\[ \hat{g}\text{-CLOSED SETS IN IDEAL TOPOLOGICAL SPACES} \]

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Abstract. Characterizations and properties of \( I^g \)-closed sets and \( I^g \)-open sets are given. A characterization of normal spaces is given in terms of \( I^g \)-open sets. Also, it is established that an \( I^g \)-closed subset of an \( I \)-compact space is \( I \)-compact.

1. Introduction and Preliminaries

An ideal \( I \) on a topological space \((X, \tau)\) is a nonempty collection of subsets of \( X \) which satisfies (i) \( A \in I \) and \( B \subseteq A \Rightarrow B \in I \) and (ii) \( A \in I \) and \( B \in I \Rightarrow A \cup B \in I \). Given a topological space \((X, \tau)\) with an ideal \( I \) on \( X \) and if \( \varphi(X) \) is the set of all subsets of \( X \), a set operator \((.,) : \varphi(X) \rightarrow \varphi(X)\), called a local function [9] of \( A \) with respect to \( \tau \) and \( I \) is defined as follows: for \( A \subseteq X \), \( A^*(I, \tau) = \{ x \in X | U \cap A \notin I \text{ for every } U \in \tau(x) \} \) where \( \tau(x) = \{ U \in \tau | x \in U \} \). We will make use of the basic facts about the local functions [8, Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator \( \text{cl}(.) \) for a topology \( \tau^*(I, \tau) \), called the \( \ast \)-topology, finer than \( \tau \) is defined by \( \text{cl}(A) = A \cup A^*(I, \tau) \) [18]. When there is no chance for confusion, we will simply write \( A^* \) for \( A^*(I, \tau) \) and \( \tau^* \) for \( \tau^*(I, \tau) \). If \( I \) is an ideal on \( X \), then \((X, \tau, I)\) is called an ideal space. \( N \) is the ideal of all nowhere dense subsets in \((X, \tau)\). A subset \( A \) of an ideal space \((X, \tau, I)\) is \( \hat{g} \)-closed [8] (resp. \( \ast \)-dense in itself [6]) if \( A^* \subseteq A \) (resp. \( A \subseteq A^* \)). A subset \( A \) of an ideal space \((X, \tau, I)\) is \( \hat{I}^g \)-closed [3] if \( A^* \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open.

By a space, we always mean a topological space \((X, \tau)\) with no separation properties assumed. If \( A \subseteq X \), \( \text{cl}(A) \) and \( \text{int}(A) \) will, respectively, denote the closure and interior of \( A \) in \((X, \tau)\). A subset \( A \) of a space \((X, \tau)\) is an \( \alpha \)-open [15] (resp. semi-open [10], preopen [12]) set if \( A \subseteq \text{int}(\text{cl}(A)) \) (resp. \( A \subseteq \text{int}(A) \)). The family of all \( \alpha \)-open sets in \((X, \tau)\), denoted by \( \tau^\alpha \), is a topology on \( X \) finer than \( \tau \). The closure of \( A \) in \((X, \tau^\alpha)\) is denoted by \( \text{cl}_\alpha(A) \). A subset \( A \) of a space \((X, \tau)\) is said to be \( g \)-closed [11] if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open. A subset \( A \) of a space \((X, \tau)\) is said to be \( g \)-open [19] if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is semi-open. A subset \( A \) of a space \((X, \tau)\) is said to be \( \hat{g} \)-open [19] if its complement is \( \hat{g} \)-closed. The family of all \( \hat{g} \)-open sets in \((X, \tau)\) is a topology on \( X \). The semi-closure [2] of a subset \( A \) of \( X \), denoted by \( \text{sc}(A) \), is defined to be the intersection of all semi-closed sets containing \( A \). An ideal \( I \) is said to be codense [4] or \( \tau \)-boundary [14] if \( \tau \cap I = \{ \emptyset \} \). \( I \) is said to be codense if every semi-open set in \( (X, \tau) \) is \( \hat{g} \)-closed. The family of all \( \hat{g} \)-open sets in \((X, \tau)\) is a topology on \( X \). Every completely codense ideal is codense but not the converse [4]. The following Lemmas will be useful in the sequel.

**Lemma 1.1.** Let \((X, \tau, I)\) be an ideal space and \( A \subseteq X \). If \( A \subseteq A^* \), then \( A^* = \text{cl}(A^*) = \text{cl}(A) = \hat{g}(A^*) \) [17, Theorem 5].

