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CONE INEQUALITIES AND STABILITY OF DIFFERENTIAL SYSTEMS

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We investigate generalizations of classes of monotone dynamical systems in a partially ordered Banach space. We establish algebraic conditions for the stability of equilibrium states of differential systems on the basis of linearization and application of derivatives of nonlinear operators with respect to a cone. Conditions for the positivity and absolute stability of a certain class of differential systems with delay are proposed. Several illustrative examples are given.

1. Introduction and Main Definitions

In systems theory and applications, one uses continuous and discrete models of dynamical objects whose states possess certain properties with respect to a certain cone of the phase space (positivity, monotonicity, cooperativity, etc.). These properties can be induced by the nature of the considered object or by the structure of the designed control system (see, e.g., [1–4]). Classes of positive and monotone systems appear in stability theory as systems of comparison [5–7]. Several models of biological and social systems possess properties of cooperativity or competition, which are determined by using a cone of nonnegative vectors [2].

In the present paper, we investigate properties of dynamical systems that generalize the notions of positivity and monotonicity with respect to a cone. We propose a classification of these systems aimed at their application to the investigation of problems of stability. We formulate an analog of the Lyapunov theorem on the stability of equilibrium states of nonlinear differential systems with respect to the first approximation using the notion of derivatives with respect to the cone of a nonlinear operator. We establish conditions for the positivity and absolute stability of a certain class of differential systems with delay.

We now give several definitions and auxiliary facts. A convex closed set $\mathcal{K}$ of a real normed space $\mathcal{E}$ is called a cone if $\mathcal{K} \cap -\mathcal{K} = \{0\}$ and $\alpha\mathcal{K} + \beta\mathcal{K} \subset \mathcal{K}$ $\forall \alpha, \beta \geq 0$. A space with cone is partially ordered: $X \overset{\mathcal{K}}{\leq} Y \Longleftrightarrow Y - X \in \mathcal{K}$. A cone $\mathcal{K}$ with nonempty set of interior points $\operatorname{int}\mathcal{K} = \{X\colon X \overset{\mathcal{K}}{>} 0\}$ is a solid cone. Nonzero elements $X \in \mathcal{K}$ are denoted by $X \overset{\mathcal{K}}{\succ} 0$. A cone $\mathcal{K}$ is called normal if the relation $0 \overset{\mathcal{K}}{\leq} X \overset{\mathcal{K}}{\leq} Y$ yields the estimate $\|X\| \leq \nu\|Y\|$, where ν is a universal constant. The least of these numbers ν is called the normality constant of the cone $\mathcal{K}$. If $\mathcal{E} = \mathcal{K} - \mathcal{K}$, then the cone $\mathcal{K}$ is reproducing.

Typical examples of normal reproducing cones in finite-dimensional spaces are a set of vectors with nonnegative elements, a circular Minkowski cone, a set of symmetric nonnegative-definite matrices, etc.

A set $\mathcal{D} \subset \mathcal{E}$ is called $\mathcal{K}$-convex if, for any pair of points $X, Y \in \mathcal{D}$, the relation $X \overset{\mathcal{K}}{\leq} Y$ implies that $(1-\gamma)X + \gamma Y \in \mathcal{D}$ for $\gamma \in (0,1)$. Any cone $\mathcal{K}$ and any convex set are $\mathcal{K}$-convex.

The dual cone $\mathcal{K}^*$ consists of linear functionals $\varphi \in \mathcal{E}^*$ that take nonnegative values on $\mathcal{K}$, and, furthermore, $\mathcal{K} = \big\{X \in \mathcal{E}\colon \varphi(X) \geq 0 \ \ \forall \varphi \in \mathcal{K}^*\big\}$. A functional $\varphi \in \mathcal{E}^*$ is called uniformly positive if there exists $\gamma > 0$ such that $\varphi(X) \geq \gamma\|X\|$ for all $x \in \mathcal{K}$. A uniformly positive functional $\varphi \in \mathcal{E}^*$ is strictly positive: $\varphi(X) > 0$ for $X \overset{\mathcal{K}}{\succ} 0$. A cone $\mathcal{K}$ admits a "plastering" $\mathcal{K}_0$ if every point $X \in \mathcal{K}$ is contained in the cone $\mathcal{K}_0$ together with

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the ball neighborhood $\|Y - X\| \le \delta\|X\|$, where $\delta > 0$ does not depend on X. The plastering of the cone $\mathcal{K}$ is equivalent to the fact that the cone $\mathcal{K}^*$ is solid. A cone $\mathcal{K}$ is normal (reproducing) if and only if the dual cone $\mathcal{K}^*$ is reproducing (normal).

