Solitary wave and other solutions for nonlinear heat equations

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Abstract: A number of explicit solutions for the heat equation with a polynomial non-linearity and for the Fisher equation is presented. An extended class of non-linear heat equations admitting solitary wave solutions is described. The generalization of the Fisher equation is proposed whose solutions propagate with arbitrary ad hoc fixed velocity

1 Introduction

Lie groups of transformations which conserve the shape of differential equations present powerfool tools for construction of their exact solutions. The related Lie algebras and their invariants are the main instruments used to find the Ansätze which make it possible to find such solutions in explicit form.

In the present paper we search for exact solutions of reaction-diffusion equations using a specific Ansätz which can be related to the classical Lie symmetry and conditional symmetry as well.

The nonlinear reaction-diffusion equations play fundamental role in a great number of various models of heat and reaction-diffusion processes, mathematical biology, chemistry, genetics and many, many others. Thus, one of the corner stones of mathematical biology

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is the Fisher equation [1]

$$u_t - u_{xx} = u(1 - u) \tag{1}$$

where $u = u(x, t)$ and subscripts denote derivatives w.r.t. the corresponding variable: $u_t = \frac{\partial u}{\partial t}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$.

Equation (1) is a particular case of the Kolmogorov-Petrovskii-Piskunov (KPP) equation [2]

$$u_t - u_{xx} = f(u) \tag{2}$$

where $f(u)$ is a sufficiently smooth function satisfying the relations $f(0) = f(1) = 0$, $f_u(0) = \alpha > 0$, $f_u(u) < \alpha$, $0 < u < 1$.

The reaction-diffusion equation with the cubic polynomial nonlinearity

$$u_t - u_{xx} = \alpha(u^3 + bu^2 + cu) \tag{3}$$

where $\alpha = \pm 1$, b and c are constants, also has a large application value and includes as particular cases the Fitzhugh-Nagumo equation [3] ($\alpha = -1, b = -c - 1, 0 < c < 1$) which is used in population genetics, the Newell-Whitehead [4] (for $c = \alpha = -1, b = 0$) and Huxley [1] (for $\alpha = b = -1, c = 0$) equations. Notice that the Fitzhugh-Nagumo equation also belongs to the Kolmogorov-Petrovskii-Piskunov type.

A nice property of equations (1)-(3) is that they admit plane wave solutions which in many cases can be found in explicit form. Existence of such solutions is caused by the symmetry w.r.t. translations $t \to t + k$, $x \to x + r$ with constant parameters k and r. For some special functions $f(u)$ equation admits more extended symmetry groups [5] and, as a result, have exact solutions of more general type than plane waves. We notice that group analysis of (2) for $f(u) = 0$ was carried out by Sophus Lie more than 130 year ago [6]. The group classification of systems of nonlinear heat equations was presented in papers [8].

The conditional (non-classical) symmetry approach [9], [10], [11] enables to construct new exact solutions of partial differential equations which cannot be found in the framework of Lie theory. In application to the equations of type (2) this approach was successfully used for the case of cubic polynomial nonlinearity (3) only [12], [13].

An effective algorithm for construction of travelling wave solutions together with a number of interesting examples was proposed in the recent paper [14]. However the nonlinear heat equations of the general type (2) were not analyzed in [14].

In our paper we present a specific "universal" Ansätz which enables to make effective reductions of an extended class of equations (2) which includes (1) and (3). Being applied to (3) this Ansätz makes it possible to obtain all exact solutions found earlier [12], [13]. Moreover, we present solutions of (3) (effectively, an infinite number of them) in explicit forms, i.e., in terms of the Jacobi elliptic functions. In addition, using the unified algebraic method [14] we select such equations of the type (2) which admit solitary wave solutions and construct these solutions explicitly. Finally, we present solutions for the Fisher equation and propose such generalization of it which admit the same exact travelling wave solution as (1.1), but *with any ad hoc given velocity of propagation* (for solutions of

(1.1) this velocity is fixed and equal to $\frac{5}{\sqrt{6}}$). In spirit of Hirota's method [15] to achieve these goals we use a special Ansätz which leads to a uniform formulation for all considered equations (which, however, is *tri-linear*). In addition to equations (1) and (3), this Ansätz makes it possible to reduce an extended class of equations of the type (2).

In the following section we discuss the Ansätz which will be used to reduce a class of nonlinear heat equations. In Section 3 we present an infinite set of exact solutions (given explicitly in terms of the Jacobi elliptic functions) for the heat equation with the cubic and cubic polynomial nonlinearity.

In Section 4 we describe plane wave solutions for special classes of equations (2). In Section 5 solitary wave solutions for equations (2) are found. Finally, in Sections 6 and 7 we present exact solutions for the Fisher equation and propose a generalization of this equation.

2   The Ansätz and related equations

We start with the reaction-diffusion equation with a power nonlinearity

$$u_t - u_{xx} = -\lambda u^n, \quad \lambda = \frac{2(n+1)}{(n-1)^2} \tag{4}$$

where n is a constant, $n \neq 1$.

For convenience we choose a special presentation for the coupling constant λ. Scaling u one can reduce λ to 1 or to -1 for $n > 1$ and $n < 1$ respectively.

For any $n \neq 1$ we set

$$u = \left(\frac{z_x}{z}\right)^k, \quad k = \frac{2}{n-1} \tag{5}$$

and transform (4) to the uniform equation

$$z\left(z_x z_{tx} - z_x z_{xxx} - (k-1)z_{xx}^2\right) = z_x^2\left(z_t - (2k+1)z_{xx}\right). \tag{6}$$

In contrast with (4) equation (6) is homogeneous with respect to the dependent variable and includes the cubic non-linearities only while (4) includes u in an arbitrary (fixed) power n. We will show that formulation (6) is very convenient for effective reductions.

Ansätz (5) is a particular case of a more general one

$$u = z_x^k \varphi(z) \tag{7}$$

where z is an unknown function of independent variables t, x and φ is a function of z. It will be shown in the following that the change (7) presents very effective tools for reductions of equations (2) which cover both classical and conditional symmetry reductions.

