REDUCTION OF IRREDUCIBLE UNITARY REPRESENTATIONS OF GENERALIZED POINCARÉ GROUPS WITH RESPECT TO THEIR SUBGROUPS

A.G. Nikitin, V.I. Fushchits, and I.I. Yurik

We consider the problem of the reduction of unitary irreducible representations of the generalized Poincaré groups $\mathcal{P}(1, n)$ with respect to their subgroups $\mathcal{P}(1, n-k)$. We find the explicit form of the unitary operator that relates the canonical basis of the representation to the $\mathcal{P}(1, n-k)$ basis. The action of the generators in the $\mathcal{P}(1, n-k)$ basis is given explicitly. The case of the inhomogeneous de Sitter group is considered in detail.

Introduction

The generalized Poincaré group $\mathcal{P}(1, n)$ is the semidirect product of the groups $SO_0(1, n)$ and T, where T is the additive group of the n-dimensional real vectors $p_0, p_1, \ldots, p_n$ and $SO_0(1, n)$ is the connected component of the identity in the group of all linear transformations of T onto T that preserve the quadratic form $p_0^2 - p_1^2 - \ldots - p_n^2$.

In [1,2] it was suggested that the groups $\mathcal{P}(1, n)$, $\mathcal{P}(1, 6)$, $\mathcal{P}(1, 4)$ should be used to describe physical systems with variable mass and spin. An example of such a physical system is one consisting of two (or three) free relativistic particles. For in this case the energy operator has the form

$$E=\sqrt{\mathbf{P}^2+M^2},\quad M=(m_1^2+\mathbf{K}^2)^{1/2}+(m_2^2+\mathbf{K}^2)^{1/2}, \tag{0.1}$$

where $\mathbf{P} = \mathbf{P}^{(1)} + \mathbf{P}^{(2)}$ is the cms momentum of the particles and $\mathbf{K}$ is the relative momentum. (For more detail about this see [2] and the literature cited there.)

It is well known [3,4] that Eq. (0.1) is obtained by the reduction of the direct product of two unitary irreducible representations of the group $\mathcal{P}(1, 3)$. Since an irreducible representation of $\mathcal{P}(1, n>3)$ is reducible with respect to $\mathcal{P}(1, 3)$, it is natural to consider the problem of the reduction of these representations with respect to the irreducible representations of the Poincaré group. [In fact one performs the reduction with respect to representations of the Lie algebra. We denote the Lie algebras and the groups corresponding to them by the same symbols. In [11], the reduction of reducible representations of $\mathcal{P}(1, n)$ with respect to $\mathcal{P}(1,3)$ was considered.]

Apart from these applications, the generalized groups $\mathcal{P}(1, 4)$, $\mathcal{P}(2, 3)$, etc, may also have a direct bearing on the problem of extending the S matrix off the mass shell [5] and of the description of particles with internal structure [2, 6]. In all these problems, the primary problem is that of the reduction of irreducible representations $\mathcal{P}(1, n)\to\mathcal{P}(1, 3)$.

In this paper, we perform the reduction of irreducible unitary representations of the group $\mathcal{P}(1, n)\to\mathcal{P}(1, n-k)$ for the case when the operator of the square of the "mass" satisfies $P_\mu P^\mu=P_0^2-P_k^2=\varkappa^2\geqslant 0$ $(k=1, 2,\ldots, n)$ and the energy operator satisfies $P_0^2>0$.

In Sec. 1, we give the necessary information about representations of the group $\mathcal{P}(1, 4)$ – the inhomogeneous de Sitter group – and we formulate the problem of the reduction $\mathcal{P}(1, 4)\to\mathcal{P}(1, 3)$. In Sec. 2, we find a unitary operator connecting the canonical basis of the group $\mathcal{P}(1, 4)$ to the $\mathcal{P}(1, 3)$ basis. Here we also give the reduction $\mathcal{P}(1, 4)\to\mathcal{P}(1, 3)\to\mathcal{P}(1, 2)$. Section 3 is devoted to the reduction $\mathcal{P}(1, n)\to\mathcal{P}(1, n-1)\to\ldots\to\mathcal{P}(1, n-k)$.

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1. Basic Definitions and Statement of the Problem

The group $\mathcal{P}(1, 4)$ is the most natural generalization of the Poincaré group $\mathcal{P}(1, 3)$, and we therefore consider in detail the reduction $\mathcal{P}(1, 4)\to\mathcal{P}(1, 3)$. Some of the results given for the group $\mathcal{P}(1, 4)$ can be readily transferred to the case of the group $\mathcal{P}(2, 3)$. The group $\mathcal{P}(1, 4)$ has three basic invariants (notation given without explanation is the same as in [1]) [1]:

$$P^2=P_\mu^2=P_0^2-P_a^2-P_4^2,\ V_1^2={}^1/_2\omega_{\mu\nu}^2,\quad V_2^2=-{}^1/_4 J_{\mu\nu}\omega^{\mu\nu},\quad \omega_{\mu\nu}={}^1/_2\varepsilon_{\mu\nu\alpha\beta\gamma}P^\alpha J^{\beta\gamma}. \tag{1.1}$$

The Lie algebra of $\mathcal{P}(1, 4)$ is generated by the operators P_μ and $J_{\mu\nu}$, which satisfy the commutation relations

$$[P_\mu, P_\nu]=0;\quad [P_\mu, J_{\alpha\beta}]=i(g_{\mu\alpha}P_\beta-g_{\mu\beta}P_\alpha);\quad [J_{\mu\nu}, J_{\alpha\beta}]=i(g_{\mu\beta}J_{\nu\alpha}+g_{\nu\alpha}J_{\mu\beta}-g_{\nu\beta}J_{\mu\alpha}-g_{\mu\alpha}J_{\nu\beta}). \tag{1.2}$$

The generators P_μ and $J_{\mu\nu}$ in the canonical basis $|\mathbf{p}, p_4, j_3, \tau_3; j, \tau, \varkappa\rangle$ have the form

$$P_0=E=\sqrt{p_a^2+p_4^2+\varkappa^2},\quad P_k=p_k,\quad J_{ab}=ip_b\frac{\partial}{\partial p_a}-ip_a\frac{\partial}{\partial p_b}+S_{ab},\quad a, b=1, 2, 3;$$
$$J_{0a}=-ip_0\frac{\partial}{\partial p_a}-\frac{S_{ab}P_b+S_{a4}P_4}{E+\varkappa}\quad J_{4a}=ip_a\frac{\partial}{\partial p_4}-ip_4\frac{\partial}{\partial p_a}+S_{4a},\quad J_{04}=-ip_0\frac{\partial}{\partial p_4}-\frac{S_{4b}P_b}{E+\varkappa} \tag{1.3}$$

where S_{kl} $(k, l = 1, 2, 3, 4)$ are the matrices of the irreducible representation $D(j, \tau)$ of the Lie algebra of $SO(4)\sim SU(2)\otimes SU(2)$. The numbers $\varkappa$, j, τ characterize the irreducible representations of the class I ($P_\mu^2 > 0$) of $\mathcal{P}(1, 4)$. In the space H of the irreducible representation of $\mathcal{P}(1, 4)$ the operators

$$J_a^2=\frac{V_1}{4\varkappa^2}+\frac{\varepsilon V_2}{2\varkappa}=j(j+1)I,\quad T_a^2=\frac{V_1}{4\varkappa^2}-\frac{\varepsilon V_2}{2\varkappa}=\tau(\tau+1)I,\quad P_\mu^2=\varkappa^2 I,\quad \varepsilon=\frac{P_0}{|P_0|} \tag{1.4}$$

are multiples of the identity operator. The matrices J_a and T_a can be expressed in terms of the matrices S_{kl} as follows:

$$J_a={}^1/_2(\varepsilon_{abc}S_{bc}+S_{4a}),\quad T_a={}^1/_2(\varepsilon_{abc}S_{bc}-S_{4a}). \tag{1.5}$$

The operators (1.3) are defined on the Gårding space $D\subset H$ (see the Appendix).

