POINCARÉ INVARIANT DIFFERENTIAL EQUATIONS FOR PARTICLES OF ARBITRARY SPIN

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Differential equations of first and second order describing the motion of a relativistic particle with arbitrary spin are derived. These equations provide the basis for an exact solution of the problem of the motion of a particle of arbitrary spin in a homogeneous magnetic field. Covariant operators for the coordinate and spin of the particle are found, and these differ from the well-known Newton–Wigner and Foldy–Wouthuysen operators. The Hamiltonian of a particle interacting with an external electromagnetic field is approximately diagonalized.

Introduction

In all manifestly covariant first-order relativistic equations describing the motion of particles with spin $s > \frac{1}{2}$ the wave equation has more components than the number $2(2s + 1)$ of possible states of the free particle–antiparticle system. This "superfluousness" is evidently one of the reasons why the Kemmer–Duffin equation [1] ($s = 1$) and the Rarita–Schwinger equation [2] ($s = {}^3/_2$), which describe the behavior of particles in external electromagnetic fields, have solutions corresponding to the motion of particles with nonzero mass with velocity greater than the velocity of light in vacuum. At the present time, only the Dirac equation, which does not have redundant components, does not lead to these unphysical consequences.

This distinguished position of the Dirac equation was the stimulus for constructing equations of motion of the form

$$i\frac{\partial}{\partial t}\Psi(t,\mathbf{x})=H(\mathbf{p},S)\Psi(t,\mathbf{x}),\quad p_a=-i\frac{\partial}{\partial x_a} \tag{0.1}$$

for particles with arbitrary spin, where the wave function Ψ has only $2(2s + 1)$ components [3,4]. Equations (0.1) are distinguished by the fact that the Hamiltonian $H(\mathbf{p}, s)$ for $s > \frac{1}{2}$ is an integrodifferential operator. The requirement that the wave function have no redundant components and the condition of hermiticity of the Hamiltonian and the other generators of the Poincaré group with respect to the ordinary scalar product

$$(\Psi_1,\Psi_2)=\int d^3x\Psi_1{}^\dagger(t,\mathbf{x})\Psi_2(t,\mathbf{x}) \tag{0.2}$$

lead to the nonlocal equations of motion (0.1) in configuration space. This circumstance (nonlocality of the corresponding Hamiltonians) makes it very difficult to use equations of the form (0.1) to describe the behavior of particles with spin $s > \frac{1}{2}$ in external electromagnetic fields. In [4], on the basis of Eqs. (0.1) a solution was found to the problem of the interaction of a particle of arbitrary spin with an external field under the assumption that the particle momentum is small compared with its rest mass, i.e., a quasirelativistic description of a particle in an external field was obtained.

The equations obtained by Weaver, Hammer, and Good [6] and Mathews and collaborators [7] lead to similar difficulties. The main difference between these equations and the equations obtained in [3,4] is that the equations of [6,7] are defined in a space with the scalar product

$$(\Psi_1,\Psi_2)=\int d^3x\Psi_1{}^\dagger(t,\mathbf{x})M\Psi_2(t,\mathbf{x}),$$

where M is an integrodifferential metric operator that depends on the momentum and the spin matrices.

Guertin [8], developing the approach of [3,4], derived equations of the form (0.1) using an indefinite metric. For $s > 1$, these are also integrodifferential equations.

The present paper is a continuation of [3,4]. On the basis of the requirement that the Hamiltonian

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H(**p**, s) in (0.1) be a differential operator of first or second order, we find all possible (to within equivalence transformations) Poincaré invariant equations for a relativistic particle of arbitrary spin that, like the Dirac equation, admit a standard introduction of an interaction with an external field. The wave function in the second-order differential equations has only $2(2s+1)$ components. For the lowest integral spins ($s = 0, 1$) these equations coincide with the well-known Tamm–Sakata–Taketani equations [9]. Thus, as in the Tamm–Sakati–Taketani formalism, the Hamiltonian $H(\mathbf{p}, s)$ is not Hermitian with respect to (0.2), but it is Hermitian in a space with indefinite metric. Thus, an indefinite metric is the price that one must pay if the Hamiltonian $H(\mathbf{p}, s)$ in Eq. (0.1) is to be a differential operator and the wave function $\Psi(t, \mathbf{x})$ is not to have redundant components.

Using the equations obtained, we solve exactly the problem of the motion of a relativistic particle of arbitrary spin in a homogeneous magnetic field. It is shown that the equations obtained do not lead to the paradox of causality violation inherent, for example, in the Rarita–Schwinger equation [2].

1. Statement of the Problem

We obtain the differential equations of motion of a particle of arbitrary spin on the basis of the following representation of the generators P_μ and $J_{\mu\nu}$ of the group $P(1, 3)$ [5]:

$$P_0=H_s,\quad P_a=p_a=-i\frac{\partial}{\partial x_a},\quad J_{ab}=x_ap_b-x_bp_a+S_{ab},\quad J_{0a}=x_0p_a-{}^1/_2[x_a, H_s]_++\lambda_a,\quad x_0=t, \tag{1.1}$$

where $[A, B]_+ = AB + BA$, H_s is an as yet unknown differential operator that includes the derivatives $\partial/\partial x_a$ to not higher than second order,

$$S_{ab}=S_c=\begin{pmatrix} \hat{s}_c & 0 \\ 0 & \hat{s}_c \end{pmatrix},\quad (a, b, c)\ \text{cyclic perm. of}\ (1,2,3), \tag{1.2}$$

$\hat{S}_c$ are the generators of the irreducible representation $D(s)$ of $O(3)$, and λ_α are certain operators whose explicit form is determined by the requirement that the generators (1.1) satisfy the Poincaré algebra $P(1, 3)$.

Equations (1.1) determine the most general form of the generators of the Poincaré group corresponding to local transformations of the $2(2s+1)$-component wave function of the particle plus antiparticle system under a rotation of the coordinate system. Representations of the form (1.1), where H_s for $s > 1$ belongs to the class of integrodifferential operators, were considered earlier in [8].