**Lemma 1.2.** Let \((X, \tau, I)\) be an ideal space. Then \( I \) is codense if and only if \( G \subseteq G^* \) for every semi-open set \( G \) in \( X \) [17, Theorem 3].

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Lemma 1.3. Let \((X,\tau,I)\) be an ideal space. If \(I\) is completely codense, then \(\tau^* \subseteq \tau^\alpha\) [17, Theorem 6].

Result 1.4. If \((X,\tau)\) is a topological space, then every closed set is \(\hat{g}\)-closed but not conversely [1, Theorem 2.3].

Lemma 1.5. If \((X,\tau,I)\) is a T\(_2\) ideal space and \(A\) is an \(I_g\)-closed set, then \(A\) is a \(*\)-closed set [13, Corollary 2.2].

Lemma 1.6. Every \(g\)-closed set is \(I_g\)-closed but not conversely [3, Theorem 2.1].

2. \(I_g\)-CLOSED SETS

Definition 2.1. A subset \(A\) of an ideal space \((X,\tau,I)\) is said to be \(I_g\)-closed if \(A^* \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open.

Definition 2.2. A subset \(A\) of an ideal space \((X,\tau,I)\) is said to be \(I_g\)-open if \(X-A\) is \(I_g\)-closed.

Theorem 2.3. If \((X,\tau,I)\) is any ideal space, then every \(I_g\)-closed set is \(I_g\)-closed but not conversely.

Example 2.4. Let \(X=\{a,b,c\}, \tau=\{\emptyset,X,\{c\}\}\) and \(I=\{\emptyset\}\). Then \(I_g\)-closed sets are \(\emptyset,X,\{a,b\}\) and \(I_g\)-closed sets \(\emptyset,X,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\}\). It is clear that \(\{a\}\) is \(I_g\)-closed but it is not \(I_g\)-closed.

The following theorem gives characterizations of \(I_g\)-closed sets.

Theorem 2.5. If \((X,\tau,I)\) is any ideal space and \(A \subseteq X\), then the following are equivalent.

(a) \(A\) is \(I_g\)-closed.

(b) \(cl^*(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open in \(X\).

(c) For all \(x \in cl^*(A), scl(\{x\}) \cap A \neq \emptyset\).

(d) \(cl^*(A)-A\) contains no nonempty semi-closed set.

(e) \(A^*-A\) contains no nonempty semi-closed set.

Proof. (a) \(\Rightarrow\) (b) If \(A\) is \(I_g\)-closed, then \(A^* \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open in \(X\) and so \(cl^*(A)=A \cup A^* \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open in \(X\). This proves (b).

(b) \(\Rightarrow\) (c) Suppose \(x \in cl^*(A)\). If \(scl(\{x\}) \cap A = \emptyset\), then \(A \subseteq X-\text{scl}(\{x\})\). By (b), \(cl^*(A) \subseteq X-scl(\{x\})\), a contradiction, since \(x \in cl^*(A)\).

(c) \(\Rightarrow\) (d) Suppose \(F \subseteq cl^*(A)-A\), \(F\) is semi-closed and \(x \in F\). Since \(F \subseteq X-A\) and \(F\) is semi-closed, then \(A \subseteq X-F\) and \(F\) is semi-closed, \(scl(\{x\}) \cap A = \emptyset\). Since \(x \in cl^*(A)\) by (c), \(scl(\{x\}) \cap A \neq \emptyset\). Therefore \(cl^*(A)-A\) contains no nonempty semi-closed set.

(d) \(\Rightarrow\) (e) Since \(cl^*(A)-A=(A \cup A^*)-A=(A \cup A^*) \cap A^c=(A \cup A^c) \cup (A^* \cap A^c)=A^* \cap A^c=A^*-A\). Therefore \(A^*-A\) contains no nonempty semi-closed set.

(e) \(\Rightarrow\) (a) Let \(A \subseteq U\) where \(U\) is semi-open set. Therefore \(X-U \subseteq X-A\) and so \(A^* \cap (X-U) \subseteq A^* \cap (X-A)=A^*-A\). Therefore \(A^* \cap (X-U) \subseteq A^*-A\). Since \(A^*\) is always closed set, so \(A^* \cap (X-U)\) is a semi-closed set contained in \(A^*-A\). Therefore \(A^* \cap (X-U)=\emptyset\) and hence \(A^* \subseteq U\). Therefore \(A\) is \(I_g\)-closed.