The basis vectors $|\mathbf{p}, p_4, j_3, \tau_3; j, \tau, \varkappa\rangle$ * are normalized in accordance with

$$\langle\mathbf{p}, p_4, j_3, \tau_3; j, \tau, \varkappa|\mathbf{p}', p_4', j_3', \tau_3'; j, \tau, \varkappa\rangle=2p_0\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta(p_4-p_4')\delta_{\tau_3\tau_3'}\delta_{j_3j_3'},$$

and the scalar product has the form

$$(\Psi_1, \Psi_2)=\int\frac{d^4p}{2p_0}\Psi_1^+(p_k, j_3, \tau_3)\Psi_2(p_k, j_3, \tau_3).$$

We shall call the basis of the irreducible representation of $\mathcal{P}(1, 4)$, in which the operators of the square of the mass, $M^2 = P_0^2 - P_a^2$, and the spin, $W^2 = W_0^2 - W_a^2$, and also the operators P_a and S_3 are diagonal the Poincaré basis and denote it by $|\mathbf{p}, m, s, s_3; j, \tau, \varkappa\rangle$.

We normalize the basis vectors in accordance with

$$\langle\mathbf{p}, m, s, s_3; j, \tau, \varkappa|\mathbf{p}', m', s', s_3'; j, \tau, \varkappa\rangle=2p_0\delta(m-m')\delta^3(\mathbf{p}-\mathbf{p}')\delta_{ss'}\delta_{s_3s_3'}, \tag{1.6}$$

and this means that

$$(\varphi_1, \varphi_2)=\sum_s\int dm\int\frac{d^3p}{2p_0}\varphi_1^+(s, s_3, m)\varphi_2(s, s_3, m).$$

The eigenvalues of the operators M^2 and W^2 correspond to irreducible representations of the group $\mathcal{P}(1, 3)$.

Our problem is to determine the spectrum of possible values of M^2 and W^2, find the explicit form of the generators $J_{\mu\nu}$ and P_μ in the $\mathcal{P}(1, 3)$ basis, and find the unitary operator that relates the basis $|\mathbf{p}, p_4, j_3, \tau_3; j, \tau, \varkappa\rangle$ to the basis $|\mathbf{p}, m, s, s_3; j, \tau, \varkappa\rangle$.

2. The Reduction $\mathcal{P}(1,4)\to\mathcal{P}(1,3)$

1. The irreducible representation (1.3) is characterized by $\varkappa^2 > 0$ and the numbers j and τ, which specify the irreducible representation of the little group $SO(4)$. On the restriction to the subgroup

* We shall say that $|p, p_4, j_3, \tau_3; j, \tau, x\rangle$ is the canonical basis.

$\mathscr{P}(1, 3)$, the space H decomposes into the direct sum of $\mathscr{P}(1, 3)$-invariant subspaces H_{p_4} (one for each value of p_4). The subspaces H_{p_4} are irreducible with respect $\mathscr{P}(1, 3)$ if and only if the representations of the little group $\mathscr{P}(1, 3)$ are irreducible. The intersection of the groups $SO(4)$ and $\mathscr{P}(1, 3)$ is the little group in $\mathscr{P}(1, 3)$ corresponding to the orbit $p_0^2 - p_a^2 = p_4^2 + \varkappa^2$, and this is the group $SO(3)$. It therefore follows that the space H is decomposed into subspaces corresponding to unitary irreducible representations of the subgroup $\mathscr{P}(1, 3)$, with the following values of the mass m and spin s:

$$\varkappa^2 \leqslant m^2 < \infty, \quad |j-\tau| \leqslant s \leqslant j+\tau. \tag{2.1}$$

The operator V_4 relating the canonical basis to the $\mathscr{P}(1, 3)$ basis is a certain matrix (which depends on the variables $\mathbf{p}$ and p_4) defined in the space of the irreducible representation of $\mathscr{P}(1, 4)$ of dimension $(2j + 1)(2\tau + 1)$, and to find its explicit form it is therefore natural to use an expansion with respect to a complete system of orthogonal projectors. We shall seek the operator V_4 in the form

$$V_4 = \sum_r \sum_l a_{rl}(\mathbf{p}, p_4) A_r B_l, \tag{2.2}$$

where

$$A_r = \prod_{r \neq r'} \frac{\frac{\mathbf{J}\cdot\mathbf{p}}{p} - r'}{r - r'}, \quad B_l = \prod_{l \neq l'} \frac{\frac{\mathbf{T}\cdot\mathbf{p}}{p} - l'}{l - l'} \quad (p = \sqrt{p_a^2}) \tag{2.3}$$

are projection operators onto the eigenspaces of the Hermitian operators $\frac{\mathbf{J}\cdot\mathbf{p}}{p}$, $\frac{\mathbf{T}\cdot\mathbf{p}}{p}$, which satisfy the conditions of orthogonality and completeness:

$$A_r A_{r'} = \delta_{rr'} A_r, \quad \sum_{r=-j}^{j} A_r = 1, \quad \frac{\mathbf{J}\cdot\mathbf{p}}{p} = \sum_{r=-j}^{j} r A_r, \quad B_l B_{l'} = \delta_{ll'} B_l, \quad \sum_{l=-\tau}^{\tau} B_l = 1, \quad \frac{\mathbf{T}\cdot\mathbf{p}}{p} = \sum_{l=-\tau}^{\tau} l B_l. \tag{2.4}$$

The inverse operator V_4^{-1} has the form

$$V_4^{-1} = \sum_r \sum_l a_{rl}^{-1}(\mathbf{p}, p_4) A_r B_l. \tag{2.5}$$

Since the generators P_0, P_a, J_{ab}, J_{0a} in the $\mathscr{P}(1, 3)$ basis have the canonical Wigner–Shirokov form, the operator V_4 must satisfy the conditions

$$V_4 P_0 V_4^{-1} = \sqrt{p_a^2 + m^2}, \quad m^2 = \varkappa^2 + p_4^2, \tag{2.6}$$

$$V_4 P_k V_4^{-1} = p_k, \tag{2.7}$$

$$V_4 J_{ab} V_4^{-1} = i p_b \frac{\partial}{\partial p_a} - i p_a \frac{\partial}{\partial p_b} + S_{ab}, \tag{2.8}$$

$$V_4 J_{0a} V_4^{-1} = -i p_0 \frac{\partial}{\partial p_a} - \frac{S_{ab} p_b}{p_0 + m} \equiv J_{0a}', \tag{2.9}$$

where $P_0, P_a, J_{ab}, J_{0a}, S_{ab}$ are from (1.3). It follows from (2.6)-(2.8) that the functions a_{rl} and a_{rl}^{-1} are scalars under three-dimensional rotations, i.e.,

$$a_{rl}(\mathbf{p}, p_4) = a_{rl}(\mathbf{p}^2, p_4), \quad a_{rl}^{-1}(\mathbf{p}, p_4) = a_{rl}^{-1}(\mathbf{p}^2, p_4). \tag{2.10}$$

