' of Eq. (16):

$$Q' = W_+ Q W_+^{-1}, \quad Q = W_+^{-1} Q' W_+, \tag{19}$$

where W_+ is the operator (15). It is convenient to decompose the operator Q', defined on the set of functions Ψ' with two nonzero components, into a complete set of matrices σ^μ:

$$Q' = \sigma^\mu Q'_\mu, \quad \sigma_a = \varepsilon_{abc} S^{bc}, \quad \sigma_0 = \frac{1}{3}\sigma_a \sigma^a. \tag{20}$$

Here S_{bc} are the matrices (13), b, c = 1, 2, 3. Then Q'_μ must be an SO of the KGF equation for a scalar function; a complete set of such operators was described in the preceding section.

Let Q'_μ be an SO of an arbitrary fixed order n for the Eqs. (1), (2). Then, by (8), (9), they depend polynomially on $\tilde{P}_\mu, \tilde{J}_{\mu\nu}$ (4), or on $P_\mu, J_{\mu\nu} - S_{\mu\nu}$ where $P_\mu, J_{\mu\nu}$ are the generators (12). But on the set of solutions of Eq. (16) the matrices $S_{\mu\nu}$ are expressible through P_μ and $J_{\mu\nu}$ [1, 4]:

$$2S_{\mu\nu}\Psi' = \frac{1}{m^2}(P_\mu W_\nu - P_\nu W_\mu + i\varepsilon_{\mu\nu\rho\sigma} W^\rho P^\sigma)\Psi', \tag{21}$$

where

$$W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^\sigma \tag{22}$$

is the Lyubanskii–Pauli vector. From this we conclude that the SO are polynomials in the generators (12).

The operator W_+ (15) obviously commutes with the operators P_μ, $J_{\mu\nu}$ (12), and hence (19) implies $Q' = Q$, i.e., all SO of arbitrary finite order n for the Dirac equation are polynomials in the generators of the Poincaré group.

One should remark that in the general case the membership of all SO of arbitrary order to the enveloping algebra generated by the generators of the symmetry group of the equation under study is by no means necessary. In particular, for the massless Dirac equation there exist first-order SO that do no belong to the relevant algebra [4].

4. Algebraic Properties of the SO of First Order. In the description of SO of an arbitrary order n a key role is played by the case $n = 1$, considered below. According to what we proved in the preceding section, a complete set of corresponding SO can be obtained by a direct sorting out of polynomials in the operators P_μ and $J_{\mu\nu}$; moreover, as we shall see below, it suffices to restrict the considerations to polynomials of degree $n \leq 3$. This leads to the known [4, 6] 26 linearly independent SO, including the generators P_μ, $J_{\mu\nu}$ (12), the identity operator I, and the following 15 operators:

$$W_\mu = \frac{i}{2}\gamma_4(p_\mu - m\gamma_\mu), \quad W_{\mu\nu} = \frac{i}{2}\gamma_4(\gamma_\mu p_\nu - \gamma_\nu p_\mu), \tag{23}$$

$$B = i\gamma_4(D - m\gamma^\mu x_\mu), \quad A_\mu = \frac{i}{2}\gamma_4\varepsilon_{\mu\nu\rho\sigma}J^{\nu\rho}\gamma^\sigma + \frac{1}{2}\gamma_\mu, \tag{24}$$

where $D = x^\mu p_\mu + \frac{3}{2}i$.

SO of higher orders are expressible through products of operators (12), (23), (24), which justifies the interest in studying the algebraic properties of this set. It turns out that the operators (12), (23), (24) include subsets that form bases of Lie algebras and superalgebras.

A direct computation yields the commutation relations

$$\begin{gathered}[P_\mu, P_\nu] = 0, \quad [P_\mu, J_{\nu\sigma}] = i(g_{\mu\nu}P_\sigma - g_{\mu\sigma}P_\nu),\\ [J_{\mu\nu}, J_{\lambda\sigma}] = i(g_{\mu\sigma}J_{\nu\lambda} + g_{\nu\lambda}J_{\mu\sigma} - g_{\mu\lambda}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\lambda});\end{gathered} \tag{25}$$