We notice that (4) is not the only nonlinear equation of type (2) which can be reduced to the tri-linear form via Ansätz (5). A more general equation (2) which admits this procedure is

$$u_t - u_{xx} = k\left(-(k+1)u^n + \lambda_1 u + \lambda_2 u^{\frac{n+1}{2}} + \lambda_3 u^{\frac{3-n}{2}} + \lambda_4 u^{2-n}\right) \tag{8}$$

where $\lambda_1, \ldots, \lambda_4$ are arbitrary constants.

Formula (8) defines special but rather extended class of the nonlinear heat equations, which includes all important models enumerated in Introduction and many others. The change (5) transforms (8) to the following form

$$z(z_x z_{tx} - z_x z_{xxx} - \lambda_3 z z_x - \lambda_4 z^2 - (k-1)z_{xx}^2)$$
$$= z_x^2(z_t + \lambda_1 z + \lambda_2 z_x - (2k+1)z_{xx}). \tag{9}$$

In contrast with (8) equation (9) is homogeneous w.r.t. the dependent variable and is much more convenient for searching for exact solutions.

3   Infinite sets of solutions

Consider a particular (but important) case of (8) which corresponds to $n = 3, \lambda_1 = \lambda_2 = \lambda_3 = 0$:

$$u_t - u_{xx} = -2u^3. \tag{10}$$

The related equation (9) takes the form

$$z(\dot{z}_x - z_{xxx}) = z_x(\dot{z} - 6z_{xx}). \tag{11}$$

Equation (11) is compatible with the condition $Xz = 0$ where

$$X = \frac{\partial}{\partial t} - \frac{3}{x}\frac{\partial}{\partial x}. \tag{12}$$

It means [10] that this equation admits conditional symmetry , thus it is reasonable to search for its solutions in the form

$$z = \varphi(y), \quad y = x^2 + 6t \tag{13}$$

where y is the invariant variable for symmetry (12). Substituting (13) into (11) we come to the third order differential equation for φ

$$\varphi\varphi_{yyy} = 3\varphi_y\varphi_{yy}. \tag{14}$$

Dividing the l.h.s. and r.h.s. of (14) by $\varphi\varphi_{yy}$ and integrating we obtain

$$\varphi_{yy} = c\varphi^3, \quad c = \pm 2 \tag{15}$$

where c is the integration constant which can be reduced to 2 (for $c > 0$) or to -2 (for $c < 0$) by scaling the dependent variable φ. We make such scaling to simplify the following formulae.

In accordance with (5), (13), any solution φ of (15) generates a solution for (10) of the following form

$$u = \frac{2x\varphi_y}{\varphi}. \tag{16}$$

An explicit solution of equation (15) for $c = 2$ is the Jacobi elliptic function

$$\varphi(y) = \mathrm{ds}\left(y, \frac{1}{\sqrt{2}}\right), \quad y = x^2 + 6t \tag{17}$$

so (16) can be represented as

$$u = u_1 = \frac{2x\mathrm{cs}\left(y, \frac{1}{\sqrt{2}}\right)}{\mathrm{dn}\left(y, \frac{1}{\sqrt{2}}\right)}. \tag{18}$$

The case $c = -2$ leads to the same solution as given in (18).
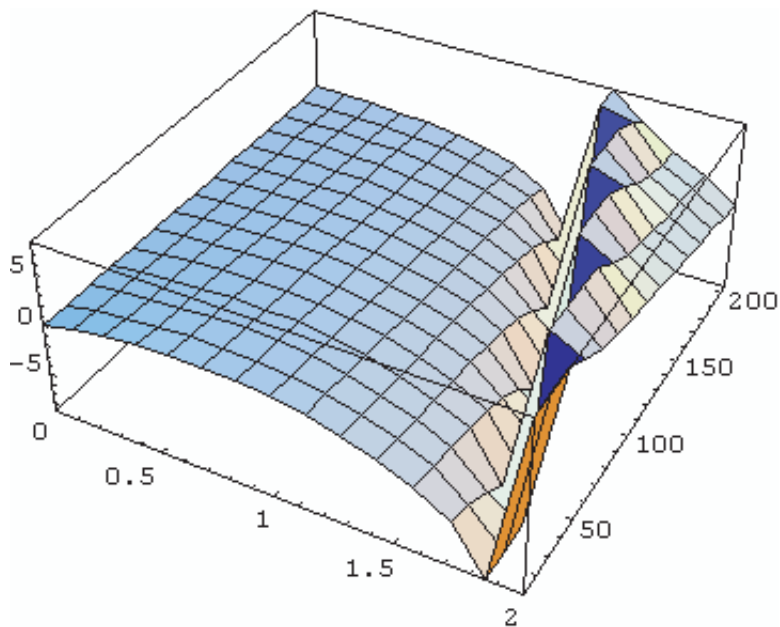The plot of this solution is given by Fig. 1.

Fig. 1 Solution (18) for equation (10), $0 < t \leq 200$.

To construct more elliptic function solutions for (10) we exploit some properties of the elliptic functions formulated in the following assertions.

Proposition 3.1. *Let $\varphi = \varphi^{(n)}$ be a solution of equation (15) for $c = 2$ or $c = -2$. Then*

$$\varphi^{(n+1)} = \frac{\varphi_y^{(n)}}{\varphi^{(n)}} \tag{19}$$

also satisfies this equation for $c = 2$.

Proof of this and the following propositions is reduced to a direct verification. We notice that equation (15) is equivalent to the following one

$$\left(\varphi_y^{(n)}\right)^2 = \left(\varphi^{(n)}\right)^4 + C_n \tag{20}$$

where C_n is the integration constant. Then $\varphi^{(n+1)}$ of (19) satisfies the equation

$$(\varphi_y^{(n+1)})^2 = (\varphi_y^{(n+1)})^4 + C_{n+1}, \quad C_{n+1} = -4C_n.$$

Proposition 3.2. *Let $\varphi^{(n)}$ be a solution of equation (20) for $C_n > 0$. Then this equation is solved also by the following function*

$$\tilde{\varphi}^{(n)} = \frac{\sqrt{C_n}}{\varphi^{(n)}}.$$

Proposition 3.3. *Let $\varphi^{(n)}$ be a solution of equation (20) for $C_n = -B_n < 0$. Then the function*

$$\hat{\varphi}^{(n)} = \frac{\sqrt{B_n}}{\varphi^{(n)}} \tag{21}$$

satisfies equation (15) for $c = -2$ and the following relation

$$\left(\hat{\varphi}_y^{(n)}\right)^2 = -\left(\hat{\varphi}^{(n)}\right)^4 + B_n^2. \tag{22}$$