Finally, the structure of the functions a_{rl} and a_{rl}^{-1} determines the relation (2.9). We write it in the form

$$[V_4^{-1}, J_{0a}] V_4 = \frac{((\mathbf{p}\times\mathbf{J})_a + (\mathbf{p}\times\mathbf{T})_a)(\varkappa - m)}{(E+m)(E+\varkappa)} + \frac{(J_a - T_a) p_4}{E + \varkappa}. \tag{2.11}$$

From Eq. (2.11), we find the conditions that the functions a_{rl} and a_{rl}^{-1} must satisfy. To calculate in explicit form the commutator on the left-hand side of Eq. (2.11), we use the relations [7]

$$\begin{aligned} \left[A_r, i\frac{\partial}{\partial p_a}\right] &= -\frac{1}{p^2}[A_r, (\mathbf{p}\times\mathbf{J})_a] = \frac{(\mathbf{p}\times\mathbf{J})_a}{2p^2}(2A_r - A_{r-1} - A_{r+1}) + \frac{i}{2p}\left(J_a - \frac{p_a}{p}\frac{\mathbf{J}\times\mathbf{p}}{p}\right)(A_{r+1} - A_{r-1}), \\ \left[B_l, i\frac{\partial}{\partial p_a}\right] &= -\frac{1}{p^2}[B_l, (\mathbf{p}\times\mathbf{T})_a] = \frac{(\mathbf{p}\times\mathbf{T})_a}{2p^2}(2B_l - B_{l-1} - B_{l+1}) + \frac{1}{2p}\left(T_a - \frac{p_a}{p}\frac{\mathbf{T}\times\mathbf{p}}{p}\right)(B_{l+1} - B_{l-1}). \end{aligned} \tag{2.12}$$

Substituting (2.2) and (2.5) into (2.11) and taking into account (2.12), we arrive at the equation

$$[V_4^{-1}, J_{0a}'] V_4 = \sum_{l,r,l',r'} \left[a_{r'l'}^{-1} A_{r'l'}, \left\{-iE\frac{\partial}{\partial p_a} - \frac{(\mathbf{p}\times\mathbf{J})_a + (\mathbf{p}\times\mathbf{T})_a}{E+m}\right\}\right] a_{rl} A_r B_l = \sum_{l,r,l',r'} \left\{ i\frac{p_a}{p}\frac{\partial a_{r'l'}^{-1}}{\partial p} A_{r'} B_{l'} a_{rl} A_r B_l - \right.$$

$$\overset{-1}{a}_{r'l'}\left[\left\{iE\frac{\partial}{\partial p_a}+\frac{(\mathbf{p}\times\mathbf{T})_a+(\mathbf{p}\times\mathbf{J})_a}{E+m}\right\},A_{r'}B_{l'}\right]\Big\}a_{rl}A_rB_l=\sum_{l',r',l,r}\left\{\frac{p_a}{p}\frac{\partial a_{rl}^{-1}}{\partial p}Ea_{rl}A_rB_l-m\overset{-1}{a}_{r'l'}a_{rl}\left(\left[A_{r'},i\frac{\partial}{\partial p_a}\right]B_{l'}+\right.\right.$$

$$\left.\left.+\left[B_{l'},i\frac{\partial}{\partial p_a}\right]A_{r'}\right)\right\}A_rB_l=\sum_{r,l}\left\{iE\frac{p_a}{p}\frac{\partial a_{rl}^{-1}}{\partial p}a_{rl}-m\left[\frac{(\mathbf{p}\times\mathbf{J})_a}{p^2}(2a_{rl}^{-1}-\overset{-1}{a}_{r+1\,l}-\overset{-1}{a}_{r-1\,l})+\frac{(\mathbf{p}\times\mathbf{T})_a}{p^2}(2a_{rl}^{-1}-\right.\right.$$

$$\left.\left.-a_{r\,l+1}-a_{r\,l-1})\right]\mp\frac{i}{2p}\left[\left(J_a-\frac{p_a}{p}\frac{\mathbf{J}\cdot\mathbf{p}}{p}\right)(\overset{-1}{a}_{r-1\,l}-\overset{-1}{a}_{r+1\,l})+\left(T_a-\frac{p_a}{p}\frac{\mathbf{T}\cdot\mathbf{p}}{p}\right)(\overset{-1}{a}_{r\,l-1}-\overset{-1}{a}_{r\,l+1})\right]\right\}A_rB_l=$$

$$=\frac{[(\mathbf{p}\times\mathbf{J})_a+(\mathbf{p}\times\mathbf{T})_a](\varkappa-m)}{(E+m)(E+\varkappa)}+\frac{(J_a-T_a)p_4}{E+m}. \tag{2.13}$$

Equating in (2.13) the coefficients of the linearly independent vectors $i\frac{p_a}{p}A_rB_l$, $T_aA_rB_l$, $J_aA_rB_l$, $(\mathbf{p}\times\mathbf{J})_aA_rB_l$, and $(\mathbf{p}\times\mathbf{T})_aA_rB_l$, we obtain

$$\frac{\partial a_{rl}^{-1}}{\partial p}a_{rl}+\frac{m}{2p}[r(\overset{-1}{a}_{r-1\,l}-\overset{-1}{a}_{r+1\,l})a_{rl}+l(\overset{-1}{a}_{r\,l-1}-\overset{-1}{a}_{r\,l+1})a_{rl}]=0,$$

$$\frac{im}{2p}(\overset{-1}{a}_{r-1\,l}-\overset{-1}{a}_{r+1\,l})a_{rl}=-\frac{p_4}{E+\varkappa},\quad\frac{im}{2p}(\overset{-1}{a}_{r\,l-1}-\overset{-1}{a}_{r\,l+1})a_{rl}=\frac{p_4}{E+\varkappa}, \tag{2.14}$$

$$\frac{m}{2p^2}(2a_{rl}^{-1}-\overset{-1}{a}_{r-1\,l}-\overset{-1}{a}_{r+1\,l})a_{rl}=\frac{m-\varkappa}{(E+m)(E+\varkappa)},\quad\frac{m}{2p}(2a_{rl}^{-1}-\overset{-1}{a}_{r\,l-1}-\overset{-1}{a}_{r\,l+1})a_{rl}=\frac{m-\varkappa}{(E+m)(E+\varkappa)}.$$