Theorem 2.6. Every \(*\)-closed set is \(I_g\)-closed but not conversely.

Proof. Let \(A\) be a \(*\)-closed, then \(A^* \subseteq A\). Let \(A \subseteq U\) where \(U\) is semi-open. Hence \(A^* \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open. Therefore \(A\) is \(I_g\)-closed.

Example 2.7. Let \(X=\{a,b,c\}, \tau=\{\emptyset,X,\{a\},\{b,c\}\}\) and \(I=\{\emptyset,\{c\}\}\). Then \(I_g\)-closed sets are powerset of \(X\) and \(*\)-closed sets are \(\emptyset,X,\{a\},\{c\},\{a,c\},\{b,c\}\). It is clear that \(\{b\}\) is \(I_g\)-closed set but it is not \(*\)-closed.
Theorem 2.8. Let \((X,\tau,I)\) be an ideal space. For every \(A \in I\), \(A\) is \(I\_g\)-closed.

Proof. Let \(A \subseteq U\) where \(U\) is semi-open. Since \(A^* = \emptyset\) for every \(A \in I\), then \(cl^*(A) = A \cup A^* = A \subseteq U\). Therefore, by Theorem 2.5, \(A\) is \(I\_g\)-closed.

Theorem 2.9. If \((X,\tau,I)\) is an ideal space, then \(A^*\) is always \(I\_g\)-closed for every subset \(A\) of \(X\).

Proof. Let \(A^* \subseteq U\) where \(U\) is semi-open. Since \((A^*)^* \subseteq A^* [8]\), we have \((A^*)^* \subseteq U\) whenever \(A^* \subseteq U\) and \(U\) is semi-open. Hence \(A^*\) is \(I\_g\)-closed.

Theorem 2.10. Let \((X,\tau,I)\) be an ideal space. Then every \(I\_g\)-closed, semi-open set is \(*\)-closed set.

Proof. Since \(A\) is \(I\_g\)-closed and semi-open. Then \(A^* \subseteq A\) whenever \(A \subseteq A\) and \(A\) is semi-open. Hence \(A\) is \(*\)-closed.

Corollary 2.11. If \((X,\tau,I)\) is a \(T_I\) ideal space and \(A\) is an \(I\_g\)-closed set, then \(A\) is \(*\)-closed set.

Corollary 2.12. Let \((X,\tau,I)\) be an ideal space and \(A\) be an \(I\_g\)-closed set. Then the following are equivalent.

a) \(A\) is a \(*\)-closed set.

b) \(cl^*(A)\) is a semi-closed set.

c) \(A^* - A\) is a semi-closed set.

Proof. (a) \(\Rightarrow\) (b) If \(A\) is \(*\)-closed, then \(A^* \subseteq A\) and so \(cl^*(A) - A = (A \cup A^*) - A = \emptyset\). Hence \(cl^*(A) - A\) is semi-closed set.

(b) \(\Rightarrow\) (c) Since \(cl^*(A) - A = A^* - A\) and so \(A^* - A\) is semi-closed set.

(c) \(\Rightarrow\) (a) If \(A^* - A\) is a semi-closed set, since \(A\) is \(I\_g\)-closed set, by Theorem 2.5, \(A^* - A = \emptyset\) and so \(A\) is \(*\)-closed.

Theorem 2.13. Let \((X,\tau,I)\) be an ideal space. Then every \(\hat{g}\)-closed set is an \(I\_g\)-closed set but not conversely.

Proof. Let \(A\) be a \(\hat{g}\)-closed set. Then \(cl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open. We have \(cl^*(A) \subseteq cl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open. Hence \(A\) is \(I\_g\)-closed.

Example 2.14. Let \(X = \{a, b, c\}\), \(\tau = \{\emptyset, X, \{a\}, \{a, c\}\}\) and \(I = \{\emptyset, \{a\}\}\). Then \(I\_g\)-closed sets are \(\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\) and \(\hat{g}\)-closed sets are \(\emptyset, X, \{b\}, \{b, c\}\). It is clear that \(\{a\}\) is \(I\_g\)-closed set but it is not \(\hat{g}\)-closed.

