the mode with $E = c_F k/\sqrt{3}$, which in the considered model is threefold degenerate in the absence of a field [4].

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RELATIVISTIC PARTICLE OF ARBITRARY SPIN IN A COULOMB FIELD AND THE FIELD OF A PLANE ELECTROMAGNETIC WAVE

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Exact solutions are found for the equations of motion of a charged relativistic particle with arbitrary spin in a Coulomb field and in the field of a plane electromagnetic wave.

1. Introduction

Exact solutions of relativistic wave equations are widely used in modern physics [1-3]. For the practically important case of the motion of a charge in a Coulomb field, such solutions have been obtained only for particles with spins 0, $\frac{1}{2}$ [2], and 1 [4, 5].

It is well known that the theory of relativistic equations for particles with higher spins encounters difficulties of a fundamental nature in the formulation of the problem of motion of a particle in an external electromagnetic field. The difficulties include superluminal signal propagation velocities and complex values of the particle energy, which are predicted by such equations, and the absence of stable solutions in the Coulomb problem (see [6] and the literature quoted there).

In [7], relativistic equations of motion for a particle with arbitrary spin were proposed; like the Dirac equation, these do not have the pathological properties listed above. These equations can be written in the form of the system

$$\left[\Gamma_\mu\pi^\mu - m + \frac{e}{4m}(1-i\Gamma_4)\left(\frac{1}{s}S_{\mu\nu} - i\Gamma_\mu\Gamma_\nu\right)F^{\mu\nu}\right]\Psi=0, \quad (\Gamma_\mu\pi^\mu+m)(1-i\Gamma_4)\left[S_{\mu\nu}S^{\mu\nu}-4s(s-1)\right]\Psi=16ms\Psi, \tag{1}$$

where $\Psi = \Psi(\hat{x})$ is a 8s-component wave function, $\hat{x}=(x_0, x_1, x_2, x_3)$, $\pi_\mu=-i\partial/\partial x^\mu-eA_\mu$, A_μ is the vector potential, $F_{\mu\nu}$ is the electromagnetic field tensor, Γ_μ are 8s × 8s matrices satisfying the Clifford algebra $\Gamma_\mu\Gamma_\nu+\Gamma_\nu\Gamma_\mu=2g_{\mu\nu}$, $\Gamma_4=\Gamma_0\Gamma_1\Gamma_2\Gamma_3$, and $S_{\mu\nu}$ are the generators of the representation $[D(1/2\ 0)\oplus D(0\ 1/2)]\otimes D(s-1/2\ 0)$ of the Lorentz group.

In the case $s = \frac{1}{2}$, the system (1) reduces to the Dirac equation for an electron, and for arbitrary (integral or half-integral) s Eqs. (1) describe the motion of a charged particle with spin s and mass m in an external electromagnetic field.

The main criterion for evaluating a particular formulation of relativistic equations for particles with arbitrary spin is the possibility of using them to solve concrete physical problems. In [7], Eqs. (1) were used to find the energy spectrum of particles of arbitrary spin in a homogeneous magnetic field. Below, we obtain solutions of Eqs. (1) for the case of a charged particle with arbitrary spin interacting with a Coulomb

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field and with the field of a plane electromagnetic wave.

2. Equation for the Radial Part of the Wave Function of a Particle of Arbitrary Spin in a Coulomb Field

To solve the problems listed above, it is convenient to go over from Eqs. (1) to a system of second-order equations for a $(2s+2)$-component wave function. Multiplying (1) by $\lambda_+ = {}^1/_2(1+i\Gamma_4)$ and $\lambda_- = {}^1/_2(1-i\Gamma_4)$ and expressing $\Psi_- = \lambda_-\Psi$ in terms of $\Psi_+ = \lambda_+\Psi$, we obtain

$$\left(\pi_\mu\pi^\mu - m^2 + \frac{e}{4sm}S_{\mu\nu}F^{\mu\nu}\right)\Psi_- = 0, \tag{2a}$$