$$\begin{gathered}[P_\mu, W_\nu] = 0, \quad [P_\mu, W_{\sigma\lambda}] = 0, \quad [W_\mu, J_{\nu\sigma}] = i(g_{\mu\nu}W_\sigma - g_{\mu\sigma}W_\nu),\\ [J_{\mu\nu}, W_{\rho\sigma}] = i(g_{\mu\sigma}W_{\nu\rho} + g_{\nu\rho}W_{\mu\sigma} - g_{\mu\rho}W_{\nu\sigma} - g_{\nu\sigma}W_{\mu\rho});\\ [W_\mu, W_\nu] = \frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}W^{\rho\sigma}, \quad [W_\mu, B] = \frac{i}{2}P_\mu + imA_\mu,\\ [W_\mu, A_\nu] = ig_{\mu\nu}B + i[J_{\mu\lambda}, W^\lambda_\nu]_+,\\ [W_\mu, W_{\rho\sigma}] = \frac{i}{2}(\varepsilon_{\mu\sigma\lambda k}P_\rho - \varepsilon_{\mu\rho\lambda k}P_\sigma)W^{\lambda k},\end{gathered} \tag{26}$$

$$\begin{gathered}[A_\mu, W_{\lambda\sigma}] = i\left\{-\frac{1}{2}\varepsilon_{\mu\sigma\lambda\rho}P^\rho - (g_{\mu\lambda}P_\sigma - g_{\mu\sigma}P_\lambda)B + [W_\mu, J_{\lambda\sigma}]_+\right\},\\ [A_\mu, B] = i\varepsilon_{\mu\nu\rho\sigma}J^{\nu\rho}A^\sigma,\\ [A_\mu, A_\nu] = -iJ_{\mu\nu} + i\varepsilon_{\mu\nu\rho\sigma}(J^{\rho\sigma}B - W^{\rho\sigma}),\\ [W_{\mu\nu}, B] = \frac{i}{2}([P_\mu, A_\nu]_+ - [P_\nu, A_\mu]_+),\\ [P_\mu, B] = 2iW_\mu, \quad [P_\mu, A_\nu] = 2iW_{\mu\nu},\\ [W_{\mu\nu}, W_{\rho\sigma}] = \frac{i}{2}(\varepsilon_{\mu\nu\rho k}P_\sigma + \varepsilon_{\rho\sigma\nu k}P_\mu - \varepsilon_{\mu\nu\sigma k}P_\rho - \varepsilon_{\rho\sigma\mu k}P_\nu)W^k.\end{gathered} \tag{27}$$

Relations (25) define the Lie algebra of the Poincaré group AP(1, 3), a basis of which is provided by the generators (12). The remaining SO of first order, given by formulas (23), (24), do not form a Lie algebra. However, one can indicate a subset of the SO (23), (24) which constitutes a basis of a Lie algebra. Such a set is provided by the operators

$$\Sigma_{\mu\nu}^{\pm} = \frac{1}{m^2}\left(\pm W_{\mu\nu} + \frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}W^{\rho\sigma}\right), \tag{28}$$

which satisfy the following commutation relations on the solution set of the Dirac equation (in (28) we take the sign + for definiteness, and we denote $\Sigma_{\mu\nu}^{+} = \Sigma_{\mu\nu}$):

$$[\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] = [\Sigma_{\mu\nu}, J_{\rho\sigma}] = i(g_{\mu\sigma}\Sigma_{\nu\rho} + g_{\nu\rho}\Sigma_{\mu\sigma} - \Sigma_{\mu\rho}g_{\nu\sigma} - g_{\mu\rho}\Sigma_{\nu\sigma}),$$

$$[\Sigma_{\mu\nu}, P_\lambda] = 0.$$

We see that the extension of the class of SO allows one to discover a wider invariance algebra of the Dirac equation than the well-known Lie algebra of the Poincaré group. The 16-dimensional algebra spanned by the basis (12), (28) includes the algebra AP(1,3) as a subalgebra [4].

The operators (12), (23), (24) include subsets that possess a structure of Lie superalgebra. To isolate such sets let us calculate anti-commutation relations (which will also be used for a constructive description of independent SO of the Dirac equation). We have

