23. A. G. Savinkov, Pis'ma Zh. Eksp. Teor. Fiz., 47, 13 (1988).
24. R. M. MacLean, Homological Algebra [Russian translation], Mir, Moscow (1966).
25. L. Crane, Commun. Math. Phys., 110, 391 (1987).
26. S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. 1, Interscience, New York (1963).
27. A. S. Goldhaber, Phys. Rev. Lett., 36, 1122 (1976).
28. S. Coleman, The Magnetic Monopole Fifty Years Later, Preprint, Harvard University, Cambridge (1981).
29. A. G. Savinkov and I. S. Shapiro, Pis'ma Zh. Eksp. Teor. Fiz., 47, 292 (1988).
30. J. L. Friedman and R. D. Sorkin, Phys. Rev. D, 20, 2511 (1979).

# NON-LIE INTEGRALS OF THE MOTION FOR PARTICLES OF ARBITRARY SPIN AND FOR SYSTEMS OF INTERACTING PARTICLES

A. G. Nikitin and V. I. Fushchich

New integrals of the motion are found for the Kemmer–Duffin–Petiau, Rarita–Schwinger, Dirac–Fierz–Pauli, and Bhabha equations describing minimal and anomalous coupling of particles of spin $s \leqq 2$ with the field of a point charge and also for a number of relativistic and quasirelativistic two- and three-particle equations. These integrals belong to the class of differential operators of order 2s with matrix coefficients and have a discrete spectrum.

It is well known that for many equations of quantum theory describing the motion of a charged particle in external fields there exist integrals of the motion that are not directly related to the geometrical symmetry of the considered system. In the case of a nonrelativistic spinless particle in the Coulomb field, there is the Runge–Lenz vector, while for the relativistic electron in the Coulomb field there are the Dirac integral [1] and Johnson–Lippmann integral [2].

These integrals of the motion make it possible to explain the degeneracy of the energy spectra of the corresponding physical objects, and the Dirac integral greatly simplifies the solution of the equation of motion by the separation of variables, giving rise to a decoupling of the equations for the radial functions into uncoupled subsystems.

The aim of the present paper is to describe additional integrals of the motion for a charged particle with spin $s \leqq 2$ in the Coulomb field, and also for systems of interacting particles. We shall see that such integrals of the motion exist for all relativistic wave equations invariant with respect to spatial inversion and for a large class of two-particle equations with spherically symmetric potential.

We obtain below new integrals of the motion for the Kemmer–Duffin, Stueckelberg, Rarita–Schwinger, Dirac–Fierz–Pauli, and Bhabha equations describing the interaction of particles of spin $s \leqq 2$ with the field of a point charge, and we give an algorithm for constructing such integrals for particles of arbitrary spin. These integrals are differential operators of order 2s with matrix coefficients and can be regarded as generalizations of the Dirac integral for the case of arbitrary s.

We find new integrals of the motion for an entire class of two-particle equations — those of Breit [3], Barut and Komy [4], Krolikowski [5], the generalized Breit equation for bound quark states [6,7], and other equations. An additional integral of the motion is also obtained for the three-particle equation of Krolokowski [8].

It must be emphasized that, in principle, the additional integrals of the motion cannot be found in the framework of the classical Lie group analysis of differential equations (for a modern exposition of the basic propositions and applications of such analysis, see [9–11]). We proceed from the generalized non-Lie approach proposed and developed in [12–14].

Institute of Mathematics, Ukrainian SSR Academy of Sciences. Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 88, No. 3, pp. 406-415, September, 1991. Original article submitted February 27, 1990.

 

## 1. Dirac Integral for the Electrons

As was first noted by Dirac [1], the Hamiltonian of a particle with spin 1/2 and charge e in the field of a point charge qe,

$$H=\gamma_0\gamma_a p_a+\gamma_0 m+V, \quad p_a=-i\frac{\partial}{\partial x_a}, \quad a=1,2,3, \tag{1.1}$$

where $\gamma_0$, $\gamma_a$ are the Dirac matrices, $V = qe^2/x$, $x = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$, commutes with the operator

$$Q=\gamma_0(2S_a J_a-{}^1/_2)\equiv\gamma_0(2\mathbf{S}\cdot\mathbf{J}-{}^1/_2), \tag{1.2}$$

where

$$J_a=\varepsilon_{abc}x_b p_c+S_a, \tag{1.3}$$

in which $S_a=\frac{i}{4}\varepsilon_{abc}\gamma_b\gamma_c$ are the spin matrices.