$$[S_{\mu\nu}S^{\mu\nu} - 4s(s+1)]\Psi_- = 0, \tag{2b}$$

$$\Psi_+ = \frac{1}{m}\Gamma_\mu\pi^\mu\Psi_-. \tag{2c}$$

Thus, the solution of Eqs. (1) reduces to finding functions Ψ_- satisfying the system (2a), (2b). The condition (2b) means that $2s-1$ components of the function Ψ_- vanish identically. The remaining $2s+1$ components form a spinor in the space of the representation $D(s\,0)$ of the Lorentz group [7]. With allowance for what we have said above, Eq. (2a) can be rewritten in the form

$$\left[\pi_\mu\pi^\mu - m^2 + \frac{e}{s}\mathbf{S}\cdot(\mathbf{H} - i\mathbf{E})\right]\Phi_s = 0, \tag{3}$$

where Φ_s is a $(2s+1)$-component function, $\mathbf{S} = (S_1, S_2, S_3)$ are $(2s+1)\times(2s+1)$ matrices realizing the irreducible representation $D(s)$ of the Lie algebra of the group $O(3)$:

$$[S_a, S_b] = i\varepsilon_{abc}S_c, \quad \mathbf{S}^2 = s(s+1), \tag{4}$$

and $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field vectors.

Equation (3) can be regarded as a generalization of the Zaitsev–Feynman–Gell-Mann equation [8] to the case of particles with arbitrary spin.

We consider Eq. (3) for the case of the Coulomb field, when $\mathbf{A} = 0$, $A_0 = ze/x$. The solutions of Eq. (4) corresponding to a state with energy ε can be expressed in the form $\Phi_s = \exp(-i\varepsilon t)\Phi_s(\mathbf{x})$. Taking into account the symmetry of the problem with respect to the group $O(3)$, we can conveniently represent the function $\Phi_s(\mathbf{x})$ as a linear combination of spherical spinors:

$$\Phi_s(\mathbf{x}) = \varphi^\lambda(x)\Omega^s_{j\,j-\lambda\,m}, \quad x = |\mathbf{x}|, \tag{5}$$

where $\Omega^s_{j\,j-\lambda\,m} = \Omega^s_{j\,j-\lambda\,m}(\mathbf{x}/x)$ form a complete set of eigenfunctions of the operators $\mathbf{J}^2$, J_3, and $\mathbf{L}^2$ ($\mathbf{J} = \mathbf{L} + \mathbf{S} = \mathbf{x}\times\mathbf{p} + \mathbf{S}$ is the operator of the total angular momentum):

$$\mathbf{J}^2\Omega^s_{j\,j-\lambda\,m} = j(j+1)\Omega^s_{j\,j-\lambda\,m}; \quad J_3\Omega^s_{j\,j-\lambda\,m} = m\Omega^s_{j\,j-\lambda\,m}; \quad \mathbf{L}^2\Omega^s_{j\,j-\lambda\,m} = (j-\lambda)(j-\lambda+1)\Omega^s_{j\,j-\lambda\,m},$$

$$m = -j, -j+1, \ldots, j; \quad \lambda = -s, -s+1, \ldots, -s+2m_{sj}; \quad m_{sj} = \begin{cases} j, & j \leqslant s, \\ s, & j \geqslant s. \end{cases}$$

Substituting (5) in (3), we arrive at the following equations for the radial functions $\varphi^\lambda(x)$:

$$D\varphi^\lambda = x^{-2}b_{\lambda\lambda'}\varphi^{\lambda'}, \tag{6}$$

where

$$D = \left(\varepsilon + \frac{\alpha}{x}\right)^2 - m^2 + \frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} - \frac{j(j+1)}{x^2}, \qquad b_{\lambda\lambda'} = [\lambda^2 - \lambda(2j+1)]\delta_{\lambda\lambda'} + \frac{i}{s}\alpha a^{sj}_{\lambda\lambda'}, \quad \alpha = ze^2, \tag{7}$$

and $a^{sj}_{\lambda\lambda'}$ are the matrix elements of the operator $\dfrac{\mathbf{S}\cdot\mathbf{x}}{x} = (ze)^{-1}x^2\mathbf{S}\cdot\mathbf{E}$ in the basis $\{\Omega^s_{j\,j-\lambda\,m}\}$:

$$\frac{\mathbf{S}\cdot\mathbf{x}}{x}\Omega^s_{j\,j-\lambda\,m} = \sum_{\lambda'} a^{sj}_{\lambda\lambda'}\Omega^s_{j\,j-\lambda'\,m}. \tag{8}$$