$$
\begin{aligned}
&[W_\mu, W_\nu]_+ = \frac{1}{2}(P_\mu P_\nu - m^2 g_{\mu\nu}),\\
&[W_\mu, W_{\rho\sigma}]_+ = \frac{1}{2}m^2(g_{\mu\sigma}P_\rho - g_{\mu\rho}P_\sigma),\\
&[W_{\mu\nu}, W_{\rho\sigma}]_+ = \frac{1}{2}(g_{\mu\rho}P_\nu P_\sigma + g_{\nu\sigma}P_\mu P_\rho - g_{\mu\sigma}P_\nu P_\rho - g_{\nu\rho}P_\mu P_\sigma),\\
&[W_\mu, B]_+ = -\frac{1}{2}[J_{\mu\nu}, P^\nu]_+,\\
&[W_\mu, A_\nu]_+ = -\frac{m}{2}\varepsilon_{\mu\nu\rho\sigma}J^{\rho\sigma} - W_{\mu\nu},\\
&[W_{\mu\nu}, B]_+ = -mJ_{\mu\nu} - \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}W^{\rho\sigma},\\
&[A_\sigma, W_{\mu\nu}]_+ = \frac{1}{2}(\varepsilon_{\sigma\mu\lambda\rho}P_\nu - \varepsilon_{\sigma\nu\lambda\rho}P_\mu)J^{\lambda\rho} + g_{\sigma\mu}W_\nu - g_{\sigma\nu}W_\mu,\\
&[A_\mu, A_\nu]_+ = \frac{1}{2}\left\{g_{\mu\nu}\left(J_{\alpha\beta}J^{\alpha\beta} + \frac{1}{2}\right) - [J_{\mu\lambda}, J^{\lambda}{}_{\nu}]_+\right\}\\
&[A_\mu, B]_+ = 0, \quad B^2 = \frac{3}{4} - \frac{1}{2}J_{\mu\nu}J^{\mu\nu}.
\end{aligned}
\tag{29}
$$

Using relations (25), (26), (28), and (29) we can isolate several different sets of SO of the Dirac equation that constitute the basis of a Lie superalgebra. Let us indicate a set which includes the algebra AP(1, 3) and a maximal number of SO operators of first order:

$$\{W_\mu, W_{\lambda\sigma}; P_\mu, J_{\mu\nu}, P_\mu P_\nu, I\}. \tag{30}$$

Here left [respectively, right] to the semicolon we wrote the odd (O) [respectively, even (E)] elements of the superalgebra. According to (25), (26), (29) the commutation and anti-commutation relations of the operators (30) are incorporated in the scheme

$$[\mathrm{E}, \mathrm{E}] \sim \mathrm{E}, \quad [\mathrm{E}, \mathrm{O}] \sim \mathrm{O}, \quad [\mathrm{O}, \mathrm{O}]_+ \sim \mathrm{E}, \tag{31}$$

which characterize a superalgebra [11]. The dimension of the superalgebra (30) equals 30.

Thus, in addition to the known relativistic invariance, the Dirac equation possesses a hidden symmetry with respect to various Lie algebras and superalgebras, which include AP(1, 3) as a subalgebra.

Let us give also a series of other relations for the operators (12), (23), (24), which hold on the solution set of the Dirac equation

$$m W_{\mu\nu} = P_\mu W_\nu - P_\nu W_\mu, \qquad B = \frac{1}{4}\varepsilon_{\mu\nu\rho\sigma} J^{\mu\nu} J^{\rho\sigma}, \tag{32}$$

$$A_\mu = \frac{1}{m}\varepsilon_{\mu\nu\rho\sigma} J^{\nu\rho} W^{\sigma} - \frac{1}{2m} P_\mu;$$

$$P_\mu P^\mu = m^2, \quad P_\mu W^\mu = 0, \quad P_\mu A^\mu = m, \quad P^\mu W_{\mu\lambda} = m W_\lambda, \quad [J_{\mu\nu}, W^\nu]_+ = mB, \tag{33}$$

$$[J_{\mu\nu}, A^\nu]_+ = 0, \quad \varepsilon_{\mu\nu\rho\sigma} W^{\nu\rho} P^\sigma = 0; \quad \varepsilon_{\mu\nu\rho\sigma} J^{\mu\nu} W^{\rho\sigma} = \frac{3}{2} m, \tag{34}$$

$$[J_{\mu\nu}, W_{\rho\sigma}]_+ - [J_{\rho\sigma}, W_{\mu\nu}]_+ = \frac{1}{4}[(\varepsilon_{\nu\rho\sigma\lambda} P_\mu + \varepsilon_{\rho\mu\nu\lambda} P_\sigma - \varepsilon_{\mu\rho\sigma\lambda} P_\nu - \varepsilon_{\sigma\mu\nu\lambda} P_\rho), A^\lambda]_+.$$