DEFINITION. We shall say that Eq. (0.1) is Poincaré invariant and describes the free motion of a particle with mass m and spin s if the operators P_a and $J_{\mu\nu}$ in (1.1) and the Hamiltonian H_s satisfy the commutation relations of the algebra $P(1, 3)$:

$$[P_\mu, P_\nu]_-=0,\quad [P_\mu, J_{\mu\nu}]_-=i(g_{\mu\nu}P_\lambda-g_{\mu\lambda}P_\nu), \tag{1.3a}$$

$$[J_{\mu\nu}, J_{\lambda\sigma}]_-=i(g_{\mu\sigma}J_{\nu\lambda}+g_{\nu\lambda}J_{\mu\sigma}-g_{\mu\lambda}J_{\nu\sigma}-g_{\nu\sigma}]_{\mu\lambda}), \tag{1.3b}$$

$$P_\mu P^\mu\equiv H_s^2-p_a^2=m^2, \tag{1.3c}$$

$$W_\mu W^\mu\Psi=m^2s(s+1)\Psi, \tag{1.3d}$$

where $[A, B]_- = AB - BA$, $g_{\mu\nu}$ is the metric tensor, $g_{\nu\nu} = (-1, 1, 1, 1)$, W_μ is the Lubański–Pauli vector

$$W_\mu={}^1/_2\varepsilon_{\mu\nu\sigma\lambda}J_{\nu\sigma}P_\lambda. \tag{1.4}$$

It follows from what we have said that if we find all such operators H_s and λ_a for which the relations (1.3) are satisfied we shall thereby have solved the problem of constructing Poincaré invariant equations of the form (0.1); for if the relations (1.3) are satisfied then the conditions of invariance of Eq. (0.1) under the Poincaré algebra $P(1, 3)$ are satisfied:

$$\left[i\frac{\partial}{\partial t}-H_s, Q_A\right]_-\Psi=0, \tag{1.5}$$

where Q_A is an arbitrary generator of the group $P(1, 3)$.

2. Differential Operators H_s of Second Order

We give the solution of our problem in the form of the following theorem.

THEOREM. All possible (to within equivalence transformations realized by numerical matrices) differential operators H_s of second order satisfying the algebra P(1, 3) (1.3) are given by the formulas

$$H_s=\sigma_1 m+\sigma_3 k_1 \mathbf{S}\cdot\mathbf{p}+\frac{1}{2m}(\sigma_1-i\sigma_2)[p^2-(k_1\mathbf{S}\cdot\mathbf{p})^2], \tag{2.1}$$

$$H_1=\sigma_1\left(m+\frac{p^2}{2m}\right)-\frac{i}{2m}\sigma_2[p^2+2k_2(\mathbf{S}\cdot\mathbf{p})^2]+\frac{1}{m}\sigma_3\sqrt{k_2(k_2-1)}(\mathbf{S}\cdot\mathbf{p})^2, \quad p^2=p_1^2+p_2^2+p_3^2, \tag{2.2}$$

$$H_1=\sigma_1\left[\left(m+\frac{p^2}{2m}-\frac{(k_3\mathbf{S}\cdot\mathbf{p})^2}{2m}\right]+\sigma_3 k_3\mathbf{S}\cdot\mathbf{p}-\frac{i}{2m}\sigma_2[p^2+(k_3-2)(\mathbf{S}\cdot\mathbf{p})^2], \tag{2.3}$$

$$H_{3/2}=\sigma_1\left(m+\frac{p^2}{2m}\right)+\frac{ik_4}{2m}\sigma_2[(\mathbf{S}\cdot\mathbf{p})^2-{}^5/_4 p^2]+\frac{1}{2m}\sqrt{k_4^2-1}\,\sigma_3 p^2, \tag{2.4}$$

$$H_{3/2}=\sigma_1\left[m+\frac{p^2}{2m}-\frac{(k_5\mathbf{S}\cdot\mathbf{p})^2}{2m}\right]+\sigma_3 k_5\mathbf{S}\cdot\mathbf{p}-\frac{i}{8m}\sigma_2[(5k_5^2-4)(\mathbf{S}\cdot\mathbf{p})^2-(9k_5^2-5)p^2], \tag{2.5}$$

where σ_a are $2(2s+1)$-row Pauli matrices that commute with S_a, and k_l ($l = 1, 2, \ldots, 5$) are arbitrary complex parameters.

The proof can be made in accordance with the scheme described in detail in [3-5]. For brevity, we omit it. We give only the explicit form of the operators λ_a for which the generators (1.1), (2.1)-(2.5) satisfy the relations (1.3) (which can be proved by direct verification).

If the Hamiltonian H_s has the form (2.1),

$$\lambda_a=\left(1-\frac{k_1}{2}\right)\left[i\sigma_3 S_a-\frac{1}{2m}(\sigma_1-i\sigma_2)(\mathbf{p}\times\mathbf{S})_a\right]. \tag{2.6}$$

If H_s is given by one of formulas (2.2)-(2.5),

$$\lambda_a=\frac{i}{2EB_s}\left\{p_a\left(2+\left[\frac{H_s}{E},\sigma_1\right]_-\right)-2\dot{x}_a H_s-E[\dot{x}_a,\sigma_1]_-\right\}+\frac{H_s}{E(E+m)}\left[S_{ab}p_b-\frac{i}{EB_s}S_{ab}p_b(\sigma_1 E+H_s)\right], \tag{2.7}$$

where $B_s=2E+[H_s,\sigma_1]_+$, $E=(p^2+m^2)^{1/2}$, $p=(p_1^2+p_2^2+p_3^2)^{1/2}$, $\dot{A}=i[H_s,A]_-$.