In other words, besides the three obvious integrals of the motion – the components $J_a$ of the vector of the angular momentum – the Dirac equation with Coulomb potential has the additional integral of the motion (1.2), which is a differential operator with matrix coefficients. Such operators are not generators of a Lie group, and therefore in principle the Dirac integral cannot be found in the framework of the classical group analysis of differential equations.

Using the identity

$$2\mathbf{S}\cdot\mathbf{J}=\mathbf{J}^2-\mathbf{L}^2+\mathbf{S}^2, \quad \mathbf{L}=\mathbf{x}\times\mathbf{p}, \tag{1.4}$$

we can readily show that in the space of square integrable functions the spectrum of the operator (1.2) is discrete and given by the formula [1]

$$Q\psi=\varepsilon(j+{}^1/_2)\psi, \quad \varepsilon=\pm1, \quad j={}^1/_2,\ {}^3/_2, \ldots. \tag{1.5}$$

By direct calculation we can verify the useful relations

$$Q^2=\mathbf{J}^2+{}^1/_4, \quad [Q, \mathbf{S}\cdot\mathbf{p}]_+\equiv Q\mathbf{S}\cdot\mathbf{p}+\mathbf{S}\cdot\mathbf{p}Q=0, \quad [Q, \mathbf{S}\cdot\mathbf{x}]_+=0. \tag{1.6}$$

Using (1.6), we readily note that the operator (1.2) is an integral of the motion not only for a particle that is coupled minimally to the Coulomb field but also for more complicated couplings. In particular, the following result, which we give without proof, holds.

PROPOSITION 1. The general form of a spherically symmetric potential $V=V(\mathbf{x})$ for which the Hamiltonian (1.1) commutes with the operator (1.2) is determined by the relation

$$V=V_1+V_2\gamma_0+V_3\gamma_a x_a+V_4\gamma_0\gamma_a x_a, \tag{1.7}$$

where $V_1$, ..., $V_4$ are arbitrary functions of x.

In the case $V_1 = qe^2/x$, $V_3 = kqe^2/x^3$, $V_2 = V_4 = 0$, the relation (1.7) specifies the potential of the anomalous Pauli coupling to the field of a point charge, while for $V_1 = V_2$, $V_3 = V_4 = 0$ it specifies the general form of interaction potential that ensures confinement in quark models using the single-particle Dirac equation [15] (we do not particularize the explicit form of $V_3$ and $V_4$, which for our purposes is not important).

One can show that the condition for symmetry of the Hamiltonian (1.1) with arbitrary potential V with respect to the group of three-dimensional rotations O(3) and with respect to the transformation of spatial inversion,

$$\psi(x_0, \mathbf{x})\to P\psi(x_0, \mathbf{x})=r\psi(x_0, -\mathbf{x}), \tag{1.8}$$

where $r = \gamma_0$, also reduces to the requirement that V have the form (1.7). In other words, the requirement of P invariance of the Hamiltonian (1.1) with arbitrary O(3)-invariant potential V is a necessary and sufficient condition for the existence of the Dirac integral for this Hamiltonian. We shall see below that symmetry with respect to the transformation of spatial inversion also entails the existence of additional integrals of the motion for other single- and two-particle equations of motion.

Thus, the Dirac integral is a symmetry operator, i.e., an operator that carries solutions into solutions – for a more rigorous definition, see [16]) for an entire class of equations of the form

$$L\psi=0,\quad L=i\frac{\partial}{\partial x_0}-H, \tag{1.9}$$

where H is the Hamiltonian specified by (1.1) and (1.7). Indeed, by virtue of the above [14,16]

$$[Q,\ L]\psi=0, \tag{1.10}$$

where $\psi$ is an arbitrary solution of Eq. (1.9).

## 2. Integrals of the Motion for Vector Particles

We show that for vector particles interacting with the field of a point charge there also exist traditional integrals of the motion, and we find them in explicit form.