5. <u>A Complete Set of SO of Arbitrary Order n Admitted by the Dirac Equation.</u> According to the analysis carried out in Sec. 3, the description of all nonequivalent SO of order n of the Dirac equation reduces to the sorting out of linearly independent combinators of the form

$$Q^{ck} = \eta^{a_1 a_2 \ldots a_c [a_{c+1} b_1] \ldots [a_k b_{k-c}]} P_{a_1} P_{a_2} \ldots P_{a_c} J_{a_{c+1} b_1} \ldots J_{a_k b_{k-c}}, \tag{35}$$

where P_a, J_{ab} are the generators (12) and $\eta^{a_1 a_2 \ldots a_c [a_{c+1} b_1] \ldots [a_k b_{k-c}]}$ are arbitrary parameters. The index k is allowed to take arbitrary integer values in the interval [0, n], and the possibility that k > n is not excluded beforehand. As it will be shown below, it suffices to take $0 \leqslant k \leqslant n+2$, $0 \leqslant c \leqslant k$.

By relations (25), (32), one can consider that the tensors $\eta^{a_1 a_2 \ldots a_c [a_{c+1} b_1] \ldots [a_k b_{k-c}]}$ possess properties 1)-3) formulated above in the explanations to formula (9); however, in general they do not possess property 4), i.e., are not basis tensors. The reason for this is that for the operators (12) (in contrast to the generators (4)) the Lyubanskii–Pauli vector (22) is not equal to zero.

To give an effective description of the linearly independent operators (35) it is convenient to decompose $\eta^{a_1 a_2 \ldots a_c [a_{c+1} b_1] \ldots [a_k b_{k-c}]}$ with respect to the basis tensors. Let us write the first terms of such a decomposition:

$$\eta^{a_1 a_2 \ldots a_c [a_{c+1} b_1] \ldots [a_k b_{k-c}]} = \lambda^{a_1 a_2 \ldots a_c [a_{c+1} b_1] \ldots [a_k b_{k-c}]} + \varepsilon^{a_{k-1} b_{k-1-c} a_k b_{k-c}} \times$$
$$\times \lambda^{a_1 a_2 \ldots a_c [a_{c+1} b_1] \ldots [a_{k-2} b_{k-2-c}]} + \varepsilon_{d_1}{}^{b_1 a_{c+1} a_1} \lambda^{d_1 a_2 a_3 \ldots a_c [a_{c+2} b_2] \ldots [a_k b_{k-c}]} +$$
$$+ \varepsilon_{d_1}{}^{b_1 a_{c+1} a_1} \lambda^{a_2 a_3 \ldots a_{c-1} [a_c d_1] [a_{c+2} b_2] \ldots [a_k b_{k-c}]} + \varepsilon_{d_1}{}^{b_1 a_{c+1} a_1} \varepsilon_{d_2}{}^{d_1 a_{c+2} b_2} \times$$
$$\times \lambda^{d_2 a_2 \ldots a_c [a_{c+3} b_3] \ldots [a_k b_{k-c}]} + \varepsilon^{a_{c+1} b_1 a_{c+2} b_2} \varepsilon^{a_{c+3} b_3 a_{c+4} b_4} \lambda^{a_1 a_2 \ldots a_c [a_{c+5} b_5] \ldots [a_k b_{k-c}]} +$$

$$+\varepsilon^{a_{c+1}b_1a_{c+2}b_2}\varepsilon_{d_1}{}^{b_3a_{c+3}a_1}\lambda^{d_1a_2\ldots a_c[a_{c+4}b_4]\ldots[a_kb_{k-c}]}+\varepsilon^{a_{c+1}b_1a_{c+2}b_2}\varepsilon_{d_1}{}^{b_3a_{c+3}a_1}\times$$
$$\times\lambda^{a_2\ldots a_{c-1}[d_1a_c][a_{c+4}b_4]\ldots[a_kb_{k-c}]}+\varepsilon_{d_1}{}^{b_1a_{c+1}a_1}\varepsilon_{d_2}{}^{d_1a_{c+2}a_2}\varepsilon_{d_3}{}^{d_2a_{c+3}b_3}\times$$
$$\times\lambda^{d_3a_2a_3\ldots a_c[a_{c+4}b_4]\ldots[a_kb_{k-c}]}+\ldots. \tag{36}$$

Here the dots denote terms that include products of three or more completely antisymmetric tensors $\varepsilon_{\mu\nu\rho\sigma}$; also symmetrization is carried out with respect to the indices $a_1, a_2, \ldots, a_c$ and the pairs of indices $[a_{c+i}, b_i]$, $i = 1, 2, \ldots, k-c$. Calculating various contractions of $\eta^{a_1\cdots}$ with $\varepsilon_{\mu\nu\rho\sigma}$ one can invert formula (36), i.e., one can express the $\lambda^{a_1\cdots}$ through $\eta^{a_1\cdots}$.