Before we turn to the solution of the system (6), it is necessary to determine the values of the coefficients $a^{sj}_{\lambda\lambda'}$. For $s=\frac{1}{2}$ and $s=1$, these coefficients are given, for example, in [9]. In the following section, we find the values of $a^{sj}_{\lambda\lambda'}$ for arbitrary spin.

3. Explicit Form of the Operator $\dfrac{\mathbf{S}\cdot\mathbf{x}}{x}$ in the Basis of Spherical Spinors

The spherical spinors $\Omega^{s}_{j\,j-\lambda\,m}$ are $(2s+1)$-component functions with components

$$(\Omega^{s}_{j\,j-\lambda\,m})^{\mu}=C^{jm}_{j-\lambda\;m-\mu\;s\;\mu}Y_{j-\lambda\;m-\mu}, \tag{9}$$

where $C^{j\lambda}_{j-\lambda\;m-\mu\;s\;\mu}$ are Clebsch–Gordan coefficients, and $Y_{j-\lambda\;m-\mu}$ are spherical functions. Substituting (9) in (8), setting $\mu=m$, $\mathbf{x}/x=\tilde{\mathbf{x}}=(0,\,0,\,1)$, and bearing in mind that [9]

$$Y_{j-\lambda\,0}(\tilde{\mathbf{x}})=\left(\frac{2j-2\lambda+1}{4\pi}\right)^{1/2},$$

we arrive at the following system of linear algebraic equations for the coefficients $a^{sj}_{\lambda\lambda'}$:

$$\sum_{\lambda'}(a^{sj}_{\lambda\lambda'}-\mu\delta_{\lambda\lambda'})\,(2j-2\lambda'+1)^{1/2}C^{j\mu}_{j-\lambda'\;0\;s\;\mu}=0, \tag{10}$$

where

$$\lambda,\lambda'=\begin{cases}-s,\,-s+1,\,\ldots,\,s, & j\geqslant s,\\ -s,\,-s+1,\,\ldots\,-s+2j, & j\leqslant s,\end{cases}\qquad \mu=\begin{cases}-s,\,-s+1,\,\ldots,\,s, & j\geqslant s,\\ -j,\,-j+1,\,\ldots\,j, & j\leqslant s.\end{cases} \tag{11}$$

We give without proof the general solution of the system (10):

$$\begin{gathered}a^{sj}_{\lambda\lambda'}=-{}^1/_2(\delta_{\lambda\lambda'+1}a^{sj}_{s+\lambda}+\delta_{\lambda\lambda'-1}a^{sj}_{s+\lambda+1}),\qquad a^{sj}_{\nu}=\frac{1}{2}\left(\frac{\nu(d_j-\nu)\,(d_s-\nu)\,(d_{js}-\nu)}{(d_{js}-2\nu+1)\,(d_{js}-2\nu+3)}\right)^{1/2}\\ \nu=s+\lambda=0,\,1,\,2,\,\ldots,\quad d_j=2j+1,\quad d_s=2s+1,\quad d_{js}=d_j+d_s.\end{gathered} \tag{12}$$

Equations (8), (11), and (12) determine the explicit form of the operator $\dfrac{\mathbf{S}\cdot\mathbf{x}}{x}$ in the basis of spherical spinors for arbitrary value of the spin. In particular, for $s,\ j\leq 3/2$ we obtain from (12)