Assume that a cone $\mathcal{K}_1$ $(\mathcal{K}_2)$ is selected in a Banach space $\mathcal{E}_1$ $(\mathcal{E}_2)$. An operator $M\colon \mathcal{E}_1 \to \mathcal{E}_2$ is called monotone if $X \overset{\mathcal{K}_1}{\le} Y \implies MX \overset{\mathcal{K}_2}{\le} MY$. The monotonicity of a linear operator M is equivalent to its positivity: $M\mathcal{K}_1 \subseteq \mathcal{K}_2$. If an operator M is positive, then the adjoint operator $M^*\colon \mathcal{E}_2^* \to \mathcal{E}_1^*$ is also positive $(M^*\mathcal{K}_2 \subseteq \mathcal{K}_1^*)$. If $M\mathcal{E}_1 \subseteq \mathcal{K}_2$, then the operator M is everywhere positive. A linear operator M is called positive invertible if $\mathcal{K}_2 \subseteq M\mathcal{K}_1$, i.e., for any $Y \in \mathcal{K}_2$, the equation $MX = Y$ has a solution $X \in \mathcal{K}_1$. Since $(M^{-1})^* = (M^*)^{-1}$, the positive invertibility of the operator M leads to the positive invertibility of the operator M^*. If $\mathcal{K}_2$ is a normal reproducing cone and $M_1 \le M \le M_2$, then the positive invertibility of operators M_1 and M_2 yields the positive invertibility of the operator M, and, furthermore, $M_2^{-1} \le M^{-1} \le M_1^{-1}$ [1].

A criterion for the positive invertibility of operators of the form $M = L - P$, where $P\mathcal{K}_1 \subseteq \mathcal{K}_2 \subseteq L\mathcal{K}_1$ and $\mathcal{K}_2$ is a normal reproducing cone, is the inequality $\rho(T) < 1$, where $\rho(T)$ is the spectral radius of the pencil of operators $T(\lambda) = P - \lambda L$ [8]. If the cone $\mathcal{K}_2$ is solid, then this inequality is equivalent to the existence of elements $X \overset{\mathcal{K}_1}{\ge} 0$ and $Y \overset{\mathcal{K}_2}{>} 0$ satisfying the equation $MX = Y$.

A linear bounded operator $F'(X_0)$ is called the Gâteaux derivative of a nonlinear operator $F\colon \mathcal{E}_1 \to \mathcal{E}_2$ at a point X_0 if the limit

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\,[F(X_0 + \varepsilon H) - FX_0] = F'(X_0)H$$

exists in the sense of strong convergence (in norm). If this relation holds only for $H \overset{\mathcal{K}_1}{\ge} 0$, then $F'(X_0)$ is the Fréchet derivative with respect to the cone $\mathcal{K}_1$ of the operator F [9]. The Fréchet derivative $F'(X_0)$ with respect to the cone $\mathcal{K}_1$ is defined by the relations

$$F(X_0 + H) - FX_0 = F'(X_0)H + R(X_0, H), \quad R(X_0, H) = o(\|H\|),$$

where $H \overset{\mathcal{K}_1}{\ge} 0$. The Fréchet derivative is also the Gâteaux derivative. If the Gâteaux derivative is continuous in the neighborhood of the point X_0, then it is the Fréchet derivative. In what follows, $F'_+(X_0)$ $(F'_-(X_0))$ denotes the Gâteaux and Fréchet derivatives with respect to the cone $\mathcal{K}_1$ $(-\mathcal{K}_1)$ at the point X_0.

2. Classification of Dynamical Systems with Respect to a Cone

Assume that a continuous (or discrete) dynamical system operates in a certain domain $\mathcal{D}$ of the Banach space $\mathcal{E}$ and its states are defined as follows:

$$X_t = \Phi(X_\tau, \tau, t) \in \mathcal{E}, \quad t \ge \tau \ge 0, \tag{2.1}$$

where Φ is the operator that determines the transition from the initial state X_τ to the state X_t and possesses the properties

$$\Phi(X_\tau, \tau, \tau) = X_\tau, \qquad \Phi(X_\tau, \tau, t + \tau) = \Phi(X_t, t, t + \tau).$$

If $\Phi(\Theta, \tau, t) = \Theta$, then $X_t \equiv \Theta$ is an equilibrium state (stationary motion) of system (2.1).

Let a set $\mathcal{K}_t$, $t \ge 0$, constant or varying according to a given law, be defined in the space $\mathcal{E}$. The set $\mathcal{K}_t$ is an invariant set of system (2.1) if $\Phi(\mathcal{K}_\tau, \tau, t) \subset \mathcal{K}_t$, i.e., $X_t \in \mathcal{K}_t$ for $t > \tau \ge 0$, provided that $X_\tau \in \mathcal{K}_\tau$. If system (2.1) has an invariant cone $\mathcal{K}_t$, then it is positive with respect to $\mathcal{K}_t$. System (2.1) is called monotone with respect to the cone $\mathcal{K}_t$ if, for any $\tau \geq 0$, one has

$$X_\tau \overset{\mathcal{K}_\tau}{\leq} Y_\tau \implies X_t \overset{\mathcal{K}_t}{\leq} Y_t, \quad t > \tau, \tag{2.2}$$

where $X_t = \Phi(X_\tau, \tau, t)$ and $Y_t = \Phi(Y_\tau, \tau, t)$. The positive (monotone) dynamical system (2.1) is defined by the operator Φ positive (monotone) with respect to the cone $\mathcal{K}_t$. Denote the classes of systems monotone and positive with respect to the cone $\mathcal{K}_t$ $(-\mathcal{K}_t)$ by $\mathcal{M}$ and $\mathcal{M}_0^+$ $(\mathcal{M}_0^-)$, respectively.