Using Propositions 3.1 and 3.2 and starting with (18) we obtain infinite sets of solutions for equation (10):

$$u_n = 2x\varphi^{(n)}, \quad n = 0, 1, 2, \cdots, \tag{23}$$

and

$$\tilde{u}_{2k+1} = \frac{2^{k+1}x}{\tilde{\varphi}^{(2k+1)}}, \quad k = 0, 1, 2, \cdots \tag{24}$$

where $\tilde{\varphi}^{(2k+1)}$ and $\varphi^{(n)}$ are defined by (21) and the following recurrence relations

$$\varphi^{(n)} = \frac{\varphi_y^{(n-1)}}{\varphi^{(n-1)}}, \quad \varphi^{(0)} = \mathrm{ds}\left(y, \frac{1}{\sqrt{2}}\right). \tag{25}$$

For $n = 1$ we have the solutions (17), while for $n = 2, 3, \cdots$ and $k = 0, 1, \cdots$ we obtain

$$u_2 = 2x\left[\frac{\mathrm{cd}\left(y, \frac{1}{\sqrt{2}}\right) - \mathrm{dc}\left(y, \frac{1}{\sqrt{2}}\right)}{\mathrm{sn}\left(y, \frac{1}{\sqrt{2}}\right)} - \mathrm{cn}\left(y, \frac{1}{\sqrt{2}}\right)\mathrm{ds}\left(y, \frac{1}{\sqrt{2}}\right)\right],$$

$$u_3 = \frac{2x\left(\mathrm{cs}^4\left(y, \frac{1}{\sqrt{2}}\right) - \mathrm{dn}^4\left(y, \frac{1}{\sqrt{2}}\right)\right)}{\mathrm{dn}\left(y, \frac{1}{\sqrt{2}}\right)\mathrm{cs}\left(y, \frac{1}{\sqrt{2}}\right)\left(\frac{9}{4}\sqrt{2}\mathrm{cn}^2\left(y, \frac{1}{\sqrt{2}}\right) - \mathrm{ds}^2\left(y, \frac{1}{\sqrt{2}}\right)\right)},$$

$$\cdots,$$

(26)

and

$$\tilde{u}_1 = \frac{2x\mathrm{dn}\left(y, \frac{1}{\sqrt{2}}\right)}{\mathrm{cs}\left(y, \frac{1}{\sqrt{2}}\right)},$$

$$\tilde{u}_3 = \frac{4x\mathrm{dn}\left(y, \frac{1}{\sqrt{2}}\right)\mathrm{cs}\left(y, \frac{1}{\sqrt{2}}\right)\left(\frac{9}{4}\sqrt{2}\mathrm{cn}^2\left(y, \frac{1}{\sqrt{2}}\right) - \mathrm{ds}^2\left(y, \frac{1}{\sqrt{2}}\right)\right)}{\mathrm{cs}^4\left(y, \frac{1}{\sqrt{2}}\right) - \mathrm{dn}^4\left(y, \frac{1}{\sqrt{2}}\right)},$$

$$\cdots.$$

(27)

Formulae (19), (26) , (27) and the recurrence relations (25) present an infinite number of exact solutions for equation (10). Moreover, taking into account the transparent invariance of (10) with respect to displacements of independent variables t and x we can write more general solutions changing $x \to x + k_1$, $t \to t + k_2$ with arbitrary constants k_1 and k_2.

We stress that all these solutions belong to the class found in [12], [13]. Here we present the explicit analytical expressions for these solutions in terms of the Jacobi elliptic functions. The plots of solution $\tilde{u}_1$ is given in Fig. 2.
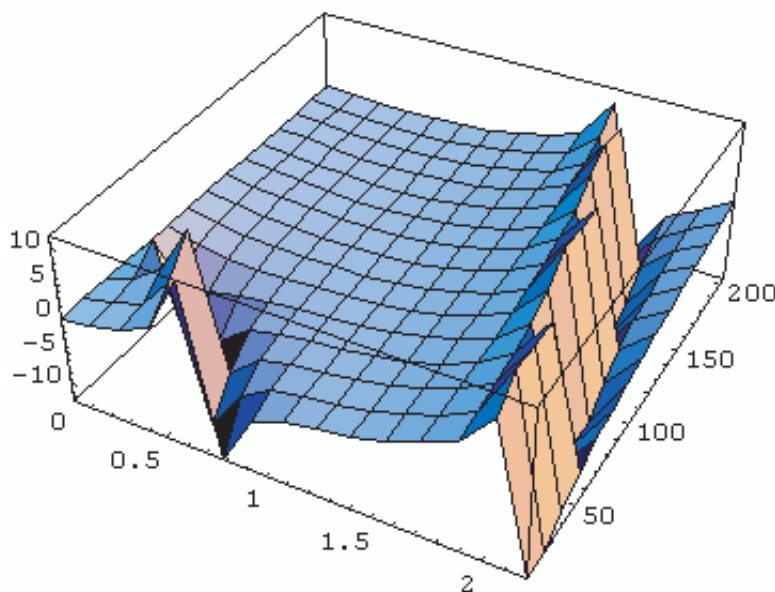

Fig. 2 Solution $\tilde{u}_1$ (27) for equation (10).

Propositions 3.1 and 3.2 make it possible to construct infinite sets of exact solutions for other equations of the type (8).

Consider first equations (8) for $n = 3$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$, i.e.,

$$u_t - u_{xx} = -2\left(u^3 + \lambda_1 u\right)$$

(28)

where without loss of generality we can set $\lambda_1 = \pm 1$. For $\lambda_1 = -1$ (28) is equivalent to

the Newell- Whitehead [4] equation up to scaling variables t and x. The Ansätze

$$u = \xi_x \varphi(\xi), \quad \xi = k_1 \cosh(x + k_2) \exp(3t), \quad \lambda_1 = 1, \tag{29}$$

$$u = \eta_x \varphi(\eta), \quad \eta = k_1 \cos(x + k_2) \exp(3t), \quad \lambda_1 = -1 \tag{30}$$

reduces (28) to the form (15). Thus repeating the arguments which follow equation (15) we come to exact solutions for (28). The explicit form of these solutions can be obtained from (18), (23)-(27) via the changes $y \to \xi$, $2x \to \xi_x$ for $\lambda_1 = 1$ and $y \to \eta$, $2x \to \eta_x$ for $\lambda_1 = -1$.