We consider the Kemmer–Duffin–Petiau (KDP) equation with anomalous coupling for a spin 1 particle in the Coulomb field:

$$[\beta^\mu\pi_\mu-m-ekS^{\mu\nu}F_{\mu\nu}]\psi\equiv L\psi=0. \tag{2.1}$$

Here, $\mu$, $\nu$ = 0, 1, 2, 3,

$$\pi_\mu=i\frac{\partial}{\partial x^\mu}-eA_\mu,\quad A_0=\frac{qe}{x},\quad A_a=0,\quad S^{\mu\nu}=i[\beta^\mu,\beta^\nu], \tag{2.2}$$

$F_{\mu\nu}$ is the tensor of the electromagnetic field,

$$F_{\mu\nu}=i[\pi_\mu,\pi_\nu],\quad F_{0a}=\frac{qex_a}{x^3},\quad F_{ab}=0,\quad a,b\neq 0, \tag{2.3}$$

$\beta^\mu$ are ten-row matrices satisfying the KDP algebra,

$$\beta^\mu\beta^\nu\beta^\lambda+\beta^\lambda\beta^\nu\beta^\mu=g^{\mu\nu}\beta^\lambda+g^{\nu\lambda}\beta^\mu, \tag{2.4}$$

and k is the constant of the anomalous coupling.

Equation (2.1) can be expressed in the Schrödinger form (1.9), where

$$H=[\beta_0,\beta_a]p_a+\beta_0 m+\frac{qe^2}{x}+\frac{iqe^2}{m}(k+\beta_0{}^2-1)\frac{\beta_a x_a}{x^3}+\frac{ikqe^2}{m^2}\left[\frac{\beta_a x_a}{x^3},\beta_b p_b\right], \tag{2.5}$$

and $\psi$ is a ten-component wave function satisfying the additional condition

$$\left(1-\beta_0{}^2+\frac{\beta_a p_a}{m}\beta_0{}^2-\frac{ikqe^2}{m^2}\beta_a\beta_0\frac{x_a}{x^3}\right)\psi=0.$$

Obvious symmetry operators of Eq. (2.1) are the generators of the group O(3) (angular momentum operators), the explicit form of which is given by (1.3) and (2.6):

$$S_a=i\varepsilon_{abc}\beta_b\beta_c. \tag{2.6}$$

These generators are integrals of the motion, since they commute with the Hamiltonian (2.5).

Using the relations (2.4), we can readily show by direct verification that the following result is true.

PROPOSITION 2. For Eqs. (2.1)–(2.3) there exists the additional integral of the motion

$$Q=(1-2\beta_0{}^2)[2(\mathbf{S}\cdot\mathbf{J})^2-2\mathbf{S}\cdot\mathbf{J}-\mathbf{J}^2], \tag{2.7}$$

where $\mathbf{J}$ and $\mathbf{S}$ are specified by (1.3) and (2.6).

The operator (2.7) commutes with both the Hamiltonian (2.5) and the operator L (2.1) and, therefore, is a symmetry operator of the equation. Of course, this result is also true in the case k = 0, i.e., in the absence of anomalous coupling.

Like the Dirac integral, the operator (2.7) has a discrete spectrum, which, in contrast to (1.5), has the form

$$Q\psi=\varepsilon j(j+1)\psi. \tag{2.8}$$

We arrive at the relation (2.8) by using the representation (1.4) for the operator $\mathbf{S}\cdot\mathbf{J}$.

The operator (2.7) does not belong to the enveloping algebra generated by the generators (1.3) and (2.6). However, the square of this operator can be expressed in terms of $\mathbf{J}^2$:

$$Q^2=(\mathbf{J}^2)^2. \tag{2.9}$$

It is interesting to note that (2.7) is also a symmetry operator for Maxwell's equations with currents and charges if they are written in the form of the system [14]

$$\begin{cases}(1-\beta_5^2)(\beta^\mu p_\mu+1)\psi=0,\\ \beta^\mu p_\mu\beta_5\psi=0,\end{cases}$$

where $\beta_5=\frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\beta^\mu\beta^\nu\beta^\rho\beta^\sigma$, $\psi$ is the column $(E_1, E_2, E_3, H_1, H_2, H_3, j_1, j_2, j_3, j_0)$, $E_a$ and $H_a$ $(a=1, 2, 3)$ are the components of the vectors of the electric and magnetic fields, $j_\mu$ ($\mu$ = 0, 1, 2, 3) are the components of the current 4-vector, and $\beta_\mu$ are the KDP matrices in the standard representation, the explicit form of which is given, for example, in [14]. Indeed, as is readily shown, $[Q, (1-\beta_5^2)(\beta^\mu p_\mu+1)]=[Q, \beta^\mu p_\mu\beta_5]=0$.