Consider the sets

$$\mathcal{K}_t^+(\Theta) = \{X \in \mathcal{E} \colon X \overset{\mathcal{K}_t}{\geq} \Theta\}, \qquad \mathcal{K}_t^-(\Theta) = \{X \in \mathcal{E} \colon X \overset{\mathcal{K}_t}{\leq} \Theta\},$$

where $\Theta \in \mathcal{E}$ and $\mathcal{K}_t$ is a certain cone. It is obvious that $\mathcal{K}_t^\pm(\Theta) = \Theta \pm \mathcal{K}_t$ and $\mathcal{K}_t^\pm(0) = \pm\mathcal{K}_t$. For the class of systems that have the invariant set $\mathcal{K}_t^+(\Theta)$ $(\mathcal{K}_t^-(\Theta))$, we use the notation $\mathcal{M}_0^+(\Theta)$ $(\mathcal{M}_0^-(\Theta))$. Denote the classes of functions with property (2.2) and additional requirements $Y_\tau \in \mathcal{K}_\tau^+(\Theta)$, $X_\tau \in \mathcal{K}_\tau^+(\Theta)$, $X_\tau \in \mathcal{K}_\tau^-(\Theta)$, and $Y_\tau \in \mathcal{K}_\tau^-(\Theta)$ by $\mathcal{M}_1^+(\Theta)$, $\mathcal{M}_2^+(\Theta)$, $\mathcal{M}_1^-(\Theta)$, and $\mathcal{M}_2^-(\Theta)$, respectively. A system of the class $\mathcal{M}_2^+(\Theta)$ $(\mathcal{M}_2^-(\Theta))$ is monotone in $\mathcal{K}_t^+(\Theta)$ $(\mathcal{K}_t^-(\Theta))$.

Denote $\mathcal{M}_k(\Theta) = \mathcal{M}_k^+(\Theta) \cap \mathcal{M}_k^-(\Theta)$, $k = 0, 1, 2$. Note that $\mathcal{M} \subseteq \mathcal{M}_1^\pm(\Theta) \subseteq \mathcal{M}_2^\pm(\Theta)$ and $\mathcal{M} \subseteq \mathcal{M}_1(\Theta) \subseteq \mathcal{M}_2(\Theta)$.

Let $\Phi'_+(\Theta, \tau, t)$ $(\Phi'_-(\Theta, \tau, t))$ be the Gâteaux derivative with respect to the cone $\mathcal{K}_\tau$ $(-\mathcal{K}_\tau)$ of the operator $\Phi(\Theta, \tau, t)$ at the point $\Theta \in \mathcal{D}$. For the classes of systems $\mathcal{M}_0^+(\Theta)$ and $\mathcal{M}_0^-(\Theta)$ that have the equilibrium state $X_t \equiv \Theta$, the following inclusions are true:

$$\Phi'_+(\Theta, \tau, t)\mathcal{K}_\tau \subseteq \mathcal{K}_t, \quad \Theta \in \mathcal{D}, \quad t \geq \tau, \tag{2.3}$$

$$\Phi'_-(\Theta, \tau, t)\mathcal{K}_\tau \subseteq \mathcal{K}_t, \quad \Theta \in \mathcal{D}, \quad t \geq \tau. \tag{2.4}$$

Lemma 2.1. *Suppose that the operator $\Phi(\Theta, \tau, t)$ is Gâteaux differentiable with respect to the cone $\mathcal{K}_\tau$ $(-\mathcal{K}_\tau)$ at every point of the $\mathcal{K}_\tau$-convex domain $\mathcal{D}$ for $t \geq \tau$. System (2.1) is monotone with respect to $\mathcal{K}_t$ if and only if conditions (2.3) [(2.4)] are satisfied.*

Proof. Assume that the Gâteaux derivative $\Phi'_+(\Theta, \tau, t)$ exists at every point $\Theta \in \mathcal{D}$. Then, using the definition of the monotonicity of a system, we obtain

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [\Phi(\Theta + \varepsilon H, \tau, t) - \Phi(\Theta, \tau, t)] = \Phi'_+(\Theta, \tau, t)H \overset{\mathcal{K}_t}{\geq} 0,$$

where $H \overset{\mathcal{K}_\tau}{\geq} 0$ and $\Theta \in \mathcal{D}$, which yields inclusion (2.3).

The sufficiency is obtained by using the Lagrange formula for the operator $\Phi(\Theta, \tau, t)$:

$$\varphi\left(\Phi(\Theta + H, \tau, t) - \Phi(\Theta, \tau, t)\right) = \varphi(\Phi'_+(\Theta + \gamma H, \tau, t)H),$$

where $0 < \gamma = \gamma(\varphi) < 1$, $\varphi \in \mathcal{E}^*$, and Θ and $\Theta + H$ are arbitrary points of a certain convex set. For this purpose, we use only positive functionals $\varphi \in \mathcal{K}_t^*$ and the property of $\mathcal{K}_\tau$-convexity of the domain $\mathcal{D}$. Moreover, $\Theta + \gamma H = (1 - \gamma)\Theta + \gamma(\Theta + H) \in \mathcal{D}$ if $\Theta \in \mathcal{D}$ and $H \in \mathcal{K}_\tau$.

By analogy, we establish the criterion for the monotonicity of a system of the form (2.4).

The lemma is proved.