Remark 1. It can be seen from Eqs. (2.1)-(2.5) that the relations (1.3) determine the Hamiltonians of the relativistic particle to within constant complex numbers k_l ($l = 1, 2, \ldots, 5$). Equation (0.1) with such Hamiltonians is invariant under the "strong reflection" transformation Θ = CPT but, in general, is not invariant under the P, C, and T transformations. The invariance of Eq. (0.1) under any of these transformations can be ensured by a special choice of the numbers k_i. For example, if we set $k_1 = 1/s$ in Eq. (2.1) for spin $s = \frac{1}{2}$, and in Eqs. (2.2)-(2.5) we set $k_2 = 1$, $k_3 = 0$, $k_4 = 1$, $k_5 = 0$, then we obtain P-, C-, and T-invariant Hamiltonians of the form

$$H_0=\sigma_1\left(m+\frac{p^2}{2m}\right)-i\sigma_2\frac{p^2}{2m}, \tag{2.8}$$

$$H_{1/2}=\sigma_1 m+2\sigma_3\mathbf{S}\cdot\mathbf{p}, \tag{2.9}$$

$$H_1=\sigma_1\left(m+\frac{p^2}{2m}\right)+i\sigma_2\left(\frac{(\mathbf{S}\cdot\mathbf{p})^2}{m}-\frac{p^2}{2m}\right), \tag{2.10}$$

$$H_{3/2}=\sigma_1\left(m+\frac{p^2}{2m}\right)+i\sigma_2\left[\frac{(\mathbf{S}\cdot\mathbf{p})^2}{2m}-\frac{5p^2}{8m}\right]. \tag{2.11}$$

The operator (2.9) coincides with the Dirac Hamiltonian, and the operators (2.8) and (2.10) with the Tamm–Sakati–Taketani Hamiltonians [9] for particles with spin $s = 0, 1$. The operator (2.1) for spin $s = \frac{1}{2}$ was considered earlier in [10].

Remark 2. All the generators of the group P(1, 3) determined by formulas (1.1), (2.1), and (2.6) belong to the class of differential operators. For $k_1 = 2$, the generators J_{0a} (1.1), (2.6) take the particularly simple form [3, 4]

$$J_{0a}=x_0 p_a-{}^1/_2[x_a,H_s]_+. \tag{2.12}$$

Remark 3. The Hamiltonians (2.1)-(2.5) and the remaining generators (1.1), (1.2), (2.6), and (2.7) of the group P(1, 3) can be reduced to the canonical Foldy–Shirokov form [11, 12, 3, 4]. This is achieved by the isometric transformation

$$P_0 \to P_0^k = V P_0 V^{-1} = \sigma_1 E, \quad P_a \to P_a^k = V P_a V^{-1} = p_a, \quad J_{ab} \to J_{ab}^k = V J_{ab} V^{-1} = x_a p_b - x_b p_a + S_{ab},$$
$$J_{0a} \to J_{0a}^k = V J_{0a} V^{-1} = x_0 p_a - {}^1/_2 [x_a, P_0^k]_+ - \sigma_1 \frac{S_{ab} p_b}{E+m}, \quad E = (m^2 + p^2)^{1/2}, \tag{2.13}$$

where the operators V have the form

$$V = V_1 V_2 V_3, \quad V_1 = \exp\left(\sigma_1 \frac{\mathbf{S}\cdot\mathbf{p}}{p} \operatorname{arth} \frac{p}{E}\right), \quad V_2 = \frac{1}{2m}[E\lambda^+ + m\lambda^- - 2\sigma_1\lambda^- \mathbf{S}\cdot\mathbf{p}],$$
$$V_3 = \exp\left[\frac{1}{2m}\sigma_1\lambda^+ (k_1 - 2)\mathbf{S}\cdot\mathbf{p}\right], \quad \lambda^\pm = {}^1/_2(1 \pm \sigma_3),$$

for the Hamiltonians (2.1) and

$$V = (E + \sigma_1 H_s)(2E^2 + E[H_s, \sigma_1]_+)^{1/2}$$

for the Hamiltonians (2.2)-(2.5).

3. Differential First-Order Hamiltonian Equations

By analogy with the Dirac electron theory, we postulate that in Eq. (0.1) the Hamiltonian $\hat{H}_s$ of a relativistic particle with arbitrary spin is a differential operator that includes derivatives with respect to the spatial variables of not higher than first order:

$$\hat{H}_s = \hat{\Gamma}_a^{(s)} p_a + \hat{\Gamma}_0^{(s)} m, \tag{3.1}$$

where $\hat{\Gamma}_\mu^{(s)}$ are certain numerical matrices.

We choose the generators of the representation of the Poincaré group which is realized on the solutions of Eq. (0.1) with the Hamiltonian (2.1) in the form

$$P_0 = \hat{H}_s, \quad P_a = p_a = -i\frac{\partial}{\partial x_a}, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}, \tag{3.2}$$

where $S_{\mu\nu}$ are matrices that form a finite-dimensional representation (in the general case reducible) of the algebra O(1, 3). Equations (3.2) specify the most general form of the generators of the group P(1, 3) corresponding to local transformations of the wave function.

Determining all possible Hamiltonians of the form (3.1) is tantamount to finding all matrices $\hat{\Gamma}_\mu^{(s)}$ and $S_{\mu\nu}$ such that the operators (3.1) and (3.2) satisfy the Poincaré algebra (1.3).

We show that the required equations of motion of a particle with spin s and mass m have the form

$$\hat{H}_s \Psi = i\frac{\partial}{\partial t}\Psi, \quad \hat{H}_s = \Gamma_0^{(s)} \cdot \Gamma_a^{(s)} p_a + \Gamma_0^{(s)} \cdot m, \tag{3.3a}$$

$$\hat{P}_s \Psi = 0, \quad \hat{P}_s = P_s + \frac{1}{2m}(1 - \Gamma_4^{(s)})[\Gamma_\mu^{(s)} p^\mu, P_s]_-, \tag{3.3b}$$

$$P_s = \frac{1}{4s}[S_{ab}^2 - 2s(s-1)], \quad S_{ab}^2 = \sum_{a,b} S_{ab} S_{ab}, \tag{3.3c}$$