Finally, we notice that Proposition 3 makes it possible to construct infinite sets of exact solutions for the equations (8) with $n = \lambda_4 = -1$, $\lambda_2 = \lambda_3 = 0$, i.e., for the equations

$$u_t - u_{xx} = 2u^3 \tag{31}$$

and

$$u_t - u_{xx} = 2\left(u^3 + \varepsilon u\right) \tag{32}$$

which differ from (10) and (28) by the sign of the l.h.s. terms. Indeed, Ansätze (18) and (29) or (30) reduce the corresponding equations (31) and (32) to the following equation for φ

$$\varphi'' = -2\varphi^3 \tag{33}$$

where the double prime denotes the second derivative w.r.t. the corresponding variable (i.e., y, ξ or η).

In accordance with Proposition 3, exact solutions for (33) have the form (21) where $\varphi^{(n)}$ are solutions of equations (25) for *even* n. The related list of exact solutions for equation (31) is given by the following formulae:

$$\hat{u}_0 = x \mathrm{sd}\left(y, \frac{1}{\sqrt{2}}\right),$$

$$\hat{u}_2 = \frac{4x \mathrm{sn}\left(y, \frac{1}{\sqrt{2}}\right)}{\mathrm{cd}\left(y, \frac{1}{\sqrt{2}}\right) - \mathrm{dc}\left(y, \frac{1}{\sqrt{2}}\right) - \mathrm{cn}\left(y, \frac{1}{\sqrt{2}}\right) \mathrm{dn}\left(y, \frac{1}{\sqrt{2}}\right) \mathrm{sn}\left(y, \frac{1}{\sqrt{2}}\right)}, \tag{34}$$

$$\dots$$

$$\hat{u}_{2k} = \frac{2^k x}{\varphi^{(2k)}}$$

where $\varphi^{(2k)}$ are defined by recurrence relations (25).

The plots of solutions (33) are given in Figs. 3, 4.

Solutions for (32) can be obtained from (34) by changing $y \to \xi$, $x \to \xi_x$ and $y \to \eta$, $x \to \eta_x$ for $\varepsilon = 1$ and $\varepsilon = -1$ respectively.

4   Solutions for arbitrary n

Let us consider equation (8) with arbitrary n and construct its exact solutions. In this section we restrict ourselves to the case $\lambda_3 = \lambda_4 = 0$ and use a reduced version of the
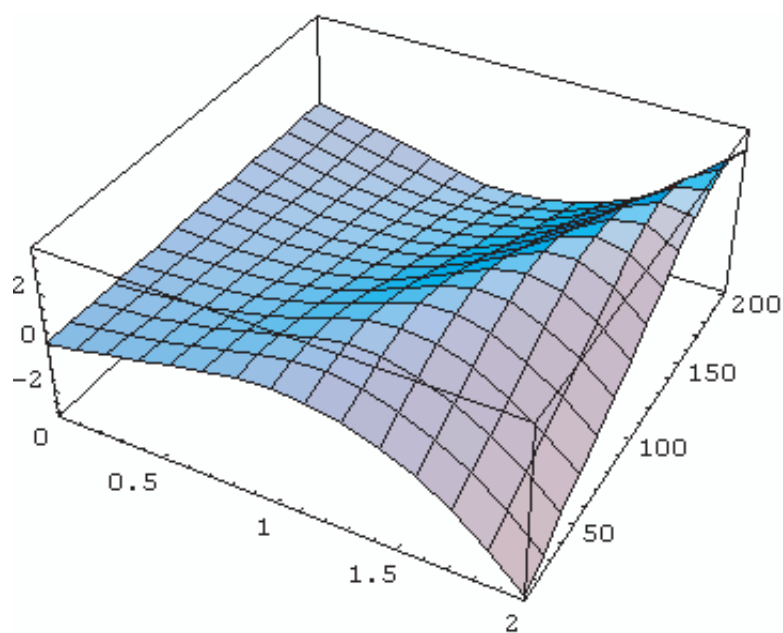
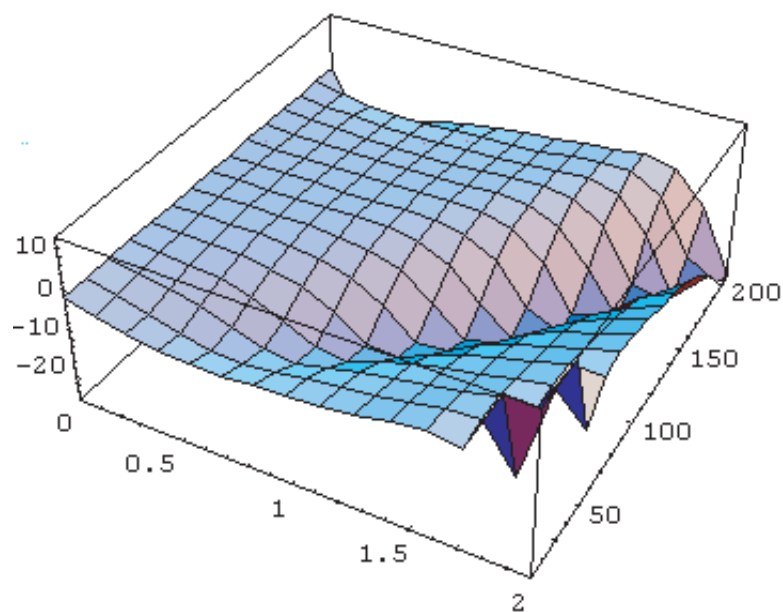**Fig. 3** Solution $\tilde{u}_0$ (27) for equation (10).