After some simple transformations, the system (2.14) is reduced to

$$E\frac{\partial a_{rl}^{-1}}{\partial p}a_{rl}+i\frac{p_4}{E+\varkappa}(r-l)=0,\quad\overset{-1}{a}_{r\pm1\,l}a_{rl}=\frac{\varkappa E+m^2\mp ipp_4}{m(E+\varkappa)}=\exp(\pm i\theta_4),$$

$$\overset{-1}{a}_{r\,l\pm1}a_{rl}=\frac{\varkappa E+m^2\pm ipp_4}{m(E+\varkappa)}=\exp(\mp\theta_4),\quad\theta_4=\operatorname{arctg}\frac{pp_4}{m^2+\varkappa E}=2\operatorname{arctg}\frac{pp_4}{(E+m)(m+\varkappa)}. \tag{2.15}$$

We show that the general solution of the system (2.15) is given by

$$a_{rl}=R_4\exp i(r-l)\theta_4, \tag{2.16}$$

where R_4 is an arbitrary function of p_4. To see this, we represent a_{rl} in the form

$$a_{rl}=B_{rl}\exp[iE(r-l)\theta_4-C_{rl}], \tag{2.17}$$

where B_{rl} and C_{rl} are functions of $\mathbf{p}^2$ and p_4, and we obtain from (2.15)

$$B_{rl}=B_{r\pm1l}=B_{rl\pm1}=B;\ C_{rl}=C_{r\pm1l}=C_{rl\pm1}=C. \tag{2.18}$$

Denoting Be^{ic} by R_4 and substituting (2.19) into (2.17), we arrive at (2.16). It is easy to see that the substitution of (2.16) into (2.17) transforms the last equation into an identity. It follows from (1.6) that $R_4=\sqrt{m/p_4}$.

Taking into account the relation

$$\frac{S_{4a}p_a}{p}=\sum(r-l)A_rB_l \tag{2.19}$$

and substituting (2.16) into (2.2), we obtain

$$V_4=\sqrt{\frac{m}{p_4}}\exp\left(i\frac{S_{4a}p_a}{p}2\operatorname{arctg}\frac{pp_4}{(E+m)(m+\varkappa)}\right). \tag{2.20}$$

Equation (2.20) is the required operator of the transformation from the canonical basis to the $\mathscr{P}(1,3)$ basis.

We now find the explicit form of the generators J_{04} and J_{4a} in the $\mathscr{P}(1,3)$ basis. Using the Hausdorff–Cambell identity

$$e^ABe^{-A}=\sum_{n=0}^{\infty}\frac{\{B,A\}^{(n)}}{n!},\quad\{B,A\}^{(n)}=[\{B,A\}^{(n-1)},A],\ \{B,A\}^{(0)}=B, \tag{2.21}$$

we obtain

$$V_4i\frac{\partial}{\partial p_a}V_4^{-1}=i\frac{\partial}{\partial p_a}-\frac{p_aS_{4b}p_bp_4}{(E+\varkappa)(E+m)Em}+\frac{p_4S_{4a}}{m(E+\varkappa)}+\frac{S_{ab}p_b(m-\varkappa)}{m(E+m)(E+\varkappa)},$$

$$V_4 S_{4a} V_4^{-1} = \frac{S_{4a}(m^2+\varkappa E)}{m(E+\varkappa)} + \frac{p_a}{p}\frac{S_{4b}p_b(m-\varkappa)}{mp(E+m)(E+\varkappa)} + \frac{S_{ab}p_b p_4}{m(E+\varkappa)}, \quad V_4 i\frac{\partial}{\partial p_4}V_4^{-1} = i\frac{\partial}{\partial p_4} - \frac{\varkappa^2}{2p_4 m^2} + \frac{\varkappa S_{4b}p_b}{Em^2} - \frac{S_{4a}p_a}{E(E+\varkappa)},$$

$$V_4 S_{ab} p_b V_4^{-1} = S_{ab}p_b + \frac{pp_4 S_{4b}p_b}{(E+\varkappa)(E+m)Em} + \frac{pp_4 S_{4a}}{m(E+\varkappa)} + \frac{S_{ab}p_b p(m-\varkappa)}{m(E+m)(E+\varkappa)}$$

Making then the change of variables $p_4 \to \varepsilon_4\sqrt{m^2-\varkappa^2}$, $\varepsilon_4=\pm 1$, we obtain the explicit form of the operators J_{04} and J_{4a} in the $\mathscr{P}(1,3)$ basis.

Thus, we have arrived at the final result.

THEOREM. The space H of a unitary irreducible representation of the group $\mathscr{P}(1,4)$ with $\varkappa^2 > 0$, $P_0 > 0$, decomposes into subspaces corresponding to unitary irreducible representations of the subgroup $\mathscr{P}(1,3)$ with the following values of the invariants M^2 and W^2: $\varkappa^2 \leqslant m^2 < \infty$, $|j-\tau| \leqslant s \leqslant j+\tau$. The operator of the transition from the basis $|\mathbf{p}, p_4, j_3, \tau_3; j, \tau, \varkappa\rangle$ to the $\mathscr{P}(1,3)$ basis is given by (2.20), and the operators $J_{\mu\nu}$ and P_μ in the $\mathscr{P}(1,3)$ basis have the form

$$P_0=\sqrt{p^2+m^2}, \quad P_a=p_a, \quad P_4=\varepsilon_4\sqrt{m^2+\varkappa^2}, \quad \varepsilon_4=\pm 1, \quad J_{ab}=ip_b\frac{\partial}{\partial p_a} - ip_a\frac{\partial}{\partial p_b} + S_{ab}, \quad J_{0a}=-ip_0\frac{\partial}{\partial p_a} - \frac{S_{ab}p_b}{E+m},$$

$$J_{04}=-iE\left\{\varepsilon_4\sqrt{1-\frac{\varkappa^2}{m^2}}, \frac{\partial}{\partial m}\right\} - \frac{\varkappa}{m}\frac{S_{4a}p_a}{m}, \tag{2.22}$$

$$J_{4a}=ip_a\left\{\varepsilon_4\sqrt{1-\frac{\varkappa^2}{m^2}}, \frac{\partial}{\partial m}\right\} - i\varepsilon m\sqrt{1-\frac{\varkappa^2}{m^2}}\frac{\partial}{\partial p_a} + \frac{\varkappa p_a S_{4b}p_b}{m^2(E+m)} + \varepsilon_4\sqrt{1-\frac{\varkappa^2}{m^2}}\frac{S_{ab}p_b}{E+m} + \frac{\varkappa}{m}S_{4a},$$

where $\{A, B\} = AB+BA$.