where $\Gamma_\mu^{(s)}, S_{ab}$ are 8s-row matrices determined by the relations

$$[\Gamma_\mu^{(s)}, \Gamma_\nu^{(s)}]_+ = 2g_{\mu\nu}, \quad \Gamma_4 = i\Gamma_0^{(s)}\Gamma_1^{(s)}\Gamma_2^{(s)}\Gamma_3^{(s)}, \quad S_{\mu\nu} = \tau_{\mu\nu} + j_{\mu\nu}, \quad [\tau_{\mu\nu}, j_{\lambda\sigma}]_- = 0, \quad \tau_{\mu\nu} = \frac{i}{2}\Gamma_\mu^{(s)}\Gamma_\nu^{(s)},$$
$$j_{ab} = j_c, \quad j_{0a} = ij_a, \quad [j_a, j_b]_- = ij_c, \quad \sum_a j_a^2 = j(j+1) = s(s-1), \tag{3.4}$$

i.e., the matrices $\Gamma_\mu^{(s)}$, as in the case of the Dirac equation, satisfy the Clifford algebra, and the matrices $S_{\mu\nu}$ are generators of the representation $[D(1/2, 0) \oplus D(0, 1/2)] \oplus D(s - 1/2, 0)$ of O(1, 3); for using (3.4), we can readily show that the Hamiltonian (3.3a) and the generators (3.2) satisfy the conditions (1.3a) and (1.3b). With regard to the condition (1.3d), it can be written in accordance with (1.4), (3.2)-(3.4) in the form (3.3b):

$$\frac{1}{2s}\left[\frac{1}{m^2} W_\mu W^\mu - s(s-1)\right]\Psi \equiv \hat{P}_s \Psi = \Psi,$$

where $\hat{P}_s$ is the operator of projection onto the subspace corresponding to fixed spin s [5].

Using the identity

$$(1+\Gamma_4^{(s)})P_s=\frac{1}{8s}[S_{\mu\nu}S^{\mu\nu}-4s(s-1)](1+\Gamma_4^{(s)}),$$

we can write Eqs. (2.3) in the manifestly covariant form

$$(\Gamma_\mu^{(s)}p^\mu-m)\Psi=0, \tag{3.5a}$$

$$(\Gamma_\mu^{(s)}p^\mu+m)(1+\Gamma_4^{(s)})(S_{\mu\nu}S^{\mu\nu}-4s(s-1)]\Psi=16ms\Psi. \tag{3.5b}$$

By virtue of what we have said above, Eqs. (3.5) are Poincaré invariant and describe the free motion of a particle with fixed spin s and mass m.

Remark 1. Equations (2.5) are also defined for the case $m = 0$. Imposing in this case on the wave function Ψ the additional Poincaré invariant condition $(1-\Gamma_4^{(s)})\Psi=0$, we obtain from (3.5) equations of motion for massless particles of arbitrary spin, these being equivalent for $s = \frac{1}{2}$ to the Weyl equation for neutrinos and for $s = 1$ to the Maxwell equations for the electromagnetic field in vacuum [13].

Remark 2. By means of the transformation $\Psi\to\Phi=W\Psi$, where

$$W=\exp\left(\frac{\Gamma_a^{(s)}p_a}{p}\operatorname{arctg}\frac{p}{m}\right)\exp\left(\Gamma_0^{(s)}\frac{j_ap_a}{p}\operatorname{arth}\frac{p}{E}\right),$$

Eqs. (2.3) and (3.5) can be reduced to the diagonal form

$$i\frac{\partial}{\partial t}\Phi=\Gamma_0^{(s)}E\Phi,\quad P_s\Phi=\Phi.$$

On the solutions of Eqs. (3.6), generators of the group P(1, 3) have the canonical form (2.1).

Note that in [14] 8s-component differential equations of first order describing the motion of a free particle with arbitrary spin s were also proposed. In contrast to (5.1) and (5.2), these equations become incompatible when allowance is made for an interaction of the particle with an external field.

4. Covariant Coordinate and Spin Operators

On the transition to a new inertial frame of reference, the operators of the physical quantities N_i (coordinate, spin, etc.) transform as follows:

$$N_i\to N_i'=\exp(iQ_l\theta_l)N_i\exp(-iQ_l\theta_l),$$

where Q_l ($l = 1, 2, \ldots, 10$) are the generators of the Poincaré group and θ_l are the transformation parameters.

One of the difficulties encountered in representations of the type (1.1) (when the generators J_{0a} cannot be expressed as a sum of commuting "spin" and "orbital" parts) is that the operator x_μ has a non-covariant transformation law under which the interval is not conserved, $x_0^2-x_a^2\neq(x_0')^2-(x_a')^2$. Therefore, x_μ cannot be interpreted as a covariant coordinate operator.

Below, we shall determine the covariant coordinate operator in the representation (1.1) and (2.1). In principle, we shall thereby solve the problem for the arbitrary representation (1.1) and (2.6), since the generators J_{0a} (2.12) and (1.1) and (2.6) are related by the equivalence transformation $J_{0a}\to VJ_{0a}V^{-1}$, where

$$V=\exp\left[(\sigma_1-i\sigma_2)(2-k_1)\frac{1}{2m}\mathbf{S}\cdot\mathbf{p}\right].$$

We go over to a representation in which the generators J_{0a} (2.12) have a locally covariant form:

$$\hat{J}_{0a}=x_0p_a-x_ap_0+S_{0a},\quad S_{0a}=i\sigma_3S_a,\quad p_0=i\frac{\partial}{\partial x_0}. \tag{4.1}$$

This is achieved by the transformation

$$\hat{J}_{0a}=VJ_{0a}V^{-1},\quad V=\exp\left[-\frac{i}{2m}(\sigma_2+i\sigma_1)(2\mathbf{S}\cdot\mathbf{p}-p_0)\right]. \tag{4.2}$$

In the representation (4.1), the covariant coordinate operator $\hat{X}_\mu$ can be chosen in the form $\hat{X}_\mu = x_\mu$.