Theorem 2.15. If \((X,\tau,I)\) is an ideal space and \(A\) is a \(*\)-dense in itself, \(I\_g\)-closed subset of \(X\), then \(A\) is \(\hat{g}\)-closed.

Proof. Suppose \(A\) is a \(*\)-dense in itself, \(I\_g\)-closed subset of \(X\). Let \(A \subseteq U\) where \(U\) is semi-open. Then by Theorem 2.5 (b), \(cl^*(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open. Since \(A\) is \(*\)-dense in itself, by Lemma 1.1, \(cl(A) = cl^*(A)\). Therefore \(cl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open. Hence \(A\) is \(\hat{g}\)-closed.

Corollary 2.16. If \((X,\tau,I)\) is any ideal space where \(I = \{\emptyset\}\), then \(A\) is \(I\_g\)-closed if and only if \(A\) is \(\hat{g}\)-closed.

Proof. From the fact that for \(I = \{\emptyset\}\), \(A^* = cl(A) \supseteq A\). Therefore \(A\) is \(*\)-dense in itself. Since \(A\) is \(I\_g\)-closed, by Theorem 2.15, \(A\) is \(\hat{g}\)-closed. Conversely, by Theorem 2.13, every \(\hat{g}\)-closed set is \(I\_g\)-closed set.

Corollary 2.17. If \((X,\tau,I)\) is any ideal space where \(I\) is codense and \(A\) is a semi-open, \(I\_g\)-closed subset of \(X\), then \(A\) is \(\hat{g}\)-closed.
Proof. By Lemma 1.2, A is \( \ast \)-dense in itself. By Theorem 2.15, A is \( \hat{g} \)-closed. \( \square \)

**Example 2.18.** Let \( X=\{a,b,c\} \), \( \tau=\{\emptyset, X, \{a, c\}\} \) and \( \mathcal{I}=\{\emptyset\} \). Then \( g \)-closed sets are \( \emptyset, X, \{a, b, c\}\) and \( I_\hat{g} \)-closed sets are \( \emptyset, X, \{b, c\}\). It is clear that \( \{a, b\}\) is \( g \)-closed set but it is not \( I_\hat{g} \)-closed.

**Example 2.19.** Let \( X=\{a,b,c\} \), \( \tau=\{\emptyset, X, \{a, c\}\} \) and \( \mathcal{I}=\{\emptyset, \{a\}\} \). Then \( g \)-closed sets are \( \emptyset, X, \{b, c\}\) and \( I_\hat{g} \)-closed sets are \( \emptyset, X, \{a\}\). It is clear that \( \{a\}\) is \( I_\hat{g} \)-closed set but it is not \( g \)-closed.

**Remark 2.20.** By Example 2.18 and Example 2.19, \( g \)-closed sets and \( I_\hat{g} \)-closed sets are independent.

**Remark 2.21.** We have the following implications for the subsets stated above.

\[
\begin{array}{ccc}
\text{closed} & \longrightarrow & \hat{g} - \text{closed} \\
\downarrow & & \downarrow \\
\ast - \text{closed} & \longrightarrow & I_\hat{g} - \text{closed}
\end{array}
\]

**Theorem 2.22.** Let \( (X, \tau, \mathcal{I}) \) be an ideal space and \( A \subseteq X \). Then A is \( I_\hat{g} \)-closed if and only if \( F=F-N \) where \( F \) is \( \ast \)-closed and \( N \) contains no nonempty semi-closed set.

**Proof.** If A is \( I_\hat{g} \)-closed, then by Theorem 2.5 (e), \( N=A^*-A \) contains no nonempty semi-closed set. If \( F=\text{cl}^\ast(A) \), then F is \( \ast \)-closed such that \( F-N=(A \cup A^*)-(A^*-A)=(A \cup A^*) \cap (A^* \cap A^*)^c=(A \cup A^*) \cap ((A^*)^c \cup A)=(A \cup A^*) \cap (A \cup (A^*)^c)=A \cup (A^* \cap (A^*)^c)=A \).