An additional integral of the motion also exists for the Stueckelberg equation [17], which describes the interaction of a quasiparticle (with possible spin values s = 0, 1) with the field of a point charge. With allowance for the anomalous Pauli coupling, this equation can be expressed in the form (2.1)–(2.3), where $\beta^\mu$ are 11 × 11 matrices (their explicit form is given, for example, in [16]), and $S^{\mu\nu}$ are the generators of the direct sum $D\left(\frac{1}{2},\frac{1}{2}\right)\oplus D(0,0)\oplus D(1,0)\oplus D(0,1)$ of irreducible representations of the Lorentz group. The integral of the motion for such an equation is specified by (2.7), where $S_a={}^1/_2\varepsilon_{abc}S_{bc}$, and $\beta_0$ and $S_{ab}$ are the corresponding 11 × 11 matrices.

The relations (2.8) and (2.9) hold for the new integral of the motion of the Stueckelberg equation.

## 3. Integrals of the Motion for Particles of Arbitrary Spin

The above results can be generalized to the case of relativistic wave equations for particles of arbitrary spin.

We consider an arbitrary equation of the form (2.1), where $S^{\mu\nu}$ are the generators of the direct sum

$$D=\sum\oplus D(m,n) \tag{3.1}$$

of finite-dimensional irreducible representations of the Lorentz group, and $\beta^\mu$ are numerical matrices that satisfy the relations

$$[\beta^\mu, S^{\nu\lambda}]=i(g^{\mu\nu}\beta^\lambda-g^{\mu\lambda}\beta^\nu) \tag{3.2}$$

and ensure relativistic invariance of Eq. (2.1).

We require that Eq. (2.1) be invariant with respect to the spatial inversion transformation (1.8), where r is a numerical matrix that must satisfy [18]

$$r^2=1,\quad r\beta^0=\beta^0 r,\quad r\beta^a=-\beta^a r,\quad rS^{ab}=S^{ab}r,\quad rS^{0a}=-S^{0a}r. \tag{3.3}$$

On the matrices $\beta^\mu$ it is customary to impose some additional restrictions to ensure the possibility of a Lagrangian formulation of Eq. (2.2) and uniqueness of the value of the spin of the particle described by it [18]. For our purposes, these additional assumptions are unimportant.

We restrict ourselves to the case when the external field reduces to the Coulomb potential (2.2). By analogy with (1.2) and (2.7), we seek the symmetry operator of the corresponding equation (2.1) in the form

$$Q=rd,\quad d=d(\mathbf{x}, \mathbf{p}, S^{\mu\nu}), \tag{3.4}$$

where r is the matrix of spatial inversion.

Requiring that Q (3.4) commutes with L (2.1), and using (3.2) and (3.3), we obtain the following equations for d:

$$[d, x]=[d, \beta_0]=0, \tag{3.5}$$

$$[d, S_{0a}p_a]_+=[d, S_{0a}x_a]_+=0. \tag{3.6}$$

It follows from (3.5) that d depends only on the matrices $S_{ab}$, $a, b\neq 0$, and $\mathbf{x}\times\mathbf{p}$, and it is sufficient to consider Eqs. (3.6) for matrices $S_{0a}$ belonging to the irreducible representation $D(m, n) \subset D$.

It is convenient to seek a solution of Eqs. (3.6) in the basis of spherical spinors (eigenvectors of the commuting operators $\mathbf{J}^2, \mathbf{S}^2, J_3$ and $\mathbf{L}^2=(\mathbf{x}\times\mathbf{p})^2$), in which the operators $S_{0a}/x_a$ and $xS_{0a}p_a$ reduce to numerical matrices. The explicit form of these matrices is given in [16].

Essentially, the transition to the basis of spherical spinors is one of the forms of realization of the algorithm proposed in [12-14] for seeking nonlocal symmetries of differential equations, the basic idea of which is to transform the equations to a representation in which the description of the symmetry reduces to a purely matrix problem.

Omitting some cumbersome calculations, we give the explicit form of the operators d for arbitrary irreducible representations D(m, n):

m + n integral:

$$d=CF\sum_{s=|m-n|}^{m+n}\sum_{\lambda=0}^{s}(-1)^{\lambda}B_\lambda^s, \quad \lambda=0, 1, \dots; \tag{3.7}$$

m + n half-integral:

$$d=CF\sum_{s=|m-n|}^{m+n}\sum_{\nu=1/2}^{s}(-1)^{s+1/2-\nu}N_\nu^s, \quad \nu=1/2, 3/2, \dots, \tag{3.8}$$

where $F=\sum_{\alpha=1}^{2(m+n)-1}(4\mathbf{J}^2+1-\alpha^2)$, $m+n\neq\frac{1}{2}$, and F = 1 for m + n = 1/2, C is an arbitrary constant, and $B_\lambda^s$ and $N_\nu^s$ are operators that satisfy the relations