By means of the inverse of the transformation (3.2), we obtain the explicit form of these operators in the original representation (2.12):

$$\hat{X}_\mu = \hat{V}^{-1} X_\mu \hat{V} = x_\mu + \frac{1}{m}(i\sigma_1 + \sigma_2)\xi_\mu, \quad \xi_a = S_a, \quad \xi_0 = {}^1/_2\sigma_3. \tag{4.3}$$

On the transition to the new inertial frame, the operators X_μ transform as the components of a four-vector and satisfy the canonical commutation relations

$$[p_\mu, X_\nu]_- = ig_{\mu\nu}, \quad [X_\mu, X_\nu]_- = 0. \tag{4.4}$$

All this enables us to conclude that X_μ (4.3) can be interpreted as covariant coordinate operator of the particle.

In the case $s = \frac{1}{2}$, the operators (4.3) take the manifestly covariant form

$$X_\mu = x_\mu + \frac{i}{2m}(1+\gamma_4)\gamma_\mu, \tag{4.5}$$

where $\gamma_4 = \sigma_3$, $\gamma_0 = \sigma_1$, $\gamma_a = -2i\sigma_2 S_a$ are Dirac matrices. By what we have said above, the operator (4.5) can be chosen as covariant coordinate operator of a Dirac particle. It is interesting to note that with this definition of the coordinate the velocity operator

$$\dot{X}_a = -i[H_{1/2}, X_a]_- = (1+\gamma_4)\gamma_0 \frac{p_a}{m}$$

(where $H_{\frac{1}{2}}$ is the Dirac Hamiltonian (2.9)) has continuous spectrum and satisfies the relation $[\dot{X}_a, \dot{X}_b] = 0$. However, $[H_{1/2}, \dot{X}_a]_- \neq 0$.

We emphasize that the operator (4.5) differs essentially from the coordinate operators proposed earlier by Newton and Wigner [15], Foldy and Wouthuysen [16], and many others [17]. The difference is that the operator (4.5) is local and transforms as a covariant four-vector, whereas the coordinate operators proposed in [15-17] belong to the class of nonlocal integrodifferential operators with noncovariant transformation law.

We give the explicit form of the covariant operator $\Sigma_{\mu\nu}$ of the spin of the particle described by Eq. (0.1) with the Hamiltonian (2.1):

$$\Sigma_{ab} = S_{ab} + \frac{1}{2m}(i\sigma_1 + \sigma_2) S_{cd} p_d, \quad (a, b, c) = (1, 2, 3), \quad \Sigma_{0a} = i\sigma_3 S_{bc} - \frac{1}{m}(i\sigma_1 + \sigma_2)[2\mathbf{S}\cdot\mathbf{p} - p_0, S_{bc}]_+.$$

By analogy with (4.1)-(4.3), we can show that the operators $\Sigma_{\mu\nu}$ transform as a covariant tensor of second rank, and the operator Σ_{ab} commutes with the Hamiltonian and is an integral of the motion.

Note also that the coordinate operator of the particle described by Eqs. (3.5) can be obtained from (4.5) by means of the substitution $\gamma_k \to \Gamma_k^{(s)}$.

5. Equation for a Charged Particle in an External Electromagnetic Field

One can show that the introduction of a minimal electromagnetic interaction directly in Eqs. (3.3) or (3.5) has the consequence that Eqs. (3.3) as well as Eqs. (3.5) become incompatible. In order to overcome this difficulty, we write (3.3) in the form of the single equation

$$\left[\hat{P}_s\left(i\frac{\partial}{\partial t} - \hat{H}_s\right) + \varkappa(1 - \hat{P}_s)\right]\Psi = 0, \tag{5.1}$$

where $\varkappa$ is an arbitrary parameter. The equivalence of (5.1) and (3.3) follows from the relations

$$\left[i\frac{\partial}{\partial t} - \hat{H}_s, \hat{P}_s\right]_- = 0, \quad \hat{P}_s\hat{P}_s = \hat{P}_s.$$

The manifestly covariant system (3.5) can, in its turn, be written in the form

$$[B_s(\Gamma_\mu^{(s)} p^\mu - m) - \varkappa(1 - B_s)]\Psi = 0, \quad B_s = \frac{1}{16ms}(\Gamma_\mu^{(s)} p^\mu + m)(1 + \Gamma_4^{(s)})[S_{\mu\nu}S^{\mu\nu} - 2s(s-1)], \tag{5.2}$$

since

$$[B_s, \Gamma_\mu^{(s)} p^\mu - m]_- \Psi = 0, \quad B_s B_s = B_s$$

In (5.1) and (5.2) we make the substitution $p_\mu \to \pi_\mu = p_\mu - eA_\mu$, where A_μ is the vector potential of the electromagnetic field, and we show that as a result (5.1) and (5.2) reduce to a system of manifestly covariant first-order differential equations describing the causal motion of a charged particle of arbitrary spin in an external field. Since Eqs. (5.1) and (5.2) ultimately lead to the same results, we consider only Eq. (5.1), which takes the form

$$\{\hat{P}_s(\pi)[\pi_0 - \hat{H}_s(\pi)] + \varkappa[1 - \hat{P}_s(\pi)]\}\Psi = 0, \tag{5.3}$$

$$\hat{H}_s(\pi) = \Gamma_0^{(s)}\Gamma_a^{(s)}\pi_a + \Gamma_0^{(s)} m, \qquad \hat{P}_s(\pi) = P_s + \frac{1}{2m}(1 - \Gamma_4^{(s)})[\Gamma_\mu^{(s)}\pi^\mu, P_s]_-. \tag{5.4}$$