Fig. 4 Solution $\tilde{u}_2$ (27) for equation (10).

related potential equation (9) i.e.,

$$(n-1)z_t = (n+3)z_{xx} - (n-1)(\lambda_1 z + \lambda_2 z_x), \tag{35}$$

$$4z_x z_{xxx} + (n-3)z_{xx}^2 - (n-1)(\lambda_1 z_x^2 + \lambda_2 z_x z_{xx}) = 0. \tag{36}$$

Any solution of the system (36) satisfies (9) with $\lambda_3 = \lambda_4 = 0$, the inverse is not true. To solve (35), (36) we introduce the new variable $y = \frac{z_{xx}}{z_x}$. Then, dividing (36) by z_x^2

and using the identity $\frac{z_{xxx}}{z_x} = y_x + y^2$ we transform this equation to the Riccatti form

$$y_x + \frac{n+1}{4}y^2 - \frac{(n-1)}{4}(\lambda_2 y + \lambda_1) = 0. \tag{37}$$

Differentiating y w.r.t. t and using (35) and (37) we obtain the following differential consequence

$$\dot{y} = Ay^3 + By^2 + Cy + D \tag{38}$$

where

$$A = \frac{1}{8}\frac{(n+3)(n-3)(n+1)}{n-1}, \quad B = \lambda_2\left(1 - \frac{3}{16}(n^2-1)\right),$$

$$C = \frac{1}{16}\left((n-1)^2\lambda_2^2 - 2(n+3)(n-3)\lambda_1\right), \quad D = \frac{(n-1)^2}{16}\lambda_1\lambda_2.$$

For arbitrary n the system of equations (37), (38) is compatible but has constant solutions only. In three exceptional cases $n = \pm3$ and $n = -1$ the compatibility conditions for (37), (38) are less restrictive in as much as the related coefficient A in (38) is equal to zero.

Let $n \neq \pm3$ and $n \neq -1$, then $y = c_1 = \text{const}$, and equations (37), (38), reduce to the only condition

$$\lambda_1 = -c_1\lambda_2 + (k+1)c_1^2, \quad k = \frac{2}{n-1}.$$

The corresponding solution for the system (35), (36) is

$$z = e^{c_1x + kc_1^2t} + c_2e^{\left(\lambda_2c_1 - (k+1)c_1^2\right)t} \tag{39}$$

and the related exact solution (5) takes the form

$$u = \frac{c_1^k}{\left(1 + c_2e^{-c_1x - \left((2k+1)c_1^2 - \lambda_2c_1\right)t}\right)^k}. \tag{40}$$

Thus we find exact solutions (40) for the equation

$$u_t - u_{xx} = -k(k+1)u^n + \lambda_2ku^{\frac{n+1}{2}} + ((k+1)c_1^2 - \lambda_2c_1)ku. \tag{41}$$

This equation belongs to the Kolmogorov-Petrovski-Piskunov type provided

$$\lambda_2 = (k+1)(c_1 + 1). \tag{42}$$

The corresponding plane wave solution (40) propagates with the velocity

$$v = k + 1 - kc_1.$$

In special cases $n = \pm3$ and $n = -1$ equations (35), (36) admit more extended classes of exact solutions. In particular, for $n = 3$ we can recover exact solutions for (3) caused by conditional symmetry and classical Lie symmetry as well. We will not study these special cases here.

5   Solitary wave solutions

Consider now the general equation (6) with arbitrary parameters λ_1, λ_2, λ_3 and λ_4. It seems to be impossible integrate in closed form the related potential equations (9). Here we search for particular solutions which belong to soliton type and so have good perspectives for various applications.

Let us consider solutions for (6) of the form $z(t,x) = U(\xi)$ where $\xi = \mu t + x$ and μ is an arbitrary (nonzero) constant. Then we come to the following ordinary differential equation for U

$$U[U'(\mu U'' - U''' - \lambda_3 U) - \lambda_4 U^2 - (k-1)(U'')^2]$$
$$= (U')^2[(\mu + \lambda_2)U' + \lambda_1 U - (2k+1)U''] \tag{43}$$

where $U' = \frac{dU}{d\xi}$.

Let us follow [14] and search for solutions for (43) in the form

$$U = \nu_0 + \nu_1 \varphi + \nu_2 \varphi^2 + \cdots, \tag{44}$$

where ν_0, ν_1, $\cdots$ are constants and φ satisfies equation of the following general form

$$\varphi' = \varepsilon \sqrt{c_0 + c_1 \varphi + c_2 \varphi^2 + \cdots} \tag{45}$$

where $\varepsilon = \pm 1$. In order (44) be compatible with (43) we have to equate separately the terms which include odd and even powers of the square root given by (45). In view of this we come to the following system

$$U'(\mu U U'' - \lambda_3 U^2) = (U')^3(\mu + \lambda_2), \tag{46}$$

$$U(U'U''' + \lambda_4 U^2 + (k-1)(U'')^2) = (U')^2((2k+1)U'' - \lambda_1 U). \tag{47}$$

Dividing any term in (46) by $\mu U^2 U'$ we come to the Riccatti equation

$$Y' - \frac{\lambda_2}{\mu} Y^2 = \frac{\lambda_3}{\mu}$$

for $Y = \frac{U'}{U}$, whose general solutions are

$$
\begin{aligned}
Y &= \sqrt{\frac{-\lambda_3}{\lambda_2}} \tanh\left(\frac{\sqrt{-\lambda_2\lambda_3}}{\mu}\xi + C\right), \\
Y &= \sqrt{\frac{-\lambda_3}{\lambda_2}} \left(\tanh\left(\frac{\sqrt{-\lambda_2\lambda_3}}{\mu}\xi + C\right)\right)^{-1}, \quad \text{if } \lambda_2\lambda_3 < 0,
\end{aligned}
\tag{48}
$$

$$y = \sqrt{\frac{\lambda_3}{\lambda_2}} \tan\left(\frac{\sqrt{\lambda_2\lambda_3}}{\mu}\xi + C\right), \qquad \text{if } \lambda_2\lambda_3 > 0, \tag{49}$$

$$y = -\frac{\mu}{\lambda_2(\xi + C)}, \qquad \text{if } \lambda_3 = 0 \tag{50}$$

where C is the integration constant.

Thus all solutions for (43) which can be obtained with using the algebraic method [14] are exhausted by hyperbolic, triangular and rational ones presented by relations (48)-(50).