Remark 2.1. One can establish that inclusion (2.3) [(2.4)], which is true for $\Theta \in \mathcal{K}_\tau \quad (\Theta \in -\mathcal{K}_\tau)$, can serve as a criterion for system (2.1) to belong to the class $\mathcal{M}_2^+(\Theta) \quad (\mathcal{M}_2^-(\Theta))$. For a system of the class $\mathcal{M}_1^+(\Theta)$ $(\mathcal{M}_1^-(\Theta),)$, both inclusions (2.3) and (2.4) hold for $\Theta \in \mathcal{K}_\tau \quad (\Theta \in -\mathcal{K}_\tau)$.

3. Differential Systems

Consider the nonlinear differential system

$$\dot{X} = F(X,t), \quad t \geq \tau \geq 0, \tag{3.1}$$

where $F(X,t)$ is a continuous operator function that guarantees the existence and uniqueness of a continuously differentiable solution $X(t) = \Phi(X_\tau, \tau, t),\ t \geq \tau,$ for any $\tau \geq 0,\ X_\tau \in \mathcal{D},$ and $\mathcal{D} \subset \mathcal{E}$ is a certain $\mathcal{K}_\tau$-convex domain.

For system (3.1), we introduce the following conditions:

$$X \overset{\mathcal{K}_t}{\geq} \Theta, \quad \varphi \in \mathcal{K}_t^*, \quad \varphi(X - \Theta) = 0 \implies \varphi\left(F(X,t)\right) \geq 0, \tag{3.2}$$

$$X \overset{\mathcal{K}_t}{\leq} Y, \quad \varphi \in \mathcal{K}_t^*, \quad \varphi(X - Y) = 0 \implies \varphi\left(F(X,t) - F(Y,t)\right) \leq 0, \tag{3.3}$$

where $t \geq 0$ and $\mathcal{K}_t^*$ is the dual cone. Let $\mathcal{F}_0^+(\Theta) \quad (\mathcal{F}_0^-(\Theta))$ denote the class of operator functions $F(X,t)$ that satisfy conditions of the type (3.2) with respect to the cone $\mathcal{K}_t \quad (-\mathcal{K}_t)$. Let $\mathcal{F}$ denote the family of operator functions that possess property (3.3). We also define the families of operator functions $\mathcal{F}_1^+(\Theta),\ \mathcal{F}_2^+(\Theta),\ \mathcal{F}_1^-(\Theta),$ and $\mathcal{F}_2^-(\Theta)$ that possess property (3.3) under the additional requirements $Y \in \mathcal{K}_t^+(\Theta),\ X \in \mathcal{K}_t^+(\Theta),\ X \in \mathcal{K}_t^-(\Theta),$ and $Y \in \mathcal{K}_t^-(\Theta),$ respectively. Denote $\mathcal{F}_k(\Theta) = \mathcal{F}_k^+(\Theta) \cap \mathcal{F}_k^-(\Theta),\ k = 0, 1, 2.$

It is obvious that $\mathcal{F},\ \mathcal{F}_0^\pm(\Theta),\ \mathcal{F}_1^\pm(\Theta),$ and $\mathcal{F}_2^\pm(\Theta)$ are wedges, and, furthermore, $\mathcal{F} \subseteq \mathcal{F}_1^\pm(\Theta) \subseteq \mathcal{F}_2^\pm(\Theta)$ and $\mathcal{F} \subseteq \mathcal{F}_1(\Theta) \subseteq \mathcal{F}_2(\Theta)$.

Lemma 3.1. *Let $\mathcal{K}_t$ be a solid cone possessing the property*

$$0 \leq \tau < t \implies \mathcal{K}_\tau \subseteq \mathcal{K}_t. \tag{3.4}$$

Then the following assertions are true:

(i) if $F \in \mathcal{F}_0^\pm(\Theta)$, then $\mathcal{K}_t^\pm(\Theta)$ is an invariant set of system (3.1);

(ii) if $F \in \mathcal{F}$, then system (3.1) is monotone with respect to $\mathcal{K}_t$;

(iii) if $F \in \mathcal{F}_k^\pm(\Theta)$, then $\mathcal{M}_0^\pm(\Theta) \subseteq \mathcal{M}_k^\pm(\Theta),\ k = 1, 2;$

(iv) if $F \in \mathcal{F}_k^\pm(\Theta)$ and $F(\Theta, t) \equiv 0$, then (3.1) is a system of the class $\mathcal{M}_k^\pm(\Theta),\ k = 1, 2.$

Proof. The proof of assertion (i) of the lemma is analogous to its proof in the case $\Theta = 0$ [7].

Parallel with (3.1), we consider the system

$$\dot{Y} = F(Y,t) + \varepsilon Q, \quad t \geq \tau \geq 0, \tag{3.5}$$

where $\varepsilon > 0$ and $Q \overset{\mathcal{K}_0}{>} 0$. If (3.1) is a system of the class $\mathcal{M}_0^{\pm}(\Theta)$, then system (3.5) is also a system of this class. Let $X(t)$ and $Y(t)$ be solutions of the corresponding systems (3.1) and (3.5) such that $X(0) \overset{\mathcal{K}_0}{\leq} Y(0)$, $X(\tau) \overset{\mathcal{K}_\tau}{\leq} Y(\tau)$, and $\varphi(X(\tau)) = \varphi(Y(\tau))$ for certain $\tau \geq 0$ and $\varphi \in \mathcal{K}_\tau^*$, $\varphi \neq 0$. If condition (3.4) is satisfied and $F \in \mathcal{F}$, then the following inequalities hold for sufficiently small $\delta > 0$:

$$\varphi(\dot{X}(\tau) - \dot{Y}(\tau)) = \varphi(F(X(\tau),\tau) - F(Y(\tau),\tau)) - \varepsilon\, \varphi(Q) < 0,$$

$$\int_{\tau}^{\tau+\delta} \varphi\left(\dot{X}(t) - \dot{Y}(t)\right) dt = \varphi\left(X(\tau+\delta) - Y(\tau+\delta)\right) \leq 0.$$