Multiplying (5.3) by $\hat{P}_s(\pi)$ and $[1 - \hat{P}_s(\pi)]$ and using the identities

$$[\pi_0 - \hat{H}_s(\pi), \hat{P}_s(\pi)]_- \hat{P}_s(\pi) = \frac{1}{4m}\Gamma_0^{(s)}(1 - \Gamma_4^{(s)})\left(\frac{1}{s}S_{\mu\nu} - i\Gamma_\mu^{(s)}\Gamma_\nu^{(s)}\right)F_{\mu\nu}\hat{P}_s(\pi), \quad \hat{P}_s(\pi)\hat{P}_s(\pi) = \hat{P}_s(\pi), \quad F_{\mu\nu} = -i[\pi_\mu, \pi_\nu]_-,$$

we arrive at the system of equations

$$i\frac{\partial}{\partial t}\Psi(t,x) = \hat{H}_s(\pi, A_0)\Psi(t,x), \quad \hat{H}_s(\pi, A_0) = \Gamma_0^{(s)}\Gamma_a^{(s)}\pi_a + \Gamma_0^{(s)} m + eA_0 + \frac{1}{4m}\Gamma_0^{(s)}(1 - \Gamma_4^{(s)})\left[\frac{1}{s}S_{\mu\nu} - i\Gamma_\mu^{(s)}\Gamma_\nu^{(s)}\right]F_{\mu\nu}, \tag{5.5}$$

$$\left\{P_s + \frac{1}{2m}(1 - \Gamma_4^{(s)})[\Gamma_\mu^{(s)}\pi^\mu, P_s]_-\right\}\Psi = 0, \tag{5.6}$$

which, like (3.3), can be written in the manifestly covariant form

$$\left[(\Gamma_\mu^{(s)}\pi^\mu - m) + \frac{1}{4m}(1 - \Gamma_4^{(s)})\left(\frac{1}{s}S_{\mu\nu} - i\Gamma_\mu^{(s)}\Gamma_\nu^{(s)}\right)F_{\mu\nu}\right]\Psi = 0, \tag{5.7}$$

$$(m + \Gamma_\mu^{(s)}\pi^\mu)(1 - \Gamma_4^{(s)})[S_{\mu\nu}S^{\mu\nu} - 4s(s-1)]\Psi = 16ms\Psi. \tag{5.8}$$

We show that Eqs. (5.7) and (5.8) do not violate causality. To this end, we make the substitution

$$\Psi(t,x) = V\Phi(t,x), \quad V = \exp\left[(1 - \Gamma_4^{(s)})\frac{1}{2m}\Gamma_\mu^s\pi_\mu\right]. \tag{5.9}$$

Substituting (5.9) in (5.7) and multiplying the result from the left by the operator

$$F = m + {}^1/_2\left(\Gamma_\mu^{(s)}\pi^\mu - \frac{1}{sm}\tilde{S}_{\mu\nu}F_{\mu\nu} - \frac{1}{m^2}\pi_\mu\pi^\mu\right)(1 - \Gamma_4^{(s)}),$$

where $\tilde{S}_{ab} = S_{ab}$, $\tilde{S}_{0a} = iS_{bc}$, we arrive at the equation

$$\left(\pi_\mu\pi^\mu - m^2 - \frac{1}{2s}\tilde{S}_{\mu\nu}F_{\mu\nu}\right)\Phi(t, \mathbf{x}) = 0. \tag{5.10}$$

From (5.8) and (5.9) we obtain an additional condition for Φ in the form

$$P_s\Phi = \Phi \quad \text{or} \quad {}^1/_2 S_{ab}{}^2\Phi = s(s+1)\Phi. \tag{5.11}$$

Equations (5.10)-(5.11) generalize the Zaitsev–Feynman–Gell-Mann equation [18] for $s = \frac{1}{2}$ to the case of a particle of arbitrary spin. The solutions $\Phi(t, x)$ of this equation describe [19] causal propagation of waves (with subluminal velocity). The solutions $\Psi(t, \mathbf{x})$ of Eqs. (5.7)-(5.8), which are related to $\Phi(t, \mathbf{x})$ by the equivalence transformation (5.9), obviously have the same properties.

Thus, we have shown that Eqs. (5.7) and (5.8) describe the motion of a charged relativistic particle with arbitrary spin in an external electromagnetic field and do not violate causality. Note also that Eqs. (5.7) and (5.8) admit a Lagrangian formulation; for choose the Lagrangian density $L(x)$ in the form

$$L(x) = \left(m\bar{\Psi}' + i\frac{\partial\bar{\Psi}'}{\partial x_\mu}\check{\Gamma}_\mu^{(s)}\right)(1 + \check{\Gamma}_4^{(s)})[S_{\mu\nu}S^{\mu\nu}\Psi - 4s(s-1)]\Psi i\check{\Gamma}_\lambda^{(s)}\frac{\partial\Psi'}{\partial x_\lambda} + 16ms\bar{\Psi}'\Psi', \tag{5.12}$$

where

$$\Psi' = \begin{pmatrix}\Psi \\ \chi\end{pmatrix}, \quad \bar{\Psi}' = \Psi'^\dagger i\check{\Gamma}_0^{(s)}\check{\Gamma}_5^{(s)},$$

Ψ and χ are 8s-component functions, and $\check{\Gamma}_\mu^{(s)}$, $\check{S}_{\mu\nu}$ are $16s \times 16s$ matrices:

$$\breve{\Gamma}_k^{(s)}=\begin{pmatrix}\Gamma_k^{(s)} & 0\\ 0 & \Gamma_k^{(s)}\end{pmatrix},\quad \breve{\Gamma}_0^{(s)}=\begin{pmatrix}\Gamma_0^{(s)} & 0\\ 0 & -\Gamma_0^{(s)}\end{pmatrix},\quad \breve{\Gamma}_5^{(s)}=\begin{pmatrix}0 & \Gamma_0^{(s)}\\ \Gamma_0^{(s)} & 0\end{pmatrix},\quad \breve{S}_{\mu\nu}=\begin{pmatrix}S_{\mu\nu} & 0\\ 0 & S_{\mu\nu}\end{pmatrix}.$$

Using the principle of least action, we obtain from (5.12) Eqs. (3.5) for the function Ψ and the equations that are the complex conjugate of (3.5) for the function χ. Making in (5.12) the minimal substitution $\frac{\partial}{\partial x_\mu}\to\frac{\partial}{\partial x_\mu}+ieA_\mu$, we arrive at Eqs. (5.7) and (5.8).

6. Expansion in Powers of 1/m

The Hamiltonian (5.5) can have both positive and negative eigenvalues. By means of a series of successive transformations, we obtain from (5.5) an equation for states with positive energy, in the same way as Foldy and Wouthuysen [16] did for the Dirac equation. Then the operator $\hat{H}_s(\pi, A_0)$ will be represented in the form of a series in powers of $1/m$, which is convenient for perturbation calculations.