Solutions (48), (49) and (50) are compatible with (47) provided

$$\mu = -\lambda_2, \ \lambda_1 = -k\frac{\lambda_3}{\lambda_2}, \ \lambda_4 = (1-k)\left(\frac{\lambda_3}{\lambda_2}\right)^2, \ \lambda_2\lambda_3 \neq 0$$

and

$$\lambda_1 = \lambda_4 = 0, \ \lambda_3 = 0$$

respectively. Using variables

$$\tau = \frac{2}{(n-1)^2}t, \ y = \frac{\sqrt{2}}{n-1}x, \ \sigma = -\lambda_2(n-1), \ \nu = \frac{\lambda_3}{\lambda_2}$$

we can rewrite the related equation (8) as follows:

$$u_\tau - u_{yy} = \left(1 + \nu u^{1-n}\right)\left(-(n+1)u^n + \nu(n-3)u + \sigma u^{\frac{n+1}{2}}\right). \tag{51}$$

The corresponding solutions (5), (48) for equation (51) have the following form

$$u = (-\nu)^{\frac{1}{n-1}}\left(\tanh\left(b\left(y - \frac{\sigma}{\sqrt{2}}t\right) + C\right)\right)^{\frac{2}{n-1}}, \tag{52}$$

$$u = (-\nu)^{\frac{1}{n-1}}\left(\tanh\left(b\left(y - \frac{\sigma}{\sqrt{2}}t\right) + C\right)\right)^{\frac{2}{1-n}} \tag{53}$$

where $\nu < 0$ and $b = (n-1)\sqrt{\frac{-\nu}{2}}$,

$$u = (\nu)^{\frac{1}{n-1}}\left(\tan\left(b\left(y - \frac{\sigma}{\sqrt{2}}t\right) + C\right)\right)^{\frac{2}{n-1}} \tag{54}$$

where $\nu > 0$ and $b = (n-1)\sqrt{\frac{\nu}{2}}$, and

$$u = 2^{n-1}\left((n-1)\left(y - \frac{\sigma}{\sqrt{2}}\tau + C\right)\right)^{\frac{2}{1-n}} \tag{55}$$

if $\nu = 0$.

For $\frac{2}{n-2} > 1$ formula (52) presents nice solitary wave solutions which propagate with the velocity $\frac{\sigma}{\sqrt{2}}$. In the case $n = 2$ we come to the bell-shaped solitary wave solution which will be discussed in Section 7.

If $\frac{2}{n-2} < -1$ then (52) is a singular solution whose physical relevance is doubtful. However, in this case equation (51) admits another solitary wave solutions which are given now by relation (53).

We see such solutions exist for the extended class of the nonlinear reaction-diffusion equations defined by formula (51).

6   Exact solutions for the Fisher equation

Let us return to Section 4 and consider in more detail the important case $n = 2$. Setting in (41) $\lambda_2 = 0$, $c_1 = -1$ and making the change

$$\tau = 6t, \quad y = \sqrt{6}x \tag{56}$$

we come to the Fisher equation (1) for $u(\tau, y)$:

$$u_\tau - u_{yy} = u(1 - u). \tag{57}$$

Thus the Fisher equation is a particular case of (41) and so our solutions (40) are valid for (57) provided we make the above mentioned changes of variables and set $c_1 = -1$ in accordance with (42). As a result we recover the well-known Ablowitz-Zeppetella [16] solution

$$u = \frac{1}{\left(1 + c_2 e^{\frac{y}{\sqrt{6}} - \frac{5\tau}{6}}\right)^2}. \tag{58}$$

This solution can be expressed via hyperbolic functions

$$u = u_1 = \frac{1}{4}\left(1 - \tanh\left(\frac{y}{2\sqrt{6}} - \frac{5}{12}\tau - c\right)\right)^2, \quad c = \frac{1}{2}\ln|c_2|, \tag{59}$$

$$u = u_2 = \frac{1}{4}\left(1 - \coth\left(\frac{y}{2\sqrt{6}} - \frac{5}{12}\tau - c\right)\right)^2 \tag{60}$$

for $c_2 > 0$ and $c_2 < 0$ respectively.

Taking into account the symmetry of (57) w.r.t. the discrete transformation $u \to 1 - u$ we obtain two more solutions: $u_3 = 1 - u_1$ and $u_4 = 1 - u_2$. Finally, bearing in mind the symmetry of (57) w.r.t. the space reflection $y \to -y$ we come to four more exact solutions by changing $y \to -y$ in u_1, u_2, u_3 and u_4.

Thus starting with our general formulae (39) and (40) we come to the family of eight exact solutions for the Fisher equation. All of them are plane waves propagating with the velocity $\pm\frac{5}{\sqrt{6}}$.

To find additional exact solutions we use the Ansätz (7) for $n = 2$, i.e.,

$$u = 3z_y^2\varphi(z) \tag{61}$$

where φ and z are functions to be found. Substituting (61) into (57) we come to the following reduced equations

$$z_\tau = 5z_{yy},$$
$$4z_y z_{yyy} - z_{yy}^2 = \tfrac{1}{2}z_y^2 \tag{62}$$

and

$$\varphi_{zz} = 3\varphi^2. \tag{63}$$

We see that φ has to satisfy the Weierstrass equation (63) which we rewrite in the following equivalent form

$$\tilde{\varphi}_z^2 = 4\tilde{\varphi}^3 - C, \ \tilde{\varphi} = \frac{1}{2}\varphi \tag{64}$$

where C is the integration constant.

Starting with (61) and choosing the following exact solutions of (62) and (64):

$$z = \exp\left(-\frac{1}{\sqrt{6}}y + \frac{5}{6}\tau\right), \ \varphi = \frac{2}{z+k} \tag{65}$$

we come to the Ablowitz-Zeppetella solutions (58) for the Fisher equation.

We notice that relations (65) present only a very particular solution of (64) which correspond to zero value of the parameter C. In addition, there exist the infinite number of other solutions corresponding to non-zero C. The related functions (61) are:

$$u = \frac{1}{2}z^2\wp(z,0,C), \ \ z = \exp\left(-\frac{1}{\sqrt{6}}y + \frac{5}{6}\tau + k\right) \tag{66}$$

where $\wp(z,0,C)$ is the Weierstrass function satisfying equation (64) for $C \neq 0$.