This means that, at time τ, the value of the function $Y(\tau+\delta) - X(\tau+\delta)$ cannot leave the cone $\mathcal{K}_\tau$. By virtue of the continuous dependence of the solution Y on ε as $\varepsilon \to 0$, we can state that system (3.1) is monotone with respect to $\mathcal{K}_t$ [assertion (ii)].

The proof of assertion (iii) is analogous to the proof of assertions (i) and (ii). If $Y(0) \overset{\mathcal{K}_0}{\geq} 0$ and $F \in \mathcal{F}_1^{+}(\Theta)$ ($X(0) \overset{\mathcal{K}_0}{\geq} 0$ and $F \in \mathcal{F}_2^{+}(\Theta)$), then $\mathcal{M}_0^{+}(\Theta) \subseteq \mathcal{M}_1^{+}(\Theta)$ ($\mathcal{M}_0^{+}(\Theta) \subseteq \mathcal{M}_2^{+}(\Theta)$). By analogy, for $X(0) \overset{\mathcal{K}_0}{\leq} 0$ and $F \in \mathcal{F}_1^{-}(\Theta)$ ($Y(0) \overset{\mathcal{K}_0}{\leq} 0$ and $F \in \mathcal{F}_2^{-}(\Theta)$), we have $\mathcal{M}_0^{-}(\Theta) \subseteq \mathcal{M}_1^{-}(\Theta)$ ($\mathcal{M}_0^{-}(\Theta) \subseteq \mathcal{M}_2^{-}(\Theta)$).

Assertion (iv) is a corollary of assertions (i) and (iii). Indeed, if $F(\Theta,t) \equiv 0$, then the inclusion $F \in \mathcal{F}_1^{\pm}(\Theta)$ ($F \in \mathcal{F}_2^{\pm}(\Theta)$) implies that $F \in \mathcal{F}_0(\Theta)$ ($F \in \mathcal{F}_0^{\pm}(\Theta)$).

The lemma is proved.

Note that, in the case of a constant cone $\mathcal{K}_t = \mathcal{K}$, one can formulate assertions converse to assertions (i)–(iv) of Lemma 3.1 [10].

In [11], a method was proposed for the construction of invariant sets of differential systems in the form

$$\mathcal{I}_t = \left\{ X \in \mathcal{E} \colon V(X,t) \overset{\mathcal{K}}{\geq} 0 \right\}, \tag{3.6}$$

where $V \colon \mathcal{E} \times [0,\infty) \to \mathcal{E}_1$ is a certain operator function and $\overset{\mathcal{K}}{\geq}$ is the inequality defined by a given cone $\mathcal{K}$ in the space $\mathcal{E}_1$.

Theorem 3.1. *Let $\mathcal{K}$ be a solid cone. Then (3.6) is an invariant set of system (3.1) if and only if, for every $t \geq 0$, the following condition is satisfied:*

$$X \in \mathcal{I}_t, \quad \varphi \in \mathcal{K}^*, \quad \varphi\left(V(X,t)\right) = 0 \implies \varphi\left(D_t V(X,t)\right) \geq 0, \tag{3.7}$$

where D_t is the operator of differentiation along solutions of system (3.1).

If $V(X,t)$ is an operator function continuous together with its partial derivatives, then, in (3.7), the operator $D_t,$ as the strong derivative of a compound function, is determined as follows:

$$D_tV(X,t) = V'_X(X,t)\,F(X,t) + V'_t(X,t), \tag{3.8}$$

where $V'_t(X,t)$ is the strong derivative of a function with respect to t and $V'_X(X,t)$ is the Gâteaux derivative with respect to X. If $V(X,t)$ is a continuous locally Lipschitzian function with respect to $X,$ then derivatives along solutions of a Dini-type system can be taken as D_t (see, e.g., [5, 6]).

Remark 3.1. Condition (3.7) is satisfied if, for a certain continuous scalar function $\alpha(X,t),$ the cone inequality

$$D_tV(X,t) + \alpha(X,t)V(X,t) \overset{\mathcal{K}}{\geq} 0, \qquad X \in \partial\mathcal{I}_t, \quad t \geq 0, \tag{3.9}$$

is true.