$$\begin{gathered}a^{1/2\,j}_{1/2\,-1/2}=a^{1/2\,j}_{-1/2\,1/2}=-{}^1/_2;\qquad a^{1j}_{10}=a^{1j}_{01}=-\sqrt{\frac{j+1}{2j+1}};\quad a^{1j}_{0\,-1}=a^{1j}_{-1\,0}=-\sqrt{\frac{j}{2j+1}},\quad j\neq 0;\\ a^{3/2\,j}_{3/2\,1/2}=a^{3/2\,j}_{1/2\,3/2}=-\frac{1}{2}\sqrt{\frac{3(j+1)}{j}};\quad a^{3/2\,j}_{-1/2\,-3/2}=a^{3/2\,j}_{-3/2\,-1/2}=-\frac{1}{2}\sqrt{\frac{3j}{j+1}};\quad a^{3/2\,j}_{1/2\,-1/2}=a^{3/2\,j}_{-1/2\,1/2}=-\frac{1}{2}\sqrt{\frac{(2j+3)\,(2j-1)}{j(j+1)}},\quad j\neq\frac{1}{2};\end{gathered} \tag{13}$$

$$\begin{gathered}a^{s\,1/2}_{1-s\,-s}=a^{s\,1/2}_{-s\,1-s}=-\frac{1}{2};\quad a^{s\,0}_{\lambda\lambda'}=0;\quad a^{s\,1}_{2-s\,1-s}=a^{s\,1}_{1-s\,2-s}=-\sqrt{\frac{s+1}{2s+1}};\quad a^{s1}_{-s\,1-s}=a^{s1}_{1-s\,-s}=-\sqrt{\frac{s}{2s+1}};\\ a^{s\,3/2}_{3-s\,2-s}=a^{s\,3/2}_{2-s\,3-s}=-\frac{1}{2}\sqrt{\frac{3(s+1)}{s}};\qquad a^{s\,3/2}_{1-s\,-s}=a^{s\,3/2}_{-s\,1-s}=-\frac{1}{2}\sqrt{\frac{3s}{2s+1}};\\ a^{s\,3/2}_{2-s\,1-s}=a^{s\,3/2}_{1-s\,2-s}=-\frac{1}{2}\sqrt{\frac{(2s+3)\,(2s-1)}{s(s+1)}},\quad s\neq{}^1/_2.\end{gathered} \tag{14}$$

The remaining coefficients $a^{sj}_{\lambda\lambda'}$ for $s\leq 3/2$ and for $j\leq 3/2$ are equal to zero.

4. Energy Levels of a Relativistic Particle with Arbitrary Spin in a Coulomb Field

We turn to the solution of Eqs. (6). The matrix $\|b^{sj}_{\lambda\lambda'}\|$ obviously commutes with the operator D (7), and, at least for $\alpha\ll 1$, can be diagonalized. This means that the system (6), (7) can be reduced to the following chain of decoupled equations:

$$D\varphi=x^{-2}b^{sj}\varphi, \tag{15}$$

where D is the operator (7), and b^{sj} are the eigenvalues of the matrix $\|b^{sj}_{\lambda\lambda'}\|$. Each of Eqs. (15) reduces, in its turn, to the well-know equation [10]

$$z\frac{d^2y}{dz^2}+\frac{dy}{dz}+\left(\beta-\frac{z}{4}-\frac{k^2}{4}\right)y=0, \tag{16}$$

where

$$y=\frac{1}{2}\left[\frac{z}{m^2-\varepsilon^2}\right]^{1/2}\varphi,\quad z=2(m^2-\varepsilon^2)^{1/2}x,\quad \beta=\frac{\varepsilon\alpha}{(m^2-\varepsilon^2)^{1/2}},\quad k^2=(2j+1)^2+4(b^{sj})^2-4\alpha^2. \tag{17}$$

The solutions of Eq. (15) can be expressed in terms of confluent hypergeometric functions, which in the case of bound states ($\varepsilon < m$) reduce to generalized Laguerre polynomials. The parameter β can take the values [10]

$$\beta=(k+1)/2+n',\quad n'=0,1,2,\ldots. \tag{18}$$

From (17) and (18), we obtain

$$\varepsilon=m\left[1+\frac{\alpha^2}{(n'+{}^1/_2+[(j+{}^1/_2)^2-\alpha^2+b^{sj}]^{1/2})^2}\right]^{-1/2}. \tag{19}$$

The expression (19) determines the energy levels of a particle of arbitrary spin in the Coulomb field. The parameter b^{sj} in (19) takes values equal to the roots of the characteristic equation for the matrix (7),

$$\det\left\|[\lambda^2-\lambda(2j+1)-b^{sj}]\delta_{\lambda\lambda'}+\frac{i}{s}\alpha a^{sj}_{\lambda\lambda'}\right\|=0, \tag{20}$$

where $a^{sj}_{\lambda\lambda'}$ are the coefficients (12).