$$\sum_{\lambda=\lambda_0}^{s}B_\lambda^s=1, \quad \lambda=\lambda_0, \lambda_0+1, \dots, \lambda_0=\frac{1}{4}[1-(-1)^{2s}]; \tag{3.9}$$

$$B_\lambda^s B_{\lambda'}^s=\delta_{\lambda\lambda'}B_\lambda^s, \quad B_\lambda^s N_{\lambda'}^s=\delta_{\lambda\lambda'}N_\lambda^s, \quad N_\lambda^s N_{\lambda'}^s=\delta_{\lambda\lambda'}(4J^2+1)B_\lambda^s, \tag{3.10}$$

$$G_s\equiv P_s(2\mathbf{S}\cdot\mathbf{J}-\mathbf{S}^2)=\sum_{\lambda=\lambda_0}^{s}(\lambda N_\lambda^s-\lambda^2 B_\lambda^s). \tag{3.11}$$

Here, $P_s$ is the projection operator

$$P_s=\prod_{s'\neq s}\frac{\mathbf{S}^2-s'(s'+1)}{s(s+1)-s'(s'+1)}, \quad |m-n|\leq s, s'\leq m+n,$$

in which S is the vector with components $S_a=1/2\varepsilon_{abc}S_{bc}$, $S_{bc}\in D(m, n)$.

For each concrete value of s, the operators $B_\lambda^s$ and $N_\lambda^s$ can be expressed in terms of $G_s$. For this, it is sufficient to raise successively both sides of Eq. (3.11) to the power n = 1, 2, ..., 2s and solve the obtained system of 2s + 1 linear algebraic equations for the 2s + 1 unknowns $B_\lambda^s$ and $N_\lambda^s$. In accordance with (3.10), the equations with the numbers n = 2k and n = 2k + 1 have the form

$$G_s^{2k}=\sum_{\lambda=\lambda_0}^{s}\Big[\sum_{m=0}^{k}\lambda^{2(k+m)}(4\mathbf{J}^2+1)^{k-m}C_{2k}^{2m}B_\lambda^s-\sum_{l=0}^{k-1}\lambda^{2(k+m)-1}(4\mathbf{J}^2+1)^{k-m-1}C_{2k}^{2l+1}N_\lambda^s\Big], \quad k\leq s;$$

$$G_s^{2k+1}=\sum_{\lambda=\lambda_0}^{s}\Big[\sum_{m=0}^{k}\lambda^{2(k+m)+1}(4\mathbf{J}^2+1)^{k-m}C_{2k+1}^{2m}N_\lambda^s-\sum_{l=0}^{k-1}\lambda^{2(k+l)}(4\mathbf{J}^2+1)C_{2k+1}^{2l+1}B_\lambda^s\Big], \quad k<s \tag{3.12}$$

($C_b^a$ is the number of combinations of b elements from $a$ possibilities), and the equation with number 2s + 1 is given by (3.8).

Let $S = (m + n)_{max}$ be the maximal value of the quantum number s that arises on reduction of the representation (3.1) with respect to the group O(3). We give solutions of Eqs. (3.7)–(3.9), (3.12) for $d = d_S$, $S \leq 2$:

$$d_{1/2}=2\mathbf{S}\cdot\mathbf{J}-{}^1/_2; \quad (3.13)$$

$$d_1=2(\mathbf{S}\cdot\mathbf{J})^2-2\mathbf{S}\cdot\mathbf{J}-\mathbf{J}^2; \quad (3.14)$$

$$d_{3/2}={}^4/_3[g^3-g^2-(7\mathbf{J}^2+\mathbf{S}^2)g+(4\mathbf{S}^2-6)\mathbf{J}^2]+3; \quad g=2\mathbf{S}\cdot\mathbf{J}-{}^3/_2; \quad (3.15)$$

$$d_2={}^2/_3[(\mathbf{S}\cdot\mathbf{J})^2-2\mathbf{S}\cdot\mathbf{J}-4J^2](\mathbf{S}\cdot\mathbf{J}-1)(\mathbf{S}\cdot\mathbf{J}-3)-\mathbf{J}^2(\mathbf{J}^2-2)+$$
$$({}^1/_3\mathbf{S}^2-2)[(4-3\mathbf{J}^2)(\mathbf{S}\cdot\mathbf{J})^2+(7\mathbf{J}^2-4)\mathbf{S}\cdot\mathbf{J}-4\mathbf{J}^2+{}^3/_8\mathbf{S}^2(4\mathbf{J}^2+1). \quad (3.16)$$

Here, **J** is the operator (1.3), and **S** are the matrices in the corresponding representation D (3.1) for $S \leq 2$.