Conversely, suppose \( A=F-N \) where \( F \) is \( \ast \)-closed and \( N \) contains no nonempty semi-closed set. Let U be a semi-open set such that \( A \subseteq U \). Then \( F-N \subseteq U \Rightarrow F \cap (X-U) \subseteq N \). Now \( A \subseteq F \) and \( F^* \subseteq F \) then \( A^* \subseteq F^* \) and so \( A^* \cap (X-U) \subseteq F^* \cap (X-U) \subseteq F \cap (X-U) \subseteq N \). By hypothesis, since \( A^* \cap (X-U) \) is semi-closed, \( A^* \cap (X-U) = \emptyset \) and so \( A^* \subseteq U \). Hence A is \( I_\hat{g} \)-closed. \( \square \)

**Theorem 2.23.** Let \( (X, \tau, \mathcal{I}) \) be an ideal space and \( A \subseteq X \). If \( A \subseteq B \subseteq A^* \), then \( A^*=B^* \) and \( B \) is \( \ast \)-dense in itself.

**Proof.** Since \( A \subseteq B \), then \( A^* \subseteq B^* \) and since \( B \subseteq A^* \), then \( B^* \subseteq (A^*)^* \subseteq A^* \). Therefore \( A^*=B^* \) and \( B \subseteq A^* \subseteq B^* \). Hence proved. \( \square \)

**Theorem 2.24.** Let \( (X, \tau, \mathcal{I}) \) be an ideal space. If \( A \) and \( B \) are subsets of \( X \) such that \( A \subseteq B \subseteq \text{cl}^\ast(A) \) and \( A \) is \( I_\hat{g} \)-closed, then \( B \) is \( I_\hat{g} \)-closed.

**Proof.** Since \( A \) is \( I_\hat{g} \)-closed, then by Theorem 2.5 (d), \( \text{cl}^\ast(A)-A \) contains no nonempty semi-closed set. Since \( \text{cl}^\ast(B)-B \subseteq \text{cl}^\ast(A)-A \) and so \( \text{cl}^\ast(B)-B \) contains no nonempty semi-closed set. Hence \( B \) is \( I_\hat{g} \)-closed. \( \square \)

**Corollary 2.25.** Let \( (X, \tau, \mathcal{I}) \) be an ideal space. If \( A \) and \( B \) are subsets of \( X \) such that \( A \subseteq B \subseteq A^* \) and \( A \) is \( I_\hat{g} \)-closed, then \( A \) and \( B \) are \( \hat{g} \)-closed sets.

**Proof.** Let \( A \) and \( B \) be subsets of \( X \) such that \( A \subseteq B \subseteq A^* \Rightarrow A \subseteq B \subseteq A^* \subseteq \text{cl}^\ast(A) \) and \( A \) is \( I_\hat{g} \)-closed. By the above Theorem, \( B \) is \( I_\hat{g} \)-closed. Since \( A \subseteq B \subseteq A^* \), then \( A^*=B^* \) and so \( A \) and \( B \) are \( \ast \)-dense in itself. By Theorem 2.15, \( A \) and \( B \) are \( \hat{g} \)-closed. \( \square \)

The following theorem gives a characterization of \( I_\hat{g} \)-open sets.

**Theorem 2.26.** Let \( (X, \tau, \mathcal{I}) \) be an ideal space and \( A \subseteq X \). Then A is \( I_\hat{g} \)-open if and only if \( F \subseteq \text{int}^\ast(A) \) whenever \( F \) is semi-closed and \( F \subseteq A \).
Proof. Suppose A is $\mathcal{I}_g$-open. If F is semi-closed and $F \subseteq A$, then $X-A \subseteq X-F$ and so $\text{cl}^*(X-A) \subseteq X-F$ by Theorem 2.5 (b). Therefore $F \subseteq X-\text{cl}^*(X-A) = \text{int}^*(A)$. Hence $F \subseteq \text{int}^*(A)$.

Conversely, suppose the condition holds. Let U be a semi-open set such that $X-A \subseteq U$. Then $X-U \subseteq A$ and so $X-U \subseteq \text{int}^*(A)$. Therefore $\text{cl}^*(X-A) \subseteq U$. By Theorem 2.5 (b), $X-A$ is $\mathcal{I}_g$-closed. Hence A is $\mathcal{I}_g$-open.

□

Corollary 2.27. Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subseteq X$. If $A$ is $\mathcal{I}_g$-open, then $F \subseteq \text{int}^*(A)$ whenever $F$ is closed and $F \subseteq A$.

The following theorem gives a property of $\mathcal{I}_g$-closed.

Theorem 2.28. Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subseteq X$. If $A$ is $\mathcal{I}_g$-open and $\text{int}^*(A) \subseteq B \subseteq A$, then B is $\mathcal{I}_g$-open.