Equation (20) is an algebraic equation of order $(2s+1)$ if $j \geq s$, or of order $2j+1$ if $j \leq s$. This equation can be solved in radicals only for $s \leq 3/2$ or $j \leq 3/2$. To analyze the spectrum (19) in the case of arbitrary s and j, it is sufficient to consider approximate solutions of Eq. (20), which can be represented in the form

$$b^{sj}=\lambda^2-\lambda d_j+b_\lambda^{sj}\alpha^2+o(\alpha^4),\quad d_j=2j+1. \tag{21}$$

Using (12), (20), and (21), we can readily calculate in explicit form the coefficients b^{sj}_λ directly for arbitrary values of s, j, and λ:

$$b_\lambda^{sj}=\frac{1}{2s^2}\left[\frac{(a_\lambda^{sj})^2}{j-\lambda+1}+\frac{(a_{\lambda+1}^{sj})^2}{\lambda-j}\right], \tag{22}$$

where a_λ^{sj} are the coefficients (12).

We substitute (21) in (19) and expand the function on the right-hand side of (19) in a series in powers of α^2. To terms of order α^4 we obtain

$$\varepsilon=m\left(1-\frac{\alpha^2}{2n^2}+\frac{\alpha^4(b_\lambda^{sj}-1)}{n^3(d_j-2\lambda)}+\frac{3}{8}\frac{\alpha^4}{n^4}\right). \tag{23}$$

The expression (23) determines the fine structure of the energy spectrum of a Coulomb particle of arbitrary spin in the Coulomb field. The parameters b^{sj}_λ in (23) can be readily calculated from (22). From (22) and (23) it can be seen that in the general case each energy level corresponding to fixed values of the quantum numbers n and j is split into k_{sj} sublevels, where $k_{sj}=2s+1$ if $j \geq s$ and $k_{sj}=2j+1$ if $j \leq s$. An exception is the case $s=\frac{1}{2}$, when $b^{sj}_\lambda=2\lambda/d_j$, and the corresponding spectrum is degenerate.

Let us consider in more detail the spectra (19) and (23) for the cases $s \leq 3/2$ and $j < 3$. Using (13) and (14), we obtain from (22)

$$b_\lambda^{0j}=0,\quad \lambda=0;\quad b_\lambda^{1/2\,j}=2\lambda/d_j,\quad \lambda=\pm{}^1/_2;\quad b_\lambda^{1j}=\lambda\frac{d_j+\lambda}{2d_j(d_j-\lambda)}-\frac{2(1-\lambda^2)}{d_j^2-1},\quad \lambda=\begin{cases}1,0,-1, & j\neq 0,\\ -1, & j=0;\end{cases}$$

$$b_\lambda^{3/2\,j}=\frac{(d_j+{}^2/_3\lambda)(\lambda^2-{}^1/_4)}{4\lambda(d_j-{}^2/_3\lambda)(d_j-{}^4/_3\lambda)}-\frac{1}{18}\frac{(d_j^2-26\lambda d_j-32)\left(\lambda^2-\frac{9}{4}\right)}{d_j(d_j+2\lambda)(d_j-4\lambda)},\quad \lambda=\begin{cases}-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2}, & j\neq\frac{1}{2},\\ -\frac{3}{2},-\frac{1}{2}, & j=\frac{1}{2};\end{cases} \tag{24}$$

$$b_\lambda^{s0}=0,\quad \lambda=-s;\quad b_\lambda^{s\,1/2}=\frac{(-1)^{s+\lambda+1}}{4s^2(2s+1)},\quad \lambda=-s,\ -s+1;\qquad b_\lambda^{s1}=\frac{1}{2s^2}\left[\lambda'\frac{d_s+\lambda'}{d_s(d_s-\lambda')}-\frac{2(1-\lambda'^2)}{d_s^2-1}\right]$$