Remark. If we set $\varkappa = 0$ ($p_4 \neq 0$) in (2.22), the operators $J_{\mu\nu}$ and P_μ take the form [1]

$$P_0=\sqrt{p^2+m^2}, \quad m^2=p_4^2, \quad J_{ab}=ip_b\frac{\partial}{\partial p_a} - ip_a\frac{\partial}{\partial p_b} + S_{ab}, \quad J_{04}=-iE\left\{\varepsilon_4\sqrt{1-\frac{\varkappa^2}{m^2}}, \frac{\partial}{\partial m}\right\},$$

$$J_{4a}=\frac{i}{2}p_a\left\{\varepsilon_4\sqrt{1-\frac{\varkappa^2}{m^2}}, \frac{\partial}{\partial m}\right\} - i\varepsilon_4\sqrt{1-\frac{\varkappa^2}{m^2}}\frac{\partial}{\partial p_a} - \varepsilon_4\frac{S_{ab}p_b}{E+m}.$$

2. In the case $\varkappa^2 < 0$ the generators of the canonical irreducible representation of the group $\mathscr{P}(1,4)$ are

$$P_0=\sqrt{p_k^2-\eta^2}, \quad P_k=p_k, \quad P_\mu P^\mu=-\eta^2, \quad J_{ab}=ip_b\frac{\partial}{\partial p_a} - ip_a\frac{\partial}{\partial p_b} + S_{ab}, \quad J_{0a}=-ip_0\frac{\partial}{\partial p_a} + S_{0a},$$

$$J_{a4}=ip_4\frac{\partial}{\partial p_a} - ip_a\frac{\partial}{\partial p_4} - \frac{S_{ab}p_b - S_{a0}P_0}{p_4+\eta}, \quad J_{04}=-ip_0\frac{\partial}{\partial p_a} - \frac{S_{0a}p_a}{p_4+\eta}$$

where $S_{\mu\nu}$ are the generators of the irreducible representation of the group $SO_0(1, 3)$. By means of the isometric transformation

$$V=\exp\left(-i\frac{S_{0a}p_a}{p}\operatorname{arcth}\frac{p}{E}\right)$$

and the subsequent change of variable $p_4 \to \varepsilon_4\sqrt{m^2-\varkappa^2}$, we obtain

$$P_0=\sqrt{p_a^2+m^2}, \quad P_a=p_a, \quad P_4=\varepsilon_4\sqrt{m^2+\eta^2},$$

$$J_{ab}=ip_b\frac{\partial}{\partial p_a} - ip_a\frac{\partial}{\partial p_b} + S_{ab}, \quad J_{0a}=-ip_0\frac{\partial}{\partial p_a} - \frac{S_{ab}p_b}{E+m}, \quad J_{04}=-ip_0\frac{\partial}{\partial p_4} + \frac{\eta}{m}\frac{S_{0a}p_a}{m},$$

$$J_{a4}=\frac{i}{2}p_a\left\{\sqrt{1+\frac{\eta^2}{m^2}}, \frac{\partial}{\partial m}\right\} - i\varepsilon_4\sqrt{m^2+\eta^2}\frac{\partial}{\partial p_a} + \frac{\eta p_a S_{ab}p_b}{m^2(E+m)} + \frac{\eta}{m}S_{0a} + \varepsilon_4\sqrt{1-\frac{\eta^2}{m^2}}\frac{S_{ab}p_b}{E+m}.$$

If $p_4^2 > \eta^2$, these equations define the representation of the group $\mathscr{P}(1, 4)$ in the $\mathscr{P}(1,3)$ basis.

3. In some physical problems in which the $\mathscr{P}(1, 3)$ symmetry is broken but there is still symmetry under the subgroup $\mathscr{P}(1, 2)$, it is convenient to use the $\mathscr{P}(1, 2)$ basis. In connection with this, it is of interest to continue the reduction with respect to the subgroup $\mathscr{P}(1, 2)$. This means transition to a basis in which the generators P_0, P_a, $J_{\alpha\beta}$ ($\alpha, \beta = 1, 2$) have the canonical form

$$P_0=E=\sqrt{p_a^2+m_3^2},\quad P_a=p_a,\quad m_3^2=m^2+p_3^2,\quad J_{12}=ip_2\frac{\partial}{\partial p_1}-ip_1\frac{\partial}{\partial p_2}+S_{12},\qquad J_{0a}=-ip_0\frac{\partial}{\partial p_a}-\frac{S_{a\beta}p_\beta}{E+m_3}.$$

Let us find the form of the remaining generators of the group $\mathscr{P}(1, 4)$. For this, it is sufficient to determine an operator V_3 satisfying the conditions

$$V_3P_0V_3^{-1}=E,\; V_3P_aV_3^{-1}=p_a, \tag{2.23}$$

$$V_3J_{0a}V_3^{-1}=-ip_0\frac{\partial}{\partial p_a}-\frac{S_{a\beta}p_\beta}{E+m}, \tag{2.24}$$

where the operators $P_0, P_a, J_{0a}, S_{a\beta}$ are defined in the $\mathscr{P}(1,3)$ basis.

We represent V_3 in the form

$$V_3=R_3\exp\left(i\frac{S_{3a}p_a}{|p|_3}\theta_3\right),\qquad |p|_3=\sqrt{p_1^2+p_2^2}, \tag{2.25}$$

where R_3 and θ_3 are certain functions of p_3, p_4 and $|p|_3$, p_3, p_4, respectively. To determine these functions, we substitute (2.25) into (2.24). Then

$$\left[V_3^{-1}, -ip_0\frac{\partial}{\partial p_a}-\frac{S_{a\beta}p_\beta}{E+m_3}\right]V_3=-\frac{S_{a3}p_3}{E+m}+\frac{(m-m_3)S_{a\beta}p_\beta}{(E+m_3)(E+m)}. \tag{2.26}$$

Using (2.21), we obtain

$$\left[V_3^{-1}, -ip_0\frac{\partial}{\partial p_a}-\frac{S_{a\beta}p_\beta}{E+m_3}\right]V_3=\frac{p_a}{|p|_3}\frac{\partial\theta_3}{\partial|p|_3}E\frac{S_{3a}p_a}{|p|_3}-m_3\frac{S_{a\beta}p_\beta}{|p|_3}(1-\cos\theta_3)+\frac{m_3}{|p|_3}\left(S_{3a}-\frac{p_a}{|p|_3}\frac{S_{3\beta}p_\beta}{|p|_3}\right)\sin\theta_3, \tag{2.27}$$

whence

$$\theta_3=2\,\mathrm{arctg}\frac{|p|p_3}{(E+m_3)(m_3+m)}. \tag{2.28}$$

We choose the factor R_3 in the form $R_3=\sqrt{m_3/p_3}$, and then the scalar product in the $\mathscr{P}(1, 2)$ basis has the form

$$(\varphi_1,\varphi_2)=\int_\varkappa^\infty dm\int_m^\infty dm_3\int\frac{d^2p}{2E}\varphi_1^+\varphi_2.$$