Proof. Since $A$ is $\mathcal{I}_g$-open, then $X-A$ is $\mathcal{I}_g$-closed. By Theorem 2.5 (d), $\text{cl}^*(X-A) - (X-A)$ contains no nonempty semi-closed set. Since $\text{int}^*(A) \subseteq \text{int}^*(B)$ which implies that $\text{cl}^*(X-B) \subseteq \text{cl}^*(X-A)$ and so $\text{cl}^*(X-B) - (X-B) \subseteq \text{cl}^*(X-A) - (X-A)$. Hence B is $\mathcal{I}_g$-open.

□

The following theorem gives a characterization of $\mathcal{I}_g$-closed sets in terms of $\mathcal{I}_g$-open sets.

Theorem 2.29. Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subseteq X$. Then the following are equivalent.

(a) A is $\mathcal{I}_g$-closed.
(b) $A \cup (X-A^*)$ is $\mathcal{I}_g$-closed.
(c) $A^* - A$ is $\mathcal{I}_g$-open.

Proof. (a) $\Rightarrow$ (b) Suppose A is $\mathcal{I}_g$-closed. If U is any semi-open set such that $A \cup (X-A^*) \subseteq U$, then $X-U \subseteq (A \cup (X-A^*)) = X \cap (A \cup (A^*)^c) = A^c \cap A^c = A$. Since A is $\mathcal{I}_g$-closed, by Theorem 2.5 (e), it follows that $X-U = \emptyset$ and so $X=U$. Therefore $A \cup (X-A^*) \subseteq U \Rightarrow A \subseteq X$ and so $(A \cup (X-A^*))^* \subseteq X^* \subseteq X=U$. Hence $A \subseteq (X-A^*)$ is $\mathcal{I}_g$-closed.

(b) $\Rightarrow$ (a) Suppose $A \cup (X-A^*)$ is $\mathcal{I}_g$-closed. If F is any semi-closed set such that $F \subseteq A^*$, then $F \subseteq A^*$ and $F \subseteq A \Rightarrow X-A^* \subseteq X-F$ and $A \subseteq X-F$. Therefore $A \cup (X-A^*) \subseteq A \cup (X-F) = X-F$ and $X-F$ is semi-open. Since $(A \cup (X-A^*))^* \subseteq X-F \Rightarrow A^* \cup (X-A^*)^* \subseteq X-F$ and so $A^* \subseteq X-F \Rightarrow F \subseteq X-A^*$. Since $F \subseteq A^*$, it follows that $F = \emptyset$. Hence A is $\mathcal{I}_g$-closed.

(b) $\Leftrightarrow$ (c) Since $X-(A^*-A) = X \cap (A^* \cap A)^c = X \cap ((A^*)^c \cup A) = (X \cap (A^*)^c) \cup (X \cap A) = A \cup (X-A^*)$.

□

Theorem 2.30. Let $(X,\tau,\mathcal{I})$ be an ideal space. Then every subset of X is $\mathcal{I}_g$-closed if and only if every semi-open set is $\ast$-closed.

Proof. Suppose every subset of X is $\mathcal{I}_g$-closed. If $U \subseteq X$ is semi-open, then U is $\mathcal{I}_g$-closed and so $U^* \subseteq U$. Hence U is $\ast$-closed. Conversely, suppose that every semi-open set is $\ast$-closed. If U is semi-open set such that $A \subseteq U \subseteq X$, then $A^* \subseteq U^* \subseteq U$ and so A is $\mathcal{I}_g$-closed.

□

The following theorem gives a characterization of normal spaces in terms of $\mathcal{I}_g$-open sets.
Theorem 2.31. Let \((X,\tau,\mathcal{I})\) be an ideal space where \(\mathcal{I}\) is completely codense. Then the following are equivalent.

(a) \(X\) is normal.
(b) For any disjoint closed sets \(A\) and \(B\), there exist disjoint \(\mathcal{I}_g\)-open sets \(U\) and \(V\) such that \(A \subseteq U\) and \(B \subseteq V\).
(c) For any closed set \(A\) and open set \(V\) containing \(A\), there exists an \(\mathcal{I}_g\)-open set \(U\) such that \(A \subseteq U \subseteq \text{cl}^\ast(U) \subseteq V\).