$$d_s=2s+1,\quad \lambda'=\lambda+s-1,\quad \lambda=-s,\ -s+1,\ -s+2.$$

From (23) and (24) we conclude that in the case s = 0 Eq. (23) gives the well-known spectrum of energies of a scalar particle (described by the Klein–Gordon equation) in a Coulomb field [11], and that in the case $s=\frac{1}{2}$ (23) is identical to the formula for the fine structure of the spectrum of the hydrogen atom [11]. For s = 1, the results given in (23) and (24) agree with those obtained earlier in [5] if we denote $\lambda=j-\nu$, $\lambda(2j+1-\lambda)=\varkappa$. Considering the spectrum (23), (24) for $j \leq 1$, we conclude that each level corresponding to fixed values of j and n is split into two sublevels if $j=\frac{1}{2}$ and three for j = 1. The magnitude of the splitting decreases with increasing spin. In the case j = 0 there is no splitting.

We now turn to the exact expression (19). For $s \leq 3/2$ or $j \leq 3/2$, Eq. (20), which determines the values of the parameter b^{sj}, can be solved exactly. For completeness, we give the corresponding solutions for $s \leq 3/2$ and $j \leq 1$:

$$b^{1/2\,j}={}^1/_4-\lambda(d_j^2-4\alpha^2)^{1/2},\quad \lambda=\pm{}^1/_2; \tag{25}$$

$$b^{1j}={}^2/_3+2\sqrt{a}\cos\left[{}^1/_3(\gamma+2\lambda\pi)\right],\quad \lambda=0,\pm1,\quad j\neq0,\quad b^{10}=2, \tag{26}$$

where $\cos\gamma=b/\sqrt{a^3}$, $b={}^2/_3\alpha^2+{}^1/_3d_j^2-{}^1/_{27}$, $a=b+{}^4/_{27}-\alpha^2$;

$$b^{3/2\,j}=b_{\varepsilon\nu}(\mu,b)={}^1/_2\left({}^5/_2+\varepsilon A+\nu\sqrt{(1+\varepsilon A)^2-2y-\varepsilon A^{-1}(2y+12b-9\mu)}\right), \tag{27}$$

where ε and ν independently take the values ±1,

$$\pm1,\ A=\sqrt{y-{}^{15}/_2\mu-2},\ y=1+{}^5/_2\mu+2\sqrt{c}\cos({}^1/_3\varkappa),\ \cos\varkappa=k/\sqrt{c^3},\ c=13\mu^2+2(\mu+b)-1,\ k=35\mu^3+33\mu^2-87\mu b+$$
$$18b^2+33b-1,\ \mu={}^4/_3[j(j+1)+2b],\ b={}^1/_2(\alpha/3)^2;$$

$$b^{s0}=s(s+1),\quad b^{s1/2}=s(s+1)-\frac{1}{2}\pm\frac{1}{2s}\sqrt{s^2(2s+1)^2-\alpha^2},\quad b^{s1}={}^2/_3+2\sqrt{p}\cos\left[{}^1/_3(\beta+2\lambda\pi)\right],\quad \lambda=0,\pm1;$$
$$\cos\beta=q/\sqrt{p^3},\quad q={}^2/_3(\alpha/s)^3+{}^1/_3d_s^2-{}^1/_{27},\quad p=q+{}^4/_{27}-(\alpha/s)^2;\quad b^{s\,3/2}=s(s+4)-{}^{33}/_4+b_{\varepsilon\nu}(\mu',b'), \tag{28}$$
$$\mu'={}^4/_3[s(s+1)+2b'],\quad b'={}^1/_8(\alpha/s)^2.$$

Substituting (25) in (19), we arrive at Sommerfeld's formula for the hydrogen atom. Equations (19) and (26)-(28) generalize Sommerfeld's formula to the case of particles with spin 1, 3/2 and to the case of particles with arbitrary spin (for $j \leq 3/2$).