Now, using (2.24) and (2.27), we can find the action of the generators J_{03}, J_{04}, J_{34} of the group $\mathscr{P}(1, 4)$ in the $\mathscr{P}(1, 2)$ basis. We have

$$J_{03}=-\frac{i}{2}E\left\{\varepsilon_3\sqrt{1-\left(\frac{m}{m_3}\right)^2},\frac{\partial}{\partial m_3}\right\}-\frac{m}{m_3}\frac{S_{3a}p_a}{m_3},\quad \varepsilon_3=p_3/|p_3|,$$

$$J_{43}=-\frac{im}{2}\left\{\varepsilon_3\varepsilon_4\sqrt{\left[1-\left(\frac{m}{m_3}\right)^2\right]\left[1-\left(\frac{\varkappa}{m}\right)^2\right]},\frac{\partial}{\partial m}\right\}+\frac{\varkappa m_3}{m^2}S_{43},$$

$$J_{04}=-\frac{i}{2}E\left[\left\{\varepsilon_3\sqrt{1-\left(\frac{m}{m_3}\right)^2},\frac{\partial}{\partial m_3}\right\}+\left\{\varepsilon_4\sqrt{1-\left\{\frac{\varkappa}{m}\right\}^2},\frac{\partial}{\partial m}\right\}\right]-\frac{\varkappa S_{4a}p_a}{mm_3}+$$
$$+\varepsilon_3\varepsilon_4\sqrt{\left[1-\left(\frac{m}{m_3}\right)^2\right]\left[1-\left(\frac{\varkappa}{m}\right)^2\right]}\frac{S_{3a}p_a}{m_3}-\varkappa\varepsilon_3\sqrt{1-\left(\frac{m}{m_3}\right)^2}E\frac{S_{43}}{m^2}.$$

3. The Reduction $\mathscr{P}(1,n)\to\mathscr{P}(1,n-1)\to\ldots\to\mathscr{P}(1,n-k)$

1. We show first how a representation of the algebra $\mathscr{P}(1,n)$ can be specified in the $\mathscr{P}(1,n-1)$ basis. The canonical irreducible representation of the generators of the group $\mathscr{P}(1,n)$ is given by

$$P_0=E=\sqrt{p_k^2+\varkappa^2},\quad P_k=p_k,\; k=1,2,\ldots,n,\; J_{ab}=ip_b\frac{\partial}{\partial p_a}-ip_a\frac{\partial}{\partial p_b}+S_{ab},\qquad a,b<n, \tag{3.1}$$

$$J_{0a}=-ip_0\frac{\partial}{\partial p_a}-\frac{S_{ab}p_b}{P_0+\varkappa}-\frac{S_{an}p_n}{P_0+\varkappa},$$

$$J_{0n}=-ip_0\frac{\partial}{\partial p_n}-\frac{S_{na}p_a}{p_0+\varkappa},\qquad J_{an}=ip_n\frac{\partial}{\partial p_a}-ip_a\frac{\partial}{\partial p_n}+S_{an}, \tag{3.2}$$

where S_{kl} are the matrices of the irreducible representation $D(m_1, m_2,\ldots, m_{[n/2]})$ of the algebra $SO(n)$, and m_i are the Gel'fand–Tsetlin numbers. The operators (3.1) are Hermitian with respect to the scalar product

$$(\Psi_1, \Psi_2) = \int \frac{d^n p}{2p_0} \Psi_1^+ \Psi_2. \tag{3.3}$$

In the $\mathcal{P}(1, n-1)$ basis, the generators (3.1) have by definition the form of a direct sum of the generators of the canonical representations of the group $\mathcal{P}(1, n-1)$. If the representation of the algebra $SO(n)$ is specified in the basis $SO(n) \supset SO(n-1) \supset \ldots$, then these generators have the form

$$P_0 = E = \sqrt{P_a^2 + m_n^2},\ m_n^2 = \varkappa^2 + p_n^2,\ P_a = p_a,\ J_{ab} = ip_b \frac{\partial}{\partial p_a} - ip_a \frac{\partial}{\partial p_b} + S_{ab},\quad J_{0a} = -ip_0 \frac{\partial}{\partial p_a} - \frac{S_{ab} p_b}{E + m_n}. \tag{3.4}$$

The problem of finding the explicit form of the generators P_μ and $J_{\mu\nu}$ in the $\mathcal{P}(1, n-1)$ basis reduces to finding an isometric operator that transforms the generator (3.1) to the form (3.4).

By analogy with Sec. 2, we shall seek the transformation operator in the form

$$V_n = R_n \exp\left(i \frac{S_{na} p_a}{|p|_n} \theta_n\right),\quad |p|_n = \left(\sum_{a<n} p_a^2\right)^{1/2}, \tag{3.5}$$

where R_n and θ_n are certain functions of p_n and p_n, $|p|_n$, respectively, that are to be found.

The operator V_n transforms (3.1) to (3.4) if

$$[V_n^{-1}, J_{0a}] V_n = \frac{S_{ab} p_b (\varkappa - m_n)}{(E + m_n)(E + \varkappa)} + \frac{S_{na} p_n}{E + \varkappa}. \tag{3.6}$$

Substituting (3.4) and (3.5) into (3.6) and using the identities

$$V_n^{-1} i \frac{\partial}{\partial p_a} V_n = i \frac{\partial}{\partial p_a} - \frac{p_a}{|p|_n} \frac{\partial \theta n}{\partial p} \frac{S_{nb} p_b}{|p|_n} + \frac{S_{ab} p_b}{|p|_n^2} (1 - \cos\theta_n) - \frac{1}{|p|_n}\left(S_{na} - \frac{p_a}{|p|_n} \frac{S_{nb} p_b}{|p|_n}\right) \sin\theta_n,$$

$$V_n S_{ab} p_b V_n^{-1} = p^2 (V_n x_a V_n^{-1} - x_a),$$

we arrive at the equation

$$\frac{p_a}{|p|_n} E \frac{\partial \theta_n}{\partial |p|_n} \frac{S_{nb} p_b}{|p|_n} - m\left[\frac{S_{ab} p_b}{|p|_n^2}(1 - \cos\theta_n) - \frac{1}{|p|_n}\left(S_{na} - \frac{p_a S_{nb} p_b}{|p|_n^2}\right)\right] = \frac{S_{ab} p_b (\varkappa - m_n)}{(E + m_n)(E + \varkappa)} + \frac{S_{na} p_n}{E + \varkappa}.$$

Equating the coefficients of the linearly independent vectors $\frac{p_a}{|p|_n} \frac{S_{nb} p_b}{|p|_n}, \frac{S_{ab} p_b}{|p|_n}$ and S_{na}, we obtain a system of equations for the required functions θ_n:

$$E \frac{\partial \theta_n}{\partial |p|_n} - \frac{m_n}{|p|_n} \sin\theta_n = 0,\quad m_n \sin\theta_n = \frac{|p|_n p_n}{E + \varkappa},\quad m_n(\cos\theta_n - 1) = \frac{p^2 (\varkappa - m_n)}{(E + m_n)(E + \varkappa)}. \tag{3.7}$$

The solution of the system (3.7) is given by

$$\theta_n = 2 \operatorname{arctg} \frac{|p|_n p_n}{(E + m_n)(m_n + \varkappa)}. \tag{3.8}$$

From the normalization condition of the basis vectors we find that the factor $R_n = \sqrt{m_n/p_n}$.