The main difficulty in diagonalizing Eqs. (5.5) and (5.6) is that it is necessary to find transformations that reduce to diagonal form two different equations simultaneously. We diagonalize first the additional condition (5.6), and then, using operators that commute with the transformed equation (5.6), we reduce Eq. (5.7) to diagonal form.

We subject the wave function $\Psi(t, \mathbf{x})$ to the transformation $\Psi\to\tilde{\Psi}=V\Psi$, where

$$V=\exp\left[\frac{1}{2m}(1-\Gamma_4^{(s)})(\Gamma_a^{(s)}\pi_a-k_1\Gamma_0^{(s)}S_a\pi_a)\right]. \tag{6.1}$$

Applying the operator (6.1) from the left to (5.5) and (5.6), we obtain an equation for $\tilde{\Psi}$:

$$H_s(\pi,A_0)\tilde{\Psi}=i\frac{\partial}{\partial t}\tilde{\Psi},\quad H_s(\pi, A_0)=\Gamma_0^{(s)}m+k_1\Gamma_4^{(s)}\mathbf{S}\cdot\pi+\frac{1}{2m}\Gamma_0^{(s)}(1-\Gamma_4^{(s)})\left[\pi^2-(k_1\mathbf{S}\cdot\mathbf{p})^2+\frac{1}{s}\mathbf{S}(\mathbf{H}-i\mathbf{E}+ik_1\mathbf{E})\right], \tag{6.2}$$

$$P_s\tilde{\Psi}=\tilde{\Psi}\ \text{ or }\ {}^1\!/_2 S_{ab}\tilde{\Psi}=s(s+1)\tilde{\Psi}, \tag{6.3}$$

where $H_a=-i[\pi_b, \pi_c]_-$ and $E_a=-[\pi_0, \pi_a]_-$ are the strengths of the magnetic and electric fields, and P_s is the projection operator (3.3c).

From (6.3) and (3.4) we conclude that the wave function $\tilde{\Psi}$ has $2(2s + 1)$ nonzero components. The matrices S_{ab} and the matrices $\Gamma_0^{(s)}, \Gamma_4^{(s)}$ which commute with them on the set of such functions can be represented in the form

$$S_{ab}\sim S_c=\begin{pmatrix}s_c & 0\\ 0 & s_c\end{pmatrix},\quad \Gamma_0^{(s)}\sim\sigma_1=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},\quad \Gamma_4^{(s)}\sim\sigma_3=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}, \tag{6.4}$$

where s_c are the generators of the representation $D(s)$ of the group $O(3)$, and I and 0 are the $(2s + 1)$-row unit and null matrix, respectively. Substituting (6.4) in (6.2), we obtain the Hamiltonian $H_s(\pi, A_0)$ in the form

$$H_s(\pi, A_0)=\sigma_1 m+k_1\sigma_3\mathbf{S}\cdot\pi+\frac{1}{2m}(\sigma_1-i\sigma_2)\left\{\pi^2-(k_1\mathbf{S}\cdot\pi)^2+\frac{e}{s}\mathbf{S}[\mathbf{H}-i(1-k_1 s)\mathbf{E}]\right\}+eA_0. \tag{6.5}$$

Equation (6.5) generalizes the Hamiltonian (2.1) for a free particle of arbitrary spin to the case of interaction with an external electromagnetic field. Thus, on the basis of the manifestly covariant equations (5.7) and (5.8) we have obtained a prescription for introducing interactions into the Poincaré invariant equations without redundant components found in Sec. 1.

We transform (6.5) to diagonal form. As in the case of the Dirac equation [16], this can be done only approximately for $\pi_\mu \ll m$. Making a series of successive transformations:

$$H_s(\pi, A_0)\to V_3V_2V_1H_s(\pi, A_0)V_1^{-1}V_2^{-1}V_3^{-1}=H_s'(\pi, A_0),\quad V_1=\exp\left(-i\sigma_2\frac{k_1\mathbf{S}\cdot\pi}{m}\right),$$

$$V_2=\exp\left\{\frac{1}{4m^2}\sigma_3\left[\pi^2-(k_1\mathbf{S}\cdot\pi)^2-\frac{e}{s}\mathbf{S}\cdot\mathbf{H}+ie\left(\frac{1}{s}-k_1\right)\mathbf{S}\cdot\mathbf{E}\right]\right\}, \tag{6.6}$$

$$V_3=\exp\left\{-\frac{i}{m^3}\left[\frac{1}{12}(k_1\mathbf{S}\cdot\pi)^3+\frac{1}{8}\left[\pi^2-(k_1\mathbf{S}\cdot\pi)^2-\frac{e}{s}\mathbf{S}\cdot\mathbf{H}+\frac{ie}{s}(1-sk_1)\mathbf{S}\cdot\mathbf{E}, \pi_0\right]_-\right]\right\}$$

and ignoring terms of order $1/m^3$ we obtain

$$H_s'(\pi, A_0)=\sigma_1\left(m+\frac{\pi^2}{2m}-\frac{e\mathbf{S}\cdot\mathbf{H}}{2sm}\right)+eA_0-\frac{e}{16s^2m^2}(\mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi}-\mathbf{S}\cdot\boldsymbol{\pi}\times\mathbf{E})-\frac{e}{24m^2s^2}\left[\frac{1}{2}Q_{ab}\frac{\partial E_a}{\partial x_b}+s(s+1)\operatorname{div}E\right]+$$

$$\frac{ie(2s-1)}{8m^2s^2}(\mathbf{S}\cdot\boldsymbol{\pi}\times\mathbf{H}-\mathbf{S}\cdot\mathbf{H}\times\boldsymbol{\pi})+\frac{e}{24m^2s^2}Q_{ab}\frac{\partial H_a}{\partial x_b}, \quad Q_{ab}=3[S_a, S_b]_+-2\delta_{ab}s(s+1). \tag{6.7}$$

On the set of functions satisfying the additional condition $\sigma_1\Phi=\Phi$ the Hamiltonian (6.7) is positive-definite and contains terms corresponding to dipole $\left(-\frac{e}{2sm}\mathbf{S}\cdot\mathbf{H}\right)$, spin-orbit $\left(-\frac{e}{16m^2s^2}(\mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi}-\mathbf{S}\cdot\boldsymbol{\pi}\times\mathbf{E})\right)$, quadrupole $\left(-\frac{e}{48s^2m^2}Q_{ab}\frac{\partial E_a}{\partial x_b}\right)$, and Darwin $\left(-\frac{e(s+1)}{24sm^2}\operatorname{div}\mathbf{E}\right)$ interactions of the particle with the field. The two last terms in (6.7) can be interpreted as magnetic spin-orbit and magnetic quadrupole interactions.