The expressions (3.4) and (3.13)–(3.16) define symmetry operators for an entire class of relativistic and T-invariant equations of the form (2.1) corresponding to $S \leq 2$. This class includes the equations discussed above in Secs. 1 and 2, the Rarita–Schwinger equations [19] in the formulation given in [20], the Dirac–Fierz–Pauli equations [21,22] describing particles with fixed values of the mass and spin, and also the Bhabha equations [23] for sets of particles with spins $s \leq S$ and masses $m_S$. The explicit form of the corresponding matrices r and S is given in [18,20,23].

We note also that the spectrum of the operators (3.13) and (3.14) is given by the expressions (1.5) and (2.8) (where $Q \to d_S$), and for the operators (3.15) and (3.16) we obtain, using (1.4),

$$d_{3/2}\psi=\varepsilon(2j-1)(2j+1)(2j+3)\psi, \quad \varepsilon=\pm1, \quad j={}^1/_2, {}^3/_2, \ldots;$$

$$d_2\psi=\varepsilon(j-1)j(j+1)(j+2)\psi, \quad \varepsilon=\pm1, \quad j=0, 1, \ldots.$$

## 4. Integrals of the Motion for Two- and Three-Particle Equations

The above results enable us to construct new integrals of the motion for equations describing systems of interacting particles.

We consider a generalized two-particle equation of the form

$$i\frac{\partial}{\partial x_0}\psi=(H^{(1)}+H^{(2)}+V)\psi, \quad (4.1)$$

where $H^{(1)}$ and $H^{(2)}$ are the single-particle Dirac Hamiltonians

$$H^{(\alpha)}=\gamma_0^{(\alpha)}\gamma_a^{(\alpha)}p_a-\gamma_0^{(\alpha)}m_{(\alpha)}, \quad \alpha=1, 2, \quad (4.2)$$

$\{\gamma_0^{(1)}, \gamma_a^{(1)}\}$ and $\{\gamma_0^{(2)}, \gamma_a^{(2)}\}$ are commuting sets of 16 × 16 Dirac matrices, and V is an interaction potential of the general form

$$V=V_A\Gamma_A+V_B'\Gamma_B{}^a x_a+V_C''\Gamma_C{}^{ab}x_a x_b. \quad (4.3)$$

Here, $\{\Gamma_A\}$ (A = 1, 2, ..., 16) is the set of matrices $\{\gamma_0^{(1)}, \gamma_0^{(2)}, \gamma_4^{(1)}\gamma_4^{(2)}, \sigma^{(1)}\sigma^{(2)}, I\}$ and all possible products of them numbered in an arbitrary order, $\sigma_a^{(\alpha)}=\frac{i}{2}\varepsilon_{abc}\gamma_b^{(\alpha)}\gamma_c^{(\alpha)}$, $\{\Gamma_B{}^a\}=\{\gamma_4^{(\alpha)}\gamma_0^{(\beta)}\sigma_a^{(\nu)}, \gamma_4^{(\alpha)}\sigma_a^{(\beta)}, \gamma_0^{(1)}\gamma_0^{(2)}\gamma_4^{(\alpha)}\sigma_a^{(\beta)}\}$, $B=1, 2, \ldots, 24$, $(\alpha, \beta, \nu)=1, 2$, $\{\Gamma_C{}^{ab}\}$ $(C=1, 2, \ldots, 8)$ is the set of matrices of the form $\Gamma_C'\sigma_a^{(1)}\sigma_b^{(2)}$, where $\{\Gamma_C'\}=\{\gamma_0^{(\alpha)}, \gamma_4^{(1)}\gamma_4^{(2)}, I, \gamma_0^{(1)}\gamma_0^{(2)}, \gamma_4^{(1)}\gamma_4^{(2)}\gamma_0^{(\alpha)}, \gamma_0^{(1)}\gamma_0^{(2)}\gamma_4^{(1)}\gamma_4^{(2)}\}$, $V_A$, $V_B'$, $V_C''$ are arbitrary functions of x.

Equation (4.3) determines the general form of the potential V for which Eq. (4.1) remains invariant with respect to the group O(3) and the spatial inversion transformation (1.8) (at the same time $r=\gamma_0^{(1)}\gamma_0^{(2)}$). Such a potential includes as special cases (obtained by a special choice of the functions $V_A$, $V_B'$, and $V_C''$) the quasirelativistic Breit potential [3], the relativistic potential of the two-particle Barut–Komy equation [4], and also the potentials used in the quark models of mesons [5–7]. The corresponding equations (4.1) are interpreted as two-particle equations in the center-of-mass system [7].