Now, using the explicit form of V_n, we can readily find expressions for the generators J_{0n} and J_{an} in the $\mathcal{P}(1, n-1)$ basis. Taking into account the identities

$$V_n i \frac{\partial}{\partial p_a} V_n^{-1} = i \frac{\partial}{\partial p_a} - \frac{p_a p_n S_{ab} p_b}{E m_n (E + m_n)(E + \varkappa)} + \frac{p_n S_{na}}{m_n (E + \varkappa)} + \frac{S_{ab} p_b (m_n - \varkappa)}{m_n (E + m_n)(E + \varkappa)},$$

$$V_n S_{na} V_n^{-1} = S_{na} \frac{m_n^2 + \varkappa E}{m_n (E + \varkappa)} + \frac{p_n S_{nb} p_b (m_n - \varkappa)}{m_n (E + m_n)(E + \varkappa)} + \frac{p_n S_{ab} p_b}{m_n (E + \varkappa)},$$

$$V_n i \frac{\partial}{\partial p_n} V_n^{-1} = i \frac{\partial}{\partial p_n} + S_{na} p_a \left(\frac{\varkappa}{E m_n^2} - \frac{1}{E(E + \varkappa)} - \frac{i\varkappa^2}{2 p_n m_n^2}\right)$$

and making the change of variables $p_n \to \varepsilon_n \sqrt{m_n^2 - \varkappa^2}$, we obtain

$$J_{na} = \frac{ip_a}{2}\left\{\varepsilon_n \sqrt{1 - \left(\frac{\varkappa}{m_n}\right)^2}, \frac{\partial}{\partial m_n}\right\} - i\varepsilon_n \sqrt{1 - \left(\frac{\varkappa}{m_n}\right)^2} \frac{\partial}{\partial p_a} +$$

$$+ \frac{p_a \varkappa S_{nb} p_b}{m_n^2 (E + m_n)} + \varepsilon_n \sqrt{1 - \left(\frac{\varkappa}{m_n}\right)^2} \frac{S_{ab} p_b}{(E + m_n)} + \frac{\varkappa}{m_n} S_{na},\quad \varepsilon_n = p_n / |p_n|, \tag{3.9}$$

$$J_{0n}=-\frac{i}{2}E\left\{\varepsilon_n\sqrt{1-\left(\frac{\varkappa}{m_n}\right)^2},\frac{\partial}{\partial m_n}\right\}-\frac{\varkappa}{m_n}\frac{S_{na}p_a}{m_n}.$$

Thus, we have found that the explicit form of the generators of the group $\mathscr{P}(1,n)$ in the $\mathscr{P}(1,n-1)$ basis is given by (3.4) and (3.9). The generators (3.4) and (3.9) are Hermitian with respect to the scalar product

$$(\varphi_1,\varphi_2)=\int\limits_{\varkappa}^{\infty}dm_n\sum_{\eta}\frac{d^{n-1}p}{2E}\varphi_1^+(\eta,m)\varphi_2(\eta,m),$$

where η is the set of numbers that characterize the irreducible representations of the group $SO(n-1)$ contained in the representation $D(m_1,m_2,\ldots,m_{\left[\frac{n}{2}\right]})$.

2. We now obtain a representation of the algebra $\mathscr{P}(1,n)$ in the $\mathscr{P}(1,n-2)$ basis. Using the above results, we conclude that the operator

$$V_{n-1}=\sqrt{\frac{m_{n-1}}{p_{n-1}}}\exp\left(2i\frac{S_{n-1\,a}p_a}{|p|_{n-1}}\operatorname{arctg}\frac{p_{n-1}|p|_{n-1}}{(E+m_{n-1})(m_n+m_{n-1})}\right), \tag{3.10}$$

where $m_{n-1}=(\varkappa^2+p_n^2+p_{n-1}^2)^{1/2}$, $|p|_{n-1}=\left(\sum\limits_{a<n-1}p_a^2\right)^{1/2}$, transforms the generators (3.4) to the form

$$P_0=E=\sqrt{p_a^2+m_{n-1}^2},\quad P_a=p_a,\quad a\leqslant n-1,\ P_n=\varepsilon_n\sqrt{m_n^2-\varkappa^2},\quad P_{n-1}=\varepsilon_{n-1}\sqrt{m_{n-1}^2-m_n^2},$$

$$J_{ab}=ip_b\frac{\partial}{\partial p_a}-ip_a\frac{\partial}{\partial p_b}+S_{ab},\quad J_{0a}=-ip_0\frac{\partial}{\partial p_a}-\frac{S_{ab}p_b}{E+m_{n-1}},\quad J_{0\,n-1}=-\frac{i}{2}E\left\{\sqrt{1-\left(\frac{m_n}{m_{n-1}}\right)^2},\frac{\partial}{\partial m_{n-1}}\right\}-\frac{m_n}{m_{n-1}}\frac{S_{n-1\,a}p_a}{m_{n-1}}. \tag{3.11}$$

To specify the form of the remaining generators in the $\mathscr{P}(1,n-1)$ basis, it is sufficient to find the generator $J_{n,n-1}$ [the others can be determined from the commutation relations (1.2)]. Using the identities

$$V_{n-1}^{-1}i\frac{\partial}{\partial p_n}V_{n-1}=i\frac{\partial}{\partial p_n}-\frac{p_np_{n-1}S_{n-1\,a}p_a}{m_nm_{n-1}^2E}-\frac{i}{2}\frac{p_n}{m_{n-1}^2},$$

$$V_{n-1}^{-1}i\frac{\partial}{\partial p_{n-1}}V_{n-1}=i\frac{\partial}{\partial p_{n-1}}+S_{n-1\,a}p_a\left(\frac{m_n}{Em_{n-1}^2}-\frac{1}{E(E+m_n)}\right)+\frac{im_n^2}{p_{n-1}m_{n-1}^2},$$

$$V_{n-1}^{-1}S_{n\,n-1}V_{n-1}=S_{n\,n-1}\frac{m_{n-1}^2+Em}{m_{n-1}(E+m_n)}+\frac{S_{na}p_ap_{n-1}}{m_{n-1}(E+m_n)}$$

we obtain

$$J_{n\,n-1}=\frac{i}{2}\left\{\sqrt{\left(1-\frac{\varkappa}{m_n}\right)\left(1-\frac{m_n}{m_{n-1}}\right)}m_{n-1},\frac{\partial}{\partial m_n}\right\}+\frac{\varkappa m_{n-1}}{m_n^2}S_{n\,n-1}. \tag{3.12}$$

The generators (3.11) and (3.12) are Hermitian with respect to the scalar product

$$(\varphi_1,\varphi_2)=\int\limits_{\varkappa}^{\infty}dm_n\int\limits_{m_n}^{\infty}dm_{n-1}\sum_{\alpha}\int\frac{d^{n-2}p}{p_0}\varphi_1^+(m_{n-1},\alpha)\varphi_2(m_{n-1},\alpha),$$

where α denotes the numbers that label the irreducible representations of the algebra $SO(n-2)$ contained in the representation $D(m_1,m_2,\ldots,m_{\left[\frac{n}{2}\right]})$ of the group $SO(n)$.