The approximate Hamiltonian (6.6) coincides with the one obtained in [20], in which the point of departure was the Zaitsev–Feynman–Gell-Mann equation (5.10). In the case $s = \frac{1}{2}$, (6.7) coincides with the Hamiltonian of Foldy and Wouthuysen [16] obtained from the Dirac equation.

7. Exact Solution of the Equations of Motion for Particles of Arbitrary Spin in a Homogeneous Magnetic Field

Let us consider the system of equations (5.5) and (5.6) for the case of a particle in a homogeneous magnetic field. Without loss of generality we can assume that the vector $\mathbf{H}$ of the strength of this field is parallel to the third projection p_3 of the particle momentum. Then the components of the electromagnetic field tensor $F_{\mu\nu}$ are

$$F_{0a}=E_a=0,\ F_{23}=H_1=0,\ F_{31}=H_2=0,\ F_{12}=H_3=H. \tag{7.1}$$

It follows from (7.1) that π_μ can be chosen in the form

$$\pi_1=p_1-eHx_2, \quad \pi_2=p_2, \quad \pi_3=p_3, \quad \pi_0=i\frac{\partial}{\partial t}. \tag{7.2}$$

Substituting (7.1) and (7.2) in (5.8), we obtain $H_s(\pi)$ in the form

$$H_s(\pi)=\Gamma_0^{(s)}\Gamma_a^{(s)}\pi_a+\Gamma_0^{(s)}m+\frac{H}{2m}\Gamma_0^{(s)}\cdot(1-\Gamma_4^{(s)})\left(i\Gamma_1^{(s)}\Gamma_2^{(s)}-\frac{1}{s}S_{12}\right) \tag{7.3}$$

We transform $H_s(\pi)$ to a form in which it contains only commuting quantities. This enables us to determine, without solving the equations of motion (5.5) and (5.6), the eigenvalue spectrum of the Hamiltonian (7.3); for as a result of the transformation

$$H_s(\pi)\rightarrow H_s'(\pi)=VH_sV^{-1},\ \hat{P}_s(\pi)\rightarrow \hat{P}_s'(\pi)=V\hat{P}_s(\pi)V^{-1}, \tag{7.4}$$

where

$$V=\lambda^++\mathscr{E}^{-1}\lambda^-\Gamma_0^{(s)}H_s(\pi), \quad \mathscr{E}=\left(\pi^2-\frac{1}{s}S_{12}H+m^2\right)^{1/2}, \quad V^{-1}=\frac{1}{m}(\lambda^-\mathscr{E}+H_s(\pi)\lambda^-\Gamma_0^{(s)}), \quad \lambda^\pm={}^1/_2(1\pm\Gamma_4^{(s)}),$$

we obtain

$$H_s'(\pi)=\Gamma_0^{(s)}\left(m^2+\pi^2-\frac{1}{s}S_{12}H\right)^{1/2}, \tag{7.5}$$

$$P_s\Phi=\Phi \text{ or } {}^1/_2S_{ab}^2\Phi=s(s+1)\Phi,\ \Phi=V\Psi. \tag{7.6}$$

The operators $\Gamma_0^{(s)}$, S_{12} and π^2 commute with one another and have the eigenvalues

$$\Gamma_0^{(s)}\Phi=\varepsilon\Phi, \quad \varepsilon=\pm1, \quad S_{12}\Phi=s_3\Phi, \quad s_3=-s, -s+1, \ldots, s, \tag{7.7}$$

$$\pi^2\Phi=[(2n+1)H+p_3^2]\Phi,\ n=0, 1, 2, \ldots . \tag{7.8}$$

Equations (7.7) follow directly from (3.4) and (7.6), and the relation (7.8) is derived, for example, in [21].

The square of the Hamiltonian (7.5) and the operators (7.7) and (7.8) have a common system of eigenfunctions $\Phi_{\varepsilon n s_3 p_3}$. Hence and from (7.7) and (7.8) we conclude that the eigenvalues of the Hamiltonian (7.5) are equal to

$$E_{\varepsilon n s_3 p_3}=\varepsilon[m^2+(2n+1-s_3/s)eH+p_3^2]^{1/2}. \tag{7.9}$$

The relation (7.9) generalizes the well-known formula [21] for the energy levels of an electron in a homogeneous magnetic field to the case of a particle with arbitrary spin s. As can be seen from (7.9), the energy values of such a particle are real for all s, whereas the Rarita–Schwinger equation for $s = {}^3/_2$ leads to complex energies when the analogous problem is solved [2].

For completeness we give the form of the eigenfunctions $\Phi_{\varepsilon n s_3 p_3}$

$$\Phi_{\varepsilon n s_3 p_3}=\Phi_\varepsilon \Phi_{s_3} \Phi_{n p_3}, \tag{7.10}$$

where Φ_{np_3} are the eigenfunctions of the operator π^2 [21]

$$\Phi_{np_3}=\exp(ip_1x_1+ip_3x_3)\exp\left[-\frac{H}{2}\left(x_2+\frac{p_1}{H}\right)\right]H_n\left[\sqrt{H}\left(x_2+\frac{p_1}{H}\right)\right], \tag{7.11}$$

H_n are Hermite polynomials, and Φ_ε, Φ_{s_3} are eigenfunctions of the operators $\Gamma_0^{(s)}$ and S_{12}, whose explicit form can be readily found for any particular representation of the matrices $\Gamma_0^{(s)}$ and S_{12}.

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