Parallel with (3.1), we consider the linear systems

$$\dot{H} = F'_+(X,t)H, \qquad X \in \mathcal{D}, \quad t \geq \tau \geq 0, \tag{3.10}$$

$$\dot{H} = F'_-(X,t)H, \qquad X \in \mathcal{D}, \quad t \geq \tau \geq 0, \tag{3.11}$$

where $F'_+(X,t)$ $(F'_-(X,t))$ is the Gâteaux derivative with respect to the cone $\mathcal{K}_t$ $(-\mathcal{K}_t)$ of the operator $F(X,t)$. We call system (3.1) cooperative with respect to $\mathcal{K}_t$ $(-\mathcal{K}_t)$ if the right-hand sides of the family of systems (3.10) [(3.11)] are operators of the class $\mathcal{F}_0^+(0)$ $(\mathcal{F}_0^-(0))$. Under the conditions of Lemma 3.1, each system (3.10) [(3.11)] corresponding to the cooperative system (3.1) with respect to $\mathcal{K}_t$ $(-\mathcal{K}_t)$ has the invariant cones $\mathcal{K}_t$ $(-\mathcal{K}_t)$. System (3.1) is a model of competition relative to $\mathcal{K}_t$ $(-\mathcal{K}_t)$ if the reverse system

$$\dot{X} = -F(X,-t), \quad t \geq \tau \geq 0,$$

is cooperative with respect to $\mathcal{K}_t$ $(-\mathcal{K}_t)$.

The following statement is true [12]:

Theorem 3.2. *Suppose that the operator $F(X,t)$ is Gâteaux differentiable with respect to the cone $\mathcal{K}_t$ $(-\mathcal{K}_t)$ at every point X of the $\mathcal{K}_t$-convex domain $\mathcal{D}$ for $t \geq 0$. System (3.1) is cooperative with respect to $\mathcal{K}_t$ $(-\mathcal{K}_t)$ if and only if $F \in \mathcal{F}$.*

Corollary 3.1. *Under the conditions of Lemma 3.1, system (3.1) is monotone if it is cooperative with respect to $\mathcal{K}_t$ or $-\mathcal{K}_t$. If the cone $\mathcal{K}_t$ is constant, then the converse statement is also true.*

Remark 3.2. It can be established that the right-hand side of system (3.10) [(3.11)] for $X \in \mathcal{K}_t^+(\Theta)$ $(X \in \mathcal{K}_t^-(\Theta))$ is a function of the class $\mathcal{F}_0^+(\Theta)$ $(\mathcal{F}_0^-(\Theta))$ if and only if $F \in \mathcal{F}_2^+(\Theta)$ $(F \in \mathcal{F}_2^-(\Theta))$. If $F \in \mathcal{F}_1^+(\Theta)$ $(F \in \mathcal{F}_1^-(\Theta)),$ then the right-hand sides of both systems (3.10) and (3.11) are functions of the class $\mathcal{F}_0^+(\Theta)$ $(\mathcal{F}_0^-(\Theta))$ for $X \in \mathcal{K}_t^+(\Theta)$ $(X \in \mathcal{K}_t^-(\Theta))$.

4. Stability of Equilibrium States of Systems of the Class $\mathcal{M}_1(\Theta)$

Consider the differential system (3.1) with isolated equilibrium state $X \equiv \Theta$:

$$\dot{X} = F(X, t), \qquad F(\Theta, t) \equiv 0, \quad t \geq \tau \geq 0. \tag{4.1}$$

Assume that $\mathcal{K}_t$ is a normal reproducing cone with bounded normality constant $\nu_t \leq \nu$.

We call the state $X \equiv \Theta$ of system (4.1) stable in $\mathcal{K}_t^+(\Theta)$ if, for any $\varepsilon > 0$ and $\tau \geq 0$, there exists $\delta > 0$ such that the fact that $X(\tau)$ belongs to the set $\mathcal{S}_\delta(\tau)$ implies that $X(t) \in \mathcal{S}_\varepsilon(t)$ for $t > \tau$, where $\mathcal{S}_\varepsilon(t) = \{X \in \mathcal{K}_t^+(\Theta)\colon \|X - \Theta\| \leq \varepsilon\}$. If, for a certain $\delta_\tau > 0$, the inclusion $X(\tau) \in \mathcal{S}_{\delta_\tau}(\tau)$ implies that $\|X(t) - \Theta\| \to 0$ as $t \to \infty$, then the state $X \equiv 0$ of the system is asymptotically stable in $\mathcal{K}_t^+(\Theta)$.

If the state $X \equiv 0$ of system (4.1) with invariant set $\mathcal{K}_t^+(\Theta)$ is Lyapunov stable (asymptotically stable), then it is stable (asymptotically stable) in $\mathcal{K}_t^+(\Theta)$.

We now prove the following statement, which was formulated in [7] for $\Theta = 0$:

Lemma 4.1. *The state $X \equiv \Theta$ of system (4.1) of the class $\mathcal{M}_1(\Theta)$ with respect to the normal reproducing cone $\mathcal{K}_t$ is Lyapunov stable (asymptotically stable) if and only if it is stable (asymptotically stable) in $\mathcal{K}_t^+(\Theta)$ and $\mathcal{K}_t^-(\Theta)$.*

Proof. For a system of the class $\mathcal{M}_1(\Theta)$, both sets $\mathcal{K}_t^+(\Theta)$ and $\mathcal{K}_t^-(\Theta)$ are invariant. Therefore, the Lyapunov stability (asymptotic stability) of the state $X \equiv \Theta$ yields its stability (asymptotic stability) in $\mathcal{K}_t^+(\Theta)$ and $\mathcal{K}_t^-(\Theta)$.