Note that our result for s = 1 differs from the result obtained earlier in [5], in which, possibly due to a misprint, the term 2/3 is absent in the expression for b^{1j} (cf. (26) and Eq. (9) in [5]; in [5] b^{1j} is denoted by the symbol λ). In addition, in [5] there are two extra roots $b^{10}=\pm\alpha$, whereas for s = 1 and j = 0 Eq. (7) becomes the identity $b^{10}\equiv 2$.

5. Particle with Arbitrary Spin in the Field of a Plane Electromagnetic Wave

We now consider Eq. (3) for the case when the external electric field is a plane wave,

$$A_\mu=A_\mu(\varphi),\quad \varphi=k_\mu x^\mu,\quad \partial_\mu A^\mu=k_\mu A^{\mu\prime}=0, \tag{29}$$

where the prime denotes differentiation with respect to φ. Substituting (29) in (3), we arrive at the equation

$$\left(-\partial_\mu\partial^\mu-2ieA_\mu\partial^\mu+e^2A_\mu A^\mu-m^2-\frac{e}{s}\mathbf{S}\cdot\mathbf{F}\right)\Phi_s=0, \tag{30}$$

where $\mathbf{F}=\mathbf{k}\times\mathbf{A}-i(k_0\mathbf{A}'-\mathbf{k}A_0')$. We seek solutions of this equation in the form

$$\Phi_s=\exp(-ip_\mu x^\mu)\Psi(\varphi),\quad p^2=m^2. \tag{31}$$

Then from (30) we obtain the following equation for $\Psi(\varphi)$:

$$2ik_\mu p^\mu\Psi'+\left[-2ep_\mu A^\mu+e^2A_\nu A^\nu-\frac{e}{s}\mathbf{S}\cdot\mathbf{F}\right]\Psi=0. \tag{32}$$

Equation (32) is readily integrated:

$$\Psi = \exp\left\{-i\int_0^{k_\mu x^\mu}\left[\frac{e}{k_\mu p^\mu}p_\nu A^\nu - \frac{e^2}{2k_\mu p^\mu}A_\nu A^\nu\right]d\varphi - \frac{ie}{2sk_\mu p^\mu}\mathbf{S}\cdot\mathbf{F}\right\}U_p, \quad (33)$$

where U_p is a constant spinor, conveniently chosen in such a way that the functions (31) determine normalized solutions of Eq. (3) in the absence of interaction (i.e., for $A_\mu = F_{\mu\nu} = 0$).

The matrices $\mathbf{S}\cdot\mathbf{F}$ in the solutions (33) satisfy the conditions

$$\prod_\lambda[(\mathbf{S}\cdot\mathbf{F})^2 - \lambda^2\mathbf{F}^2] = 0; \quad \mathbf{S}\cdot\mathbf{F}\prod_\nu[(\mathbf{S}\cdot\mathbf{F})^2 - \nu^2\mathbf{F}^2] = 0, \quad (34)$$

where $\lambda = 1/2, 3/2, \ldots, s$; $\nu = 1, 2, \ldots, s$. Since by definition $\mathbf{F}^2 = k_\mu k^\mu A_\nu A^\nu = 0$, the condition (34) can be written in the form

$$(\mathbf{S}\cdot\mathbf{F})^{2s+1} = 0. \quad (35)$$

By virtue of (35), the solutions (31) and (33) reduce to the form

$$\Phi_s = \sum_{n=0}^{2s}\left(\frac{ie\mathbf{S}\cdot\mathbf{F}}{2sk_\mu p^\mu}\right)^n \exp(iS)\,U_p,$$

where S is the classical action for a particle moving in the field of an electromagnetic wave:

$$S = -p_\nu x^\nu - \int_0^{k_\nu x^\nu}\left[\frac{e}{k_\mu p^\mu}p_\nu A^\nu - \frac{e^2}{2k_\mu p^\mu}A_\nu A^\nu\right]d\varphi.$$

We see that, in contrast to Volkov's well-known solution [12] for an electron in the field of a plane electromagnetic wave, the solutions of the equations for particles with arbitrary spin depend on the field intensity as polynomials of degree 2s, and not linearly.

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