Let us substitute (36) in (35) and agree to sort out the possible values of k in their increasing order. To the first term in the right-hand side of (36) there corresponds an SO of the form

$$Q_1=\sum_{c=0}^{k}\lambda_1^{a_1a_2\ldots a_c[a_{c+1}b_1]\ldots[a_kb_{k-c}]}P_{a_1}P_{a_2}\ldots P_{a_c}J_{a_{c+1}b_1}\ldots J_{a_kb_{k-c}}, \tag{37}$$

where P_a and J_{ab} are the generators (12). The order of this operator (which differs from (9) only by the substitution $\hat{J}_{ab}\to J_{ab}$) is equal to k; the number of SO of order n is given by formula (7).

Using relations (22), (32)-(34) we obtain the following representation for the operators corresponding to the second, third, fourth, and fifth-to-seventh terms in the right-hand side of (36):

$$Q_2=B\sum_{c=0}^{k-2}\lambda_2^{a_1a_2\ldots a_c[a_{c+1}b_1]\ldots[a_{k-2}b_{k-2-c}]}P_{a_1}P_{a_2}\ldots P_{a_c}J_{a_{c+1}b_1}\ldots J_{a_{k-2}b_{k-2-c}};$$
$$Q_3=\sum_{c=1}^{k-2}\lambda_3^{a_1a_2\ldots a_c[a_{c+1}b_1]\ldots[a_{k-2}b_{k-2-c}]}W_{a_1}P_{a_2}P_{a_3}\ldots P_{a_c}J_{a_{c+1}b_1}\ldots J_{a_{k-2}b_{k-2-c}};$$
$$Q_4=\sum_{c=0}^{k-4}\lambda_4^{a_1a_2\ldots a_c[a_{c+1}b_1]\ldots[a_{k-3}b_{k-3-c}]}P_{a_1}\ldots P_{a_c}W_{a_{c+1}b_1}J_{a_{c+2}b_2}\ldots J_{a_{k-3}b_{k-3-c}};$$
$$Q_5=\sum_{c=1}^{k-3}\lambda_5^{a_1a_2\ldots a_c[a_{c+1}b_1]\ldots[a_{k-3}b_{k-3-c}]}A_{a_1}P_{a_2}\ldots P_{a_c}J_{a_{c+1}b_1}\ldots J_{a_{k-3}b_{k-3-c}}+$$
$$+\sum_{i=0}^{\left[\frac{k-5}{2}\right]}\sum_{c=0}^{k-5-2i}\lambda_6^{a_1a_2\ldots a_c[a_{c+1}b_1]\ldots[a_{k-4-2i}b_{k-4-2i-c}]}P_{a_1}\ldots P_{a_{c+1}}A_{b_1}J_{a_{c+2}b_2}J_{a_{c+3}b_3}\ldots \tag{38}$$
$$\ldots J_{a_{k-4-2i}b_{k-4-2i-c}}(J_{\mu\nu}J^{\mu\nu})^i+\sum_{i=0}^{\left[\frac{k}{2}-3\right]}\sum_{c=0}^{k-6-2i}\lambda_7^{a_1a_2\ldots a_c[a_{c+1}b_1]\ldots[a_{k-6-2i}b_{k-6-2i-c}]}\times$$
$$\times(P_{a_c}P_{a_{c+1}b}P^b+P_{a_{c+1}}J_{a_cb}P^b)A_{b_1}P_{a_1}P_{a_2}\ldots P_{a_{c-1}}J_{a_{c+2}b_2}\ldots J_{a_{k-6-2i}b_{k-6-2i-c}}\times$$
$$\times(J_{\mu\nu}J^{\mu\nu})^i+\sum_{c=1}^{k-4}\lambda_8^{a_1a_2\ldots a_c[a_{c+1}b_1]\ldots[a_{k-4}b_{k-4-c}]}\varepsilon_{a_1\mu\nu\sigma}J^{\mu\nu}A^\sigma P_{a_2}\ldots P_{a_c}J_{a_{c+1}b_1}\ldots$$
$$\ldots J_{a_{k-4}b_{k-4-c}}.$$