For Eq. (4.1) with arbitrary potential of the form (4.3), there exists an obvious vector integral of the motion — the operator of the total angular momentum (1.3), where

$$S_a=S_a^{(1)}+S_a^{(2)},\quad S_a^{(\alpha)}=\frac{i}{4}\varepsilon_{abc}\gamma_b^{(\alpha)}\gamma_c^{(\alpha)},\quad \alpha=1,2. \tag{4.4}$$

However, as for the single-particle relativistic equations considered above, one can find an additional integral of the motion of Eq. (4.1). It has the form

$$\hat{Q}=\gamma_0^{(1)}\gamma_0^{(2)}d_1, \tag{4.5}$$

where $d_1$ is given by the expressions (3.14), (1.3), and (4.4).

It can shown by direct verification that the operator (4.5) commutes with the Hamiltonians $H^{(1)}$ and $H^{(2)}$ (4.2) with any potential of the form (4.3). Such a verification is readily done by using the following representation for $\hat{Q}$:

$$\hat{Q}=\gamma_0^{(1)}\gamma_0^{(2)}([Q_{(1)},Q_{(2)}]_+-{}^1/_2), \tag{4.6}$$

where $Q_{(\alpha)}$ are operators whose explicit form can be obtained from (3.13) by the substitution $\mathbf{S}\to\mathbf{S}^{(\alpha)}$ ($\mathbf{S}^{(\alpha)}$ are specified in (4.4), $\mathbf{J}$ in (1.3) and (4.4)). These operators satisfy the conditions

$$[Q_{(\alpha)},\boldsymbol{\sigma}^{(\alpha)}\cdot\mathbf{p}]_+=\boldsymbol{\sigma}^{(\alpha')}\cdot\mathbf{p},\quad [Q_{(\alpha)},\boldsymbol{\sigma}^{(\alpha)}\cdot\mathbf{x}]_+=\boldsymbol{\sigma}^{(\alpha')}\cdot\mathbf{x},$$

$$[Q_{(\alpha)},\boldsymbol{\sigma}^{(\alpha')}\cdot\mathbf{p}]=[Q_{(\alpha)},\boldsymbol{\sigma}^{(\alpha')}\cdot\mathbf{x}]=[Q_{(\alpha)},x]=0,\quad \alpha'\neq\alpha.$$

Thus, for any two-particle equation of the form (4.1) there exists an additional integral of the motion, which is specified by (4.5) and (4.6). One can show that in the space of square integrable functions the spectrum of the operator (4.5) is discrete and given by (2.8).

We give one further integral of the motion for the three-particle Krolikowski equation [8]:

$$Q=\gamma_0^{(1)}\gamma_0^{(2)}\gamma_0^{(3)}d_{3/2}.$$

Here, $d_{3/2}$ is the operator (3.14) for $S_a=\frac{i}{4}\varepsilon_{abc}(\gamma_b^{(1)}\gamma_c^{(1)}+\gamma_b^{(2)}\gamma_c^{(2)}+\gamma_b^{(3)}\gamma_c^{(3)})$, $\{\gamma_\mu^{(1)}\}$, $\{\gamma_\mu^{(2)}\}$ and $\{\gamma_\mu^{(3)}\}$ are three sets of 64 × 64 commuting Dirac matrices.

## 5. Conclusions

We have seen that additional integrals of the motion of Dirac type exist for a large class of single-particle and two-particle equations. The obtained integrals of the motion can be used in the solution of the corresponding equations by separation of the variables, in the construction of orthogonal bases, and for other purposes.

In deriving the new integrals of the motion, we have made essential use of the symmetry of the equations with respect to the group O(3) and the spatial inversion transformation P. The obtained results can be generalized to the case of arbitrary O(3)- and P-invariant equations, which need not necessarily satisfy the condition of relativistic invariance. In particular such integrals of the motion can be obtained for Galileo-invariant wave equations [24–26,16] and for equations of the form (2.1) with arbitrary O(3)- and P-invariant potential $A_0$.

It should be emphasized once more that, in principle, the obtained integrals of the motion cannot be obtained by the methods of classical group analysis of differential equations. Essentially "non-Lie elements" of our approach are the high order of the operators of differentiation that occur in the symmetry operators and the reduction of the problem to the finding of a general solution of the anticommutation relations (3.6).