In order to solutions (61) be bounded it is sufficient to restrict ourselves to the case when $-\frac{1}{\sqrt{6}}y + \frac{5}{6}\tau + k > 0$. Such conditions can be satisfied, e.g., for arbitrary positive y and negative τ and k. The related solutions can be interpreted as ones describing the history of the process because the time variable takes arbitrary negative values. The graphics of solutions (61) for some values of the parameter C are given by Figures 5-7.
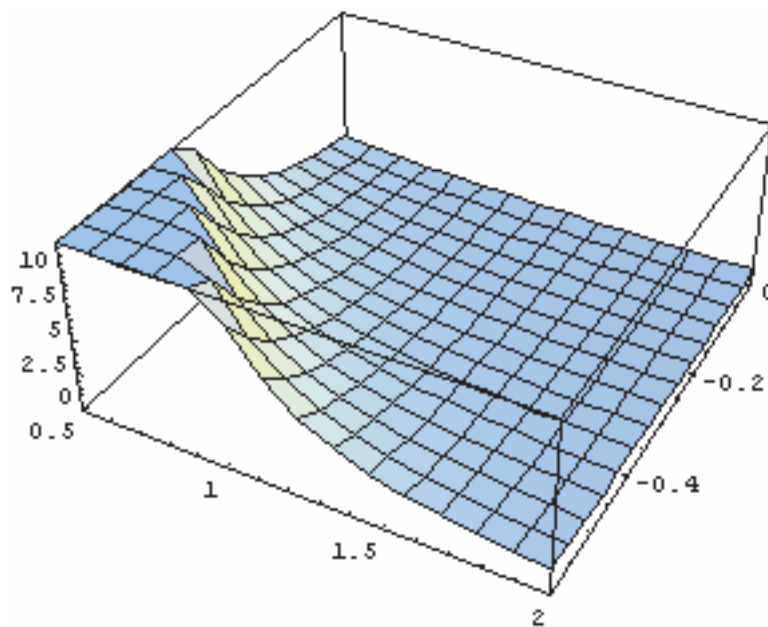

Fig. 5 Solution (66) with $k = 0, C = 10^2$ for the Fisher equation (57).

Thus the Fisher equation admits the infinite set of exact solutions which include the Ablowitz-Zeppetella solutions (58) and also solutions (66) enumerated by two parameters, $C \neq 0$ and k. All these solutions are plane waves propagating with the same velocity $v = \frac{5}{\sqrt{6}}$.
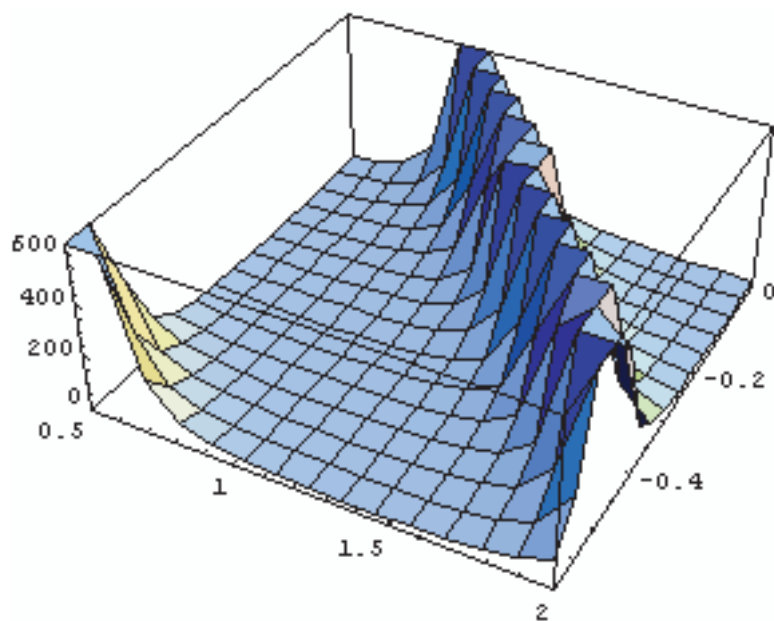
Fig. 6 Solution (66) with $k = 0, C = 10^4$ for the Fisher equation (57).
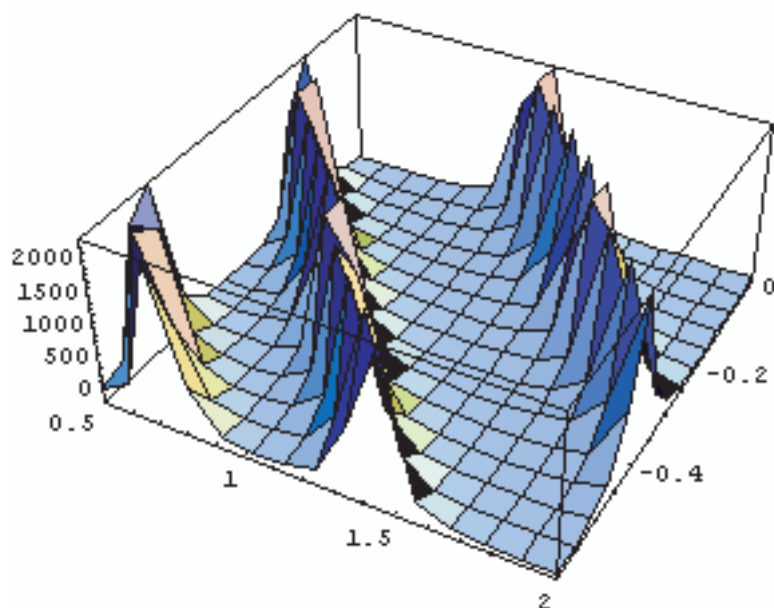

Fig. 7 Solution (66) with $k = 0, C = 10^6$ for the Fisher equation (57).

In the following section we consider generalized Fisher equations which admit exact solutions with arbitrary propagation velocities.