Assume that the state $X \equiv \Theta$ of system (4.1) is stable in $\mathcal{K}_t^+(\Theta)$ and $\mathcal{K}_t^-(\Theta)$. Consider an arbitrary solution $X(t)$ of it with initial condition $X(\tau) = X_\tau$. Since the cone $\mathcal{K}_\tau$ is reproducing and unflattened, the following relations hold for certain $X_\tau^\pm \in \mathcal{K}_\tau^\pm(\Theta)$ and γ_τ:

$$X_\tau^- \overset{\mathcal{K}_\tau}{\leq} X_\tau \overset{\mathcal{K}_\tau}{\leq} X_\tau^+, \qquad \|X_\tau^\pm - \Theta\| \leq \gamma_\tau \|X_\tau - \Theta\|.$$

Using the definitions of the classes $\mathcal{M}_1^\pm(\Theta)$, we obtain

$$X_-(t) \overset{\mathcal{K}_t}{\leq} X(t) \overset{\mathcal{K}_t}{\leq} X_+(t), \quad t \geq \tau,$$

where $X_\pm(t)$ are solutions of the system with the corresponding initial conditions $X_\pm(\tau) = X_\tau^\pm$.

By virtue of the stability of the state $X \equiv \Theta$ of the system in $\mathcal{K}_t^\pm(\Theta)$, for any $\varepsilon > 0$ we choose $\delta_\pm > 0$ so that the inequality $\|X_0^\pm - \Theta\| \leq \delta_\pm$ yields $\|X_\pm(t) - \Theta\| \leq \varepsilon(2\nu + 1)^{-1}$ for $t > \tau$. Then

$$\|X(t) - \Theta\| \leq (\nu_t + 1)\|X_-(t) - \Theta\| + \nu_t \|X_+(t) - \Theta\| \leq \varepsilon,$$

provided that $\|X_\tau\| \leq \delta = \gamma_\tau^{-1} \min\{\delta_+, \delta_-\}$. Here, $0 < \nu_t \leq \nu$ $(\gamma_\tau > 0)$ is the constant of normality (unflattenedness) of the cone $\mathcal{K}_t$ $(\mathcal{K}_\tau)$. Moreover, $\|X(t) - \Theta\| \to 0$ if $\|X_\pm(t) - \Theta\| \to 0$, $t \to \infty$, i.e., the state $X \equiv \Theta$ of system (4.1) is Lyapunov asymptotically stable.

The lemma is proved.

Lemma 4.2. *Suppose that the normal solid cone $\mathcal{K}_t$ satisfies condition (3.4) and is an invariant set of the linear system*

$$\dot{X} = MX, \quad t \geq \tau \geq 0. \tag{4.2}$$

Also assume that the system of inequalities

$$Z \overset{\mathcal{K}_\tau}{>} 0, \qquad -MZ = Y \overset{\mathcal{K}_\tau}{>} 0 \tag{4.3}$$

is consistent. Then system (4.2) is asymptotically stable.

Proof. Consider an arbitrary solution $X(t) = e^{M(t-\tau)}X_\tau$. First, let $X_\tau \overset{\mathcal{K}_\tau}{\geq} 0$. Since Z and Y are interior points of the cone $\mathcal{K}_\tau$ and $\mathcal{K}_t$ is an invariant set of system (4.2), the following inequalities hold for a certain $\alpha > 0$:

$$Y \overset{\mathcal{K}_\tau}{\geq} \alpha Z \overset{\mathcal{K}_\tau}{\geq} \alpha^2 X_\tau, \qquad 0 \overset{\mathcal{K}_t}{\leq} X(t) \overset{\mathcal{K}_t}{\leq} \alpha^{-1} Z(t),$$

where the function $Z(t) = e^{M(t-\tau)}Z$ satisfies the relations

$$\dot{Z}(t) + \alpha Z(t) = G(t), \qquad G(t) = e^{M(t-\tau)}(-Y + \alpha Z) \overset{\mathcal{K}_t}{\leq} 0,$$

$$0 \overset{\mathcal{K}_t}{\leq} Z(t) = e^{-\alpha(t-\tau)}Z + \int_\tau^t e^{-\alpha(t-s)}G(s)ds \overset{\mathcal{K}_t}{\leq} e^{-\alpha(t-\tau)}Z.$$

In the last inequality, property (3.4) of the cone $\mathcal{K}_t$ has been used. It follows from the normality of this cone that

$$\|Z(t)\| \leq \nu e^{-\alpha(t-\tau)}\|Z\|, \qquad \|X(t)\| \leq \alpha^{-1}\nu^2 e^{-\alpha(t-\tau)}\|Z\|.$$

In the general case $X_\tau = X_\tau^+ - X_\tau^-$, where $X_\tau^\pm \in \mathcal{K}_\tau$, we obtain an analogous inequality:

$$\|X(t)\| \leq \nu^2 \left(\alpha_+^{-1} e^{-\alpha_+(t-\tau)} + \alpha_-^{-1} e^{-\alpha_-(t-\tau)}\right)\|Z\|,$$

where $\alpha_\pm > 0$ are certain numbers.

Therefore, every solution of system (4.2) is bounded and tends to zero as $t \to \infty$, which is equivalent to the asymptotic stability of this system.

The lemma is proved.