Here $W_\mu, W_{\mu\nu}, B, A_\mu$ are the operators (23), (24), $r'_\mu, J_{\mu\nu}$ are the generators (12), $\lambda_6^{a_1\cdots}, \lambda_7^{a_1\cdots}$ are arbitrary irreducible tensors, and $\lambda_i^{a_1\cdots}$, $i=8,1,2,\ldots,5$ are arbitrary basic tensors satisfying the conditions

$$\lambda_3^{a_1 a_2 \ldots a_c [a_{c+1} b_1] \ldots [a_f b_{f-c}]} g_{b_1 b_2} g_{a_{c+1} a_{c+2}} = 0,$$
$$\lambda_3^{a_1 a_2 \ldots a_c [a_{c+1} b_1] \ldots [a_f b_{f-c}]} g_{a_1 a_{c+1}} g_{a_2 a_{c+2}} = 0, \tag{39}$$
$$\lambda_3^{a_0 [a_1 b_1][a_2 b_2] \ldots [a_f b_f]} g_{a_1 a_2} = 0;$$

$$\lambda_4^{a_1 a_2 \ldots a_c [a_{c+1} b_1] \ldots [a_f b_{f-c}]} g_{a_1 b_1} g_{a_2 b_2} \ldots g_{a_{f-c} b_{f-c}} = 0, \quad c \geqslant \frac{f}{2}; \tag{40}$$

$$\lambda_4^{a_1 a_2 \ldots a_c [a_{c+1} b_1] \ldots [a_f b_{f-c}]} g_{a_{c+1} a_{c+2}} g_{b_1 b_2} g_{a_{c+3} a_{c+4}} g_{b_3 b_4} \ldots g_{a_{f-1} a_f} g_{b_{f-c-1} b_{f-c}} = 0; \tag{41}$$

$$\lambda_\alpha^{a_1 a_2 \ldots a_c [a_{c+1} b_1] \ldots [a_f b_{f-c}]} g_{a_1 b_1} g_{a_2 b_2} \ldots g_{a_c b_c} = 0, \quad c \leqslant \frac{f}{2}, \quad \alpha = 5, 8. \tag{42}$$

The terms denoted by dots in (36) may be omitted: the corresponding SO include products of operators (23), (24), which allows one to reduce them to the form (37) or (38) with smaller k with the aid of the relations (26), (29).

Therefore, any SO of order n for the Dirac equation can be represented either in the form (37), or as a product of the SO (37) by one of the operators (23), (24). Moreover, in (37), (38) one has to put k = n for Q_1, k = n + 1 for Q_2, Q_3, and k = n + 2 for Q_4, Q_5.

Summing the independent components of the tensors $\lambda_i^{a_1 \ldots}$ in (37), (38), it is not difficult to calculate the number of linearly independent SO of order n. For operators (37) this number ($N_1^{(n)}$) is given by formula (7), while for the operators (38) we obtain

$$N_2^{(n)} = \frac{1}{4!} n(n+1)(2n+1)(n^2+n+2), \tag{43}$$

$$N_3^{(n)} = \frac{1}{6} n(n+1)(5n^2-3n+13) - n, \tag{44}$$

$$N_4^{(n)} = N_1^{(n)} - \frac{1}{6} n(2n^2+9n+13) - \frac{1}{2}[1+(-1)^n], \tag{45}$$

$$N_5^{(n)} = \frac{1}{6} n(n+1)(n+3)(n^2+n+1). \tag{46}$$

The total number of symmetry operators of order n is obtained by adding the numbers (7), (43)-(46):

$$N^{(n)} = \sum_{i=1}^{5} N_i^{(n)} = 5N_n^{(1)} - \frac{1}{6}(2n+1)(13n^2+19n+18) - \frac{1}{2}[1+(-1)^n]. \tag{47}$$

In particular, $N^{(0)} = 1$, $N^{(1)} = 25$, $N^{(2)} = 154$, $N^{(3)} = 601$.

Let us formulate the results obtained as the following assertion.

THEOREM. The Dirac equation admits $N^{(n)}$ linearly independent SO of order n, where $N^{(n)}$ is given by formula (47), and the explicit form of the corresponding operators is given by formulas (37), (38).