7   Generalizations of the Fisher equations

Let us consider equation (51) for $n = 2$, which takes the following form

$$u_\tau - u_{yy} = (u + \nu)(-3u + \sigma\sqrt{u} - \nu). \tag{67}$$

Relation (67) is a formal generalization of the Fisher equation in as much as in the case $\sigma = 0$ (67) is equivalent to (1). However, for $\sigma \neq 0$ equation (67) admits soliton solutions (52) and (or) solutions (53)-(55) and so has absolutely another nature then (1). Nevertheless for small σ it would be interesting to treat (67) as a perturbed version of (1).

Consider equation (67) in more detail. Let $\nu < 0$ then scaling dependent and independent variables we can reduce its value to the following one

$$\nu \to \nu' = -\frac{3}{2}, \text{ if } \nu < 0. \tag{68}$$

Setting then

$$\tilde{u} = \frac{3}{2} - u, \ \sigma = 3\varepsilon, \ t = 3\tau, \ x = -\sqrt{3}y \tag{69}$$

we come to the following relation

$$\tilde{u}_t - \tilde{u}_{xx} = \tilde{u}\left(1 - \tilde{u} + \varepsilon\left(\frac{3}{2} - \tilde{u}\right)^{\frac{1}{2}}\right). \tag{70}$$

In the limiting case $\varepsilon \to 0$ equation (70) reduces to the Fisher equation in the canonical formulation (1).

In accordance with Section 5 equation (67) admits nice bell-shaped traveling wave solution (52) which transforms via changes (68), (69) to the following form

$$\tilde{u} = \frac{3}{2\cosh^2\left(\frac{1}{2}\left(x - \frac{\varepsilon}{\sqrt{6}}t\right) + C\right)}. \tag{71}$$

Consider now equation (67) for $\nu\sigma = 0 = 0$ and set $u = \frac{\tilde{u}}{3}$. As a result we reduce (67) to the simplest form

$$\tilde{u}_\tau - \tilde{u}_{yy} = -\tilde{u}^2. \tag{72}$$

The Ansätz

$$u = \frac{z_x^2}{6z^2} + \frac{1}{3}\left(1 + \varepsilon\sqrt{\frac{3}{2}}\right)\frac{z_{xx}}{z}, \quad \varepsilon = \pm 1$$

leads to the following reduced equations

$$z_{xxx} = 0, \quad z_t = \kappa z_{xx}, \quad \kappa = 5(3 \pm \sqrt{6}).$$

Thus we have $z = \frac{x^2}{2} + \kappa t$ and the related exact solution for (72) is

$$u = \frac{(3 \pm \sqrt{6})x^2 + 10(12 \pm 5\sqrt{6})t}{3\left(x^2 + 10(3 \pm \sqrt{6})t\right)^2}.$$

We notice that this solution can be found also using the classical Lie reduction.

Finally, let us consider one more generalization of the Fisher equation given by relation (41) for $n = 2$. Using notations (56) we rewrite it in the following form

$$\tilde{u}_\tau - \tilde{u}_{yy} = \tilde{u}(-c_1 + (c_1 + 1)\tilde{u}^{\frac{1}{2}} - \tilde{u}). \tag{73}$$

For $c_1 = -1$ equation (73) reduces to the Fisher equation (57).

In accordance with the results presented in Section 4 equation (73) admits exact solutions (40) which in our notations can be rewritten as

$$
\begin{aligned}
\tilde{u}_1 &= \tfrac{1}{4}\left(1 + \tanh\left(\tfrac{c_1}{2\sqrt{6}}y + \tfrac{c_1(2c_1-3)}{12}\tau - c\right)\right)^2, \\
\tilde{u}_2 &= \tfrac{1}{4}\left(1 + \coth\left(\tfrac{c_1}{2\sqrt{6}}y + \tfrac{c_1(2c_1-3)}{12}\tau - c\right)\right)^2.
\end{aligned}
\tag{74}
$$

Two more solutions can be obtained by changing $y \to -y$ in (74).

Formulae (74) present the analogies of solutions (59), (60) for equation (73). In contrast with (59), (60) these solutions describe a wave whose propagation velocity is equal to $\frac{2c_1-3}{\sqrt{6}}$ Thus changing parameter c_1 in (73) we can obtain solutions (74) with any velocity of propagation given *ad hoc*. In other words we always can take this velocity in accordance with experimental data.

Thus equation (73) reduces to the Fisher equation if the parameter c_1 is equal to -1. Moreover, both equations (57) and (73) admit the analogous exact solutions, (59), (60) and (74), which, however, have different propagation velocities.

8   Discussion

There exist well known regular approaches to search for exact solutions of nonlinear partial differential equations - the classical Lie approach [6], the conditional (non-classical) symmetries method [9], [11], [10], [12], generalized conditional symmetries [17], etc. These approaches present effective tools for finding special Ansätze which make it possible to reduce the equation of interest and find its particular solutions.

However, sometimes it seems that the Ansätze by themselves are more fundamental than the related symmetries. First, historically, the most famous Ansätze (like the Cole-Hopt one for the Burgers equation) was found without a scope of a symmetry approach. Secondly, some of Ansätze are effective in rather extended classes of problems characterized by absolutely different symmetries. In addition, in some cases the direct search for the Ansätz is a more straight-forward and effective procedure than search for (conditional) symmetries. We remind that the conditional symmetry approach presupposes search for solutions of nonlinear determining equations which in many cases are not simpler than the equation whose symmetries are investigated [18].

The present paper is based on using special Ansätze (7) which appear to be very effective for the extended class of nonlinear reaction-diffusion equations. In particular, they make it possible to find new exact solutions for the very well studied heat equations with quadratic polynomial non-linearities. Moreover, such Anaätze can be used to reduce wave equations of another type, e.g., hyperbolic equations. We plane to discuss the related results elsewhere.

We propose a generalization of the Fisher equation which preserves the type of its exact solutions, but predicts another propagation velocity. This property distinguishes (73) from numerous other generalizations of the Fisher equation refer, e.g., to [19] and

references cited therein.

Finally, we find soliton solutions for a number of nonlinear equations (8). To make this we use the algebraic method [14] which however was applied not directly to the equation of interest (8) but to the potential equation (9). By this we extend the class of non-integrable equations which have soliton solutions to the case of appropriate quasi-linear heat equations (2).

We stress that all these results were obtained with essential using the Ansätz (7). It seems to be an intriguing problem to find a regular way for searching such "universal" Ansätze.

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