3. Similarly, a representation of the algebra $\mathscr{P}(1,n)$ in the $\mathscr{P}(1,n-3)$ basis can be determined. Subjecting the generators (3.11) and (3.12) to the transformation

$$V_{n-2}=\sqrt{\frac{m_{n-2}}{p_{n-2}}}\exp\left(2i\frac{S_{n-2a}p_a}{|p|_{n-2}}\operatorname{arctg}\frac{|p|_{n-2}p_{n-2}}{(E+m_{n-2})(m_{n-2}+m_{n-1})}\right),$$

$$|p|_{n-2}=\left(\sum_{a<n-2}p_a^2\right)^{1/2},\quad m_{n-2}=(\varkappa^2+p_n^2+p_{n-1}^2-p_{n-2}^2)^{1/2}$$

and remembering that (3.12) and (3.2) commute, we obtain

$$P_0=E,\quad P_a=p_a,\quad P_n=\varepsilon_n\sqrt{m_n^2-\varkappa^2},\quad P_{n-1}=\varepsilon_{n-1}\sqrt{m_{n-1}^2-m_n^2},$$

$$J_{ab}=i\frac{\partial}{\partial p_a}p_b-i\frac{\partial}{\partial p_b}p_a+S_{ab};\quad a,b<n-2,\quad P_{n-2}=\varepsilon_{n-2}\sqrt{m_{n-2}^2-m_{n-1}^2},$$

$$J_{0\,n-2}=-\frac{i}{2}E\left\{\frac{P_{n-2}}{m_{n-2}},\frac{\partial}{\partial m_{n-2}}\right\}-\frac{m_{n-1}}{m_{n-2}}\frac{S_{n-2\,a}p_a}{m_{n-2}},\quad J_{n-1\,n-2}=\frac{i}{2}\left\{\frac{P_{n-1}P_{n-2}}{m_{n-1}},\frac{\partial}{\partial m_n}\right\}+\frac{m_nm_{n-2}}{m_{n-1}^2}S_{n\,n-2}, \tag{3.13}$$

$$J_{n\,n-1}=\frac{i}{2}\left\{\frac{P_nP_{n-1}}{m_n},\frac{\partial}{\partial m_{n-1}}\right\}+\frac{\varkappa m_{n-1}}{m_n^2}S_{n\,n-1}.$$

4. Subjecting the generators (3.13) successively to the transformations

$$V_{n-l}=\sqrt{\frac{m_{n-l}}{p_{n-l}}}\exp\left(2i\frac{S_{n-l\,a}p_a}{|p|_{n-l}}\operatorname{arctg}\frac{|p|_{n-l}p_{n-l}}{(E+m_{n-l})(m_{n-l}+m_{n-l+1})}\right),$$

where

$$|p|_{n-l}=\Big(\sum_{a<n-l}p_a^2\Big)^{1/2},\quad m_{n-l}=\Big(\varkappa^2+\sum_{\alpha=1}^{l}p_{n-\alpha}^2\Big)^{1/2},\quad l=3,4,\ldots,$$

and using the results of § 1-3, we obtain

$$P_0=E,\quad P_a=p_a,\quad P_{n-\alpha}=\varepsilon_{n-\alpha}\sqrt{m_{n-\alpha}^2-m_{n-\alpha+1}^2},\quad \alpha<k,$$

$$J_{0a}=-ip_0\frac{\partial}{\partial p_a}-\frac{S_{ab}p_b}{E+m_{n-k+1}},\quad a,b\leqslant n-k,\ J_{ab}=i\frac{\partial}{\partial p_a}p_b-i\frac{\partial}{\partial p_b}p_a+S_{ab},\tag{3.14}$$

$$J_{0\,n-\alpha}=-\frac{i}{2}E\left\{\frac{P_{n-\alpha}}{m_{n-\alpha}},\frac{\partial}{\partial m_{n-\alpha}}\right\}-\frac{m_{n-\alpha+1}}{m_{n-\alpha}}\frac{S_{n-\alpha\,a}p_a}{m_{n-\alpha}}\quad J_{n-\alpha,n-\alpha+1}=\frac{i}{2}\left\{\frac{P_{n-\alpha}P_{n-\alpha+1}}{m_{n-\alpha}},\frac{\partial}{\partial m_n}\right\}+\frac{\varkappa m_{n-\alpha+1}}{m_{n-\alpha}^2}S_{n-\alpha\,n-\alpha+1},$$

The generators (3.14) are Hermitian with respect to the scalar product

$$(\varphi_1,\varphi_2)=\int dm_{n-k}\int dm_{n-k+1}\ldots\int dm_n\sum_{\lambda}\frac{d^{n-l}p}{p_0}\varphi_1^+\varphi_2,$$

where λ is the set of numbers that characterize the representations of the group $SO(n-k)$ in the representation $D(m_1,m_2,\ldots,m_{[\frac{n}{2}]})$ of the group $SO(n)$.

Thus, the generators of the group $\mathscr{P}(1,n)$ in the $\mathscr{P}(1,n-k)$ basis have the form (3.14). The operator of the transformation from (3.1) to (3.14) is given by $V=\prod_{l=1}^{k}V_{n-l}$.

Appendix

In D one can introduce a topology with the countable system of norms

$$(\varphi_1,\varphi_2)_n=(\varphi_1,(\Delta+1)^n\varphi_2),\qquad \Delta=\sum_{\mu}p_\mu^2+\frac{1}{2}\sum_{\mu,\nu}J_{\mu\nu}^2,$$

where (.,.) is the scalar product in the space H with respect to which $J_{\mu\nu}$ and P_μ are Hermitian, in such a way that, completing D with respect to this norm, we obtain a space Ψ which has the following remarkable properties: 1) Ψ is dense in H; 2) the enveloping algebra $E(\mathscr{P}(1,4))$ is the algebra of continuous (with respect to the topology of Ψ) operators on Ψ; 3) Ψ is nuclear.

We give the proof that Ψ is nuclear. Using the results of [8] and the fact that the group $\mathscr{P}(1,4)$ can be obtained by contracting the group SO_0 (1,5) in the Inönü–Wigner sense [9], it is sufficient to show that there exists an operator X belonging to $E(SO_0(1,5))$ for which $X^* = X^{**}$ and X^{-1} is nuclear.

Consider the operator $A = (C + 1)^n$, where C is the Casimir operator of the group SO_0 (1,5) of second order. It follows from Nelson's theorem [10] that C and C^n are essentially self-adjoint, and therefore $A^* = A^{**}$.

We show further that A^{-1} is a Hilbert–Schmidt operator. Obviously

$$A^{-1}=\sum_i\frac{1}{(c_i-1)^n}P_i,$$

where P_i are projectors onto the subspaces H_i (H_i is the eigenspace of the Casimir operator C with eigenvalue c_i). In addition, it is easy to show that for sufficiently large n

$$\sum\left(\frac{1}{(c_i+1)^n}\dim H_i\right)^2<\infty.$$

Thus, A^{-1} is a Hilbert–Schmidt operator. Since the square of a Hilbert–Schmidt operator is always essentially self-adjoint, we can take X to be the operator $(A^{-1})^2$. Thus, Ψ is nuclear. It follows from the properties 1–3 that in our case the nuclear spectral theorem applies and the vectors $|\mathbf{p}, p_4, j_3, \tau_3; j, \tau, \varkappa\rangle$ of the canonical basis belong to the space Ψ^* ($\Psi \subset H \subset \Psi^*$).

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