5. Conclusion. We determined the number and explicit form of all linearly independent SO of arbitrary finite order of the Dirac equation. These SO are given up to arbitrary parameters, which represent basis tensors satisfying conditions (39)-(42). Decomposing these tensors into irreducible ones, it is not difficult to obtain a representation of SO depending on indecomposable sets of parameters.

Let us list the linearly independent SO of second and third order obtained from the general formulas (37)-(42) (the SO of order zero reduce to the identity matrix, and the SO of order one were listed in (12), (23), (24)):

$$
\begin{gathered}
n=2:\ \lambda_1^{ab}P_aP_b,\ \ \widetilde{\lambda}_1^{ab}J_{ac}J^c_b,\ \ \lambda_1^{ab}P_aJ_{bc},\ \ \lambda_1^aJ_{ab}P^b,\\
\lambda_1^{[ab][cd]}J_{ab}J_{cd},\ \ \lambda_1J_{ab}J^{ab},\ \ \lambda_2^aP_aB,\ \ \lambda_2^{[ab]}BJ_{ab},\\
\lambda_3^{ab}P_aW_c,\ \ \lambda_3^{a[bc]}W_aJ_{bc},\ \ \lambda_4^{a[bc]}P_aW_{bc},\ \ \lambda_4^{ab}W_{ac}J^c{}_b,\\
\lambda_4^{[ab][cd]}W_{ab}J_{cd},\ \ \lambda_5^{ab}P_aA_b,\ \ \lambda_5^{a[bc]}A_aJ_{bc},\\
\lambda_6^{[ab]}P_aA_b,\ \ \lambda_8^a\varepsilon_{abcd}J^{bc}A^d;\\
n=3:\ \lambda_1^{abc}P_aP_bP_c,\ \ \lambda_1^{ab[cd]}P_aP_bJ_{cd},\ \ \lambda_1^aP_aJ_{bc}J^{bc},\\
\lambda_1^{a[bc][de]}P_aJ_{oc}J_{de},\ \ \lambda_1^{[ab][cd][ef]}J_{ab}J_{cd}J_{ef},\\
\lambda_1^{ab}P_aJ_{bc}P^c,\ \ \lambda_1^{a[bc]}J_{ad}P^dJ_{bc},\ \ \lambda_1^{[ab]}J_{ab}J_{cd}J^{cd},\\
\widetilde{\lambda}_1^{abc}P_aJ_{bk}J^k_c,\ \ \lambda_2^{ab}P_aP_bB,\ \ \lambda_1^{ab[cd]}J_{ak}J^k_bJ_{cd},\\
\lambda_2^{a[bc]}P_aJ_{bc}B;\qquad \lambda_2^{[ab][cd]}J_{ab}J_{cd}B,\ \widetilde{\lambda}_2^aJ_{ab}P^bB,\\
\lambda_2J_{ab}J^{ab}B,\ \ \widetilde{\lambda}_2^{ab}J_{ac}J^c_bB,\ \ \lambda_3^{ab[cd]}W_aP_bJ_{cd},\\
\lambda_3^{abc}W_aP_bP_c,\ \ \lambda_3^{a[bc][de]}W_aJ_{bc}J_{de},\ \ \lambda_3^{ab}W_aJ_{bc}P^c,\\
\widetilde{\lambda}_3^{abc}W_aJ_{bk}J^k_c,\ \ \lambda_4^{[ab][cd][ef]}W_{ab}J_{cd}J_{ef},\\
\lambda_4^{ab[cd]}P_aP_bW_{cd},\ \ \lambda_4^{a[bc][ae]}P_aJ_{bc}W_{de},\\
\lambda_4^{a[bc]}J_{ad}P^dW_{bc},\ \ \lambda_4^{[ab]}W_{ab}J_{cd}J^{cd},
\end{gathered}
\tag{48}
$$

$$
\begin{gathered}
\lambda_4^{abc}P_aJ_{bk}W^k_c,\ \ \widetilde{\lambda}_4^{ab[cd]}J_{ak}J^k_bW_{cd},\ \ \lambda_5^{abc}P_aP_bA_c,\\
\lambda_5^{ab[cd]}P_aA_bJ_{cd},\ \ \lambda_5^{a[bc][de]}A_aJ_{bc}J_{de},\ \ \lambda_5^{ab}A_aJ_{bc}P^c,\\
\lambda_5^{abc}A_aJ_{bk}J^k_c,\ \ \lambda_5^aA_aJ_{bc}J^{bc},\ \ \lambda_6^{a[bc]}P_aP_bA_c,\\
\lambda_6^{[ab][cd]}P_aA_bJ_{cd},\ \ \lambda_7^{[ab]}A_aJ_{bc}P^c,\\
\lambda_8^{ab}\varepsilon_{aken}J^{ke}A^nP_b,\ \ \lambda_8^{a[bc]}\varepsilon_{aken}J^{ke}A^nJ_{bc}.
\end{gathered}
\tag{49}
$$