Numerous examples of non-Lie symmetry of the basic equations of quantum theory are given in [16].

LITERATURE CITED


1. P. A. M. Dirac, The Principles of Quantum Mechanics, Oxford (1958).
2. M. H. Johnson and B. A. Lippmann, Phys. Rev., 77, 702 (1950).
3. G. Breit, Phys. Rev., 34, 553 (1929).

4. A. O. Barut and S. Komy, Fortschr. Phys., 33, 309 (1985).
5. W. Krolikowski and I. Rzewuski, Acta Phys. Pol., 7, 487 (1976).
6. A. A. Khelashvili, Teor. Mat. Fiz., 51, 201 (1982).
7. R. W. Childers, Phys. Rev. D, 26, 2902 (1982).
8. W. Krolikowski, Acta Phys. Pol. B, 15, 927 (1984).
9. L. V. Ovsyannikov, Group Analysis of Differential Equations [in Russian], Nauka, Moscow (1978).
10. N. Kh. Ibragimov, Transformation Groups in Mathematical Physics [in Russian], Nauka, Moscow (1983).
11. P. Olver, Applications of Lie Groups to Differential Equations [Russian translation], Mir, Moscow (1989).
12. V. I. Fushchich, Dokl. Akad. Nauk SSSR, 246, 846 (1979).
13. V. I. Fushchich and A. G. Nikitin, Fiz. Elem. Chastits At. Yadra, 14, 5 (1983).
14. W. I. Fushchich and A. G. Nikitin, Symmetries of Maxwell's Equations, Reidel, Dordrecht (1987); (shortened version: The Symmetry of Maxwell's Equations [in Russian], Naukova Dumka, Kiev (1983)); V. I. Fushchich, L. F. Barannik, and A. F. Barannik, The Subgroup Structure of the Galileo and Poincaré Groups and Reductions in Nonlinear Equations [in Russian], Naukova Dumka, Kiev (1991).
15. M. L. de Amaral and N. Zagury, Phys. Rev. D, 26, 3119 (1982).
16. V. I. Fushchich and A. G. Nikitin, The Symmetry of the Equations of Quantum Mechanics [in Russian], Nauka, Moscow (1990).
17. E. Stueckelberg, Helv. Phys. Acta, 11, 225 (1938).
18. I. M. Gel'fand, R. A. Minlos, and Z. Ya. Shapiro, Representations of the Rotation and Lorentz Groups and their Applications, Pergamon Press, Oxford (1963).
19. W. Rarita and J. Schwinger, Phys. Rev., 60, 61 (1941).
20. H. A. Bethe and E. E. Salpeter, Quantum Mechanics of One and Two Electron Atoms, Springer, Berlin (1957).
21. P. A. M. Dirac, Proc. R. Soc. London, Ser. A, 155, 447 (1936).
22. M. Fierz and W. Pauli, Proc. R. Soc. London, Ser. A, 173, 211 (1939).
23. R. A. Kraicik and M. M. Nieto, Phys. Rev. D, 10, 4049 (1974).
24. J.-M. Levy-Leblond, Commun. Math. Phys., 6, 286 (1967).
25. W. J. Hurley, Phys. Rev. D, 3, 2339 (1971).
26. A. G. Nikitin and V. I. Fushchich, Teor. Mat. Fiz., 44, 34 (1980).

# COLLECTIVE TWO-PHOTON PROCESSES IN THE PRESENCE OF A THERMAL ELECTROMAGNETIC FIELD

N. A. Enaki and O. B. Prepelitsa

A study is made of a lumped system of emitters (nuclei, atoms, molecules) with dipole-forbidden transitions between the first two energy levels in interaction with a thermal electromagnetic field. Elimination of the boson operators leads to an equation for the statistical operator of the system of emitters describing two-photon interaction with the electromagnetic field. A Fokker–Planck equation is also obtained and solved in the stationary case, and the result is used to investigate the equilibrium fluctuations in the populations of the atomic levels. The kinetics of the system and the statistical properties of the electromagnetic field are considered.

## 1. Introduction

Much attention is currently being devoted to two-photon generation and absorption of the electromagnetic field in multilevel systems [1–4].

The present paper is devoted to the possibility of collectivization of a lumped system (nuclei, atoms, molecules) in two-photon transitions in the presence of an external

Institute of Applied Physics, Academy of Sciences of the Moldavian Republic. Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 88, No. 3, pp. 416-429, September, 1991. Original article submitted August 31, 1990.