Proof. (a) \(\Rightarrow\) (b) The proof follows from the fact that every open set is \(\mathcal{I}_g\)-open.
(b) \(\Rightarrow\) (c) Suppose \(A\) is closed and \(V\) is an open set containing \(A\). Since \(A\) and \(X-V\) are disjoint closed sets, there exist disjoint \(\mathcal{I}_g\)-open sets \(U\) and \(W\) such that \(A \subseteq U\) and \(X-V \subseteq W\). Since \(X-V\) is semi-closed and \(W\) is \(\mathcal{I}_g\)-open, \(X-V \subseteq \text{int}^\ast(W)\) and so \(X-\text{int}^\ast(W) \subseteq V\). Again \(U \cap W = \emptyset \Rightarrow U \cap \text{int}^\ast(W) = \emptyset\) and so \(U \subseteq X-\text{int}^\ast(W) \Rightarrow \text{cl}^\ast(U) \subseteq X-\text{int}^\ast(W) \subseteq V\). \(U\) is the required \(\mathcal{I}_g\)-open set with \(A \subseteq U \subseteq \text{cl}^\ast(U) \subseteq V\).
(c) \(\Rightarrow\) (a) Let \(A\) and \(B\) be two disjoint closed subsets of \(X\). By hypothesis, there exists an \(\mathcal{I}_g\)-open set \(U\) such that \(A \subseteq U \subseteq \text{cl}^\ast(U) \subseteq X-B\). Since \(U\) is \(\mathcal{I}_g\)-open, \(A \subseteq \text{int}^\ast(U)\). Since \(X\) is completely codense, by Lemma 1.3, \(\tau^\ast \subseteq \tau^a\) and so \(\text{int}^\ast(U)\) and \(X-\text{cl}^\ast(U)\in \tau^a\). Hence \(A \subseteq \text{int}^\ast(U) \subseteq \text{int}(\text{cl}(\text{int}(\text{int}(U))))=G\) and \(B \subseteq X-\text{cl}^\ast(U) \subseteq \text{int}(\text{cl}(X-\text{cl}^\ast(U)))=H\). \(G\) and \(H\) are the required disjoint open sets containing \(A\) and \(B\) respectively, which proves (a). \(\square\)

A subset \(A\) of an ideal space \((X,\tau,\mathcal{I})\) is said to be an \(\alpha gs\)-closed set [16] if \(\text{cl}_{\alpha}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open. The complement of \(gs\)-closed is said to be an \(\alpha gs\)-open set. If \(\mathcal{I}=\mathcal{N}\), then \(\mathcal{I}_g\)-closed sets coincide with \(gs\)-closed sets and so we have the following Corollary.

Corollary 2.32. Let \((X,\tau,\mathcal{I})\) be an ideal space where \(\mathcal{I}=\mathcal{N}\). Then the following are equivalent.

(a) \(X\) is normal.
(b) For any disjoint closed sets \(A\) and \(B\), there exist disjoint \(\alpha gs\)-open sets \(U\) and \(V\) such that \(A \subseteq U\) and \(B \subseteq V\).
(c) For any closed set \(A\) and open set \(V\) containing \(A\), there exists an \(\alpha gs\)-open set \(U\) such that \(A \subseteq U \subseteq \text{cl}_{\alpha}(U) \subseteq V\).

A subset \(A\) of an ideal space is said to be \(I\)-compact [5] or compact modulo \(I\) [14] if for every open cover \(\{U_\alpha \mid \alpha \in \Delta\}\) of \(A\), there exists a finite subset \(\Delta_0\) of \(\Delta\) such that \(A-\cup \{U_\alpha \mid \alpha \in \Delta_0\} \in \mathcal{I}\). The space \((X,\tau,\mathcal{I})\) is \(I\)-compact if \(X\) is \(I\)-compact as a subset.

Theorem 2.33. Let \((X,\tau,\mathcal{I})\) be an ideal space. If \(A\) is an \(\mathcal{I}_g\)-closed subset of \(X\), then \(A\) is \(I\)-compact [13, Theorem 2.17].

Corollary 2.34. Let \((X,\tau,\mathcal{I})\) be an ideal space. If \(A\) is an \(\mathcal{I}_g\)-closed subset of \(X\), then \(A\) is \(I\)-compact.

Proof. The proof follows from the fact that every \(\mathcal{I}_g\)-closed set is \(\mathcal{I}_g\)-closed. \(\square\)

References


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