Here P_a, J_{ab}, W_a, W_{ab}, A_b, B are the operators (12), (23), (24), and $\lambda_i^{a\cdots}$ are arbitrary irreducible tensors. As is readily verified, the numbers of the operators (48), (49) coincide with those given by formula (47).

We should mention that the set of SO (48) differs from that found in [10], where part of the SO are linearly dependent on the solution set of Eqs. (1), (10).

In addition to the applications mentioned in the introduction, the SO operators found above may be used to construct superalgebras of hidden symmetries of the Dirac equation. An example of such a superalgebra in the class of differential operators of second order was considered in Sec. 4. Let us indicate a chain of superalgebras in the class of order n.

Let $\{Q^k\}$, $k=1,2,\ldots,n$ be subsets of SO of order k of the Dirac equation which satisfy the supplementary condition $[\widetilde{Q}^k,P_\mu]=0$. By our theorem,

$$\{\widetilde{Q}^k\}=\{q^{a_1a_2\cdots a_k},\ q_2^{a_1a_2\cdots a_k},\ q_3^{a_1a_2\cdots a_{k-1}[a_ka_{k+1}]}\},$$

where

$$
\begin{gathered}
q_1^{a_1a_2\cdots a_{k'}}=p^{a_1}p^{a_2}\ldots p^{a_{k'}},\ q_2^{a_1a_2\cdots a_k}=W^{a_1}P^{a_2}P^{a_3}\ldots P^{a_k},\\
q_3^{a_1a_2\cdots a_{k-1}[a_ka_{k+1}]}=p^{a_1}p^{a_2}\ldots p^{a_{k-1}}W^{a_ka_{k+1}}
\end{gathered}
$$

represent SO that are irreducible tensors. Regarding $q_2^{a_1}$ and $q_3^{a_1\cdots}$ as odd, and $q_1^{a_1\cdots}$ and P_μ, $J_{\mu\nu}$ as even, and using relations (25), (29), we convince ourselves that the commutation and anticommutation relations for these operators correspond to the scheme (31) characterizing a superalgebra, for any $k \le n$ and $k' \le 2n$.

LITERATURE CITED

1. V. I. Fushchich and A. G. Nikitin, Symmetry of the Equations of Quantum Mechanics [in Russian], Nauka, Moscow (1990).
2. L. V. Ovsyannikov, Group Analysis of Differential Equations [in Russian], Nauka, Moscow (1978).
3. R. L. Anderson and N. H. Ibragimov, Bäcklund Transformations in Applications, SIAM Studies in Appl. Math., No. 1, Philadelphia (1979).
4. W. I. Fushchich and A. G. Nikitin, "On the new invariance algebras and superalgebras of relativistic wave equations," J. Phys., A: Math. Gen., 20, No. 3, 537-549 (1987).
5. W. I. Fushchich and A. G. Nikitin, "On superalgebras of symmetry operators of relativistic wave equations," in: Selected Topics in QFT and Math. Phys., World Scientific, Singapore (1990), pp. 385-391.
6. V. N. Shapovalov and G. G. Ékle, Algebraic Properties of the Dirac Equation [in Russian], Kalmyts. Univ., Élista (1972).
7. W. Miller, Jr., Symmetry and Separation of Variables, Addison-Wesley, Reading, Massachusetts (1977).
8. A. G. Nikitin and A. I. Prilipko, "Generalized Killing tensors and symmetry of the Klein–Gordon–Fock equation," Preprint 90.23, Inst. Mat. Akad. Nauk Ukr. SSR, Kiev (1990).
9. A. G. Nikitin, "Generalized Killing tensors of arbitrary rank and order," Ukr. Mat. Zh., 43, No. 6, 786-795 (1991).
10. E. G. Kalnis, W. Miller, Jr., and G. C. Williams, "Matrix operator symmetries of the Dirac equation and separation of variables," J. Math. Phys., 27, No. 7, 1893-1990 (1986).
11. P. West, Introduction to Supersymmetry and Supergravitation, World Scientific, Singapore (1986).