Let us formulate conditions for the asymptotic stability of the equilibrium state $X \equiv \Theta$ of the nonlinear autonomous system

$$\dot{X} = F(X), \qquad F(\Theta) = 0, \quad t \geq \tau \geq 0. \tag{4.4}$$

Assume that the cone $\mathcal{K}$ is constant and there exist the Fréchet derivatives $F'_\pm(X)$ with respect to the cones $\pm\mathcal{K}$ of the operator F at the point $X = \Theta$.

Theorem 4.1. *Suppose that system (4.4) belongs to the class $\mathcal{M}_1(\Theta)$ with respect to the normal reproducing cone $\mathcal{K}$ and there exist the Fréchet derivatives $F'_\pm(\Theta)$ with respect to $\pm\mathcal{K}$, respectively. The state $X \equiv \Theta$ of system (4.4) is Lyapunov asymptotically stable if the operators $-F'_\pm(\Theta)$ are positive invertible:*

$$\mathcal{K} \subseteq -F'_+(\Theta)\mathcal{K}, \qquad \mathcal{K} \subseteq -F'_-(\Theta)\mathcal{K}. \tag{4.5}$$

Moreover, if the cone $\mathcal{K}$ *is solid, then conditions (4.5) are equivalent to the solvability of the system of inequalities*

$$H_- \overset{\mathcal{K}}{\leq} 0 \overset{\mathcal{K}}{\leq} H_+, \qquad F'_+(\Theta)H_+ \overset{\mathcal{K}}{<} 0 \overset{\mathcal{K}}{<} F'_-(\Theta)H_- \tag{4.6}$$

with respect to H_+ *and* H_-.

Proof. System (4.4) of the class $\mathcal{M}_1(\Theta)$ has the invariant sets $\mathcal{K}^{\pm}(\Theta)$. For $X = \Theta + H \in \mathcal{K}^{\pm}(\Theta)$, this system is represented as follows:

$$\dot{H} = F'_{\pm}(\Theta)H + R_{\pm}(\Theta, H), \quad H \in \pm\mathcal{K}. \tag{4.7}$$

According to Lemma 4.1, the asymptotic stability of the states $H \equiv 0$ of systems (4.7) in $\pm\mathcal{K}$ yields the Lyapunov asymptotic stability of the state $X \equiv \Theta$ of the original system (4.4). In order to use the Lyapunov theorem on stability with respect to the first approximation, we establish the asymptotic stability of the linear systems

$$\dot{H} = F'_{\pm}(\Theta)H, \quad H \in \mathcal{E}. \tag{4.8}$$

Systems (4.8) are positive with respect to $\mathcal{K}$ and $-\mathcal{K}$. Indeed, taking into account the relations

$$F(\Theta + \varepsilon H) = \varepsilon F'_+(\Theta)H + R_+(\Theta, \varepsilon H), \qquad \frac{R(\Theta, \varepsilon H)}{\varepsilon\|H\|} \underset{\varepsilon\to 0}{\to} 0,$$

and the fact that $F \in \mathcal{F}_1^{\pm}(\Theta)$ (see Sec. 3), we get

$$H \in \pm\mathcal{K}, \quad \varphi \in \pm\mathcal{K}^*, \quad \varphi(H) = 0 \implies \frac{\varphi(F'_{\pm}(\Theta)H)}{\|H\|} + \frac{\varphi(R_{\pm}(\Theta, \varepsilon H))}{\varepsilon\|H\|} \geq 0.$$

This implies that $\varphi(F'_{\pm}(\Theta)H) \geq 0$, i.e., the conditions of the positivity of systems (4.8) are satisfied.

For positive systems (4.8), conditions (4.5) and (4.6) are equivalent. By virtue of Lemma 4.2, systems (4.8) are asymptotically stable. Moreover, the state $X \equiv \Theta$ of the original system (4.4) is also Lyapunov asymptotically stable.

The theorem is proved.

Conjecture 4.1. *Let system (4.4) belong to the class* $\mathcal{M}_1(\Theta)$ *with respect to the normal solid cone* $\mathcal{K}$ *and let the system of cone inequalities*

$$X_- \overset{\mathcal{K}}{\leq} \Theta \overset{\mathcal{K}}{\leq} X_+, \qquad F(X_+) \overset{\mathcal{K}}{<} 0 \overset{\mathcal{K}}{<} F(X_-) \tag{4.9}$$

be consistent. Then the state $X \equiv \Theta$ *of system (4.4) is Lyapunov asymptotically stable.*

Example 4.1. Consider the nonlinear differential system

$$\dot{x} = Ax + b \odot \sin|x|, \quad x \in R^n, \tag{4.10}$$

where $A \in R^{n\times n}$, $b \in R^n$, and $\odot$ and $|\cdot|$ are the elementwise operations of taking the product and the modulus, respectively. Assume that all off-diagonal elements of the matrix A are nonnegative. Then (4.10) is a system of the class $\mathcal{M}$ with respect to the cone of nonnegative vectors $\mathcal{K} = R^n_+$.

We determine the Fréchet derivatives of the vector function f with respect to the cones $\pm\mathcal{K}$ at the point $\theta = 0$:

$$f'_+(0) = A + \mathrm{diag}\{b\}, \qquad f'_-(0) = A - \mathrm{diag}\{b\}.$$

By virtue of Theorem 4.1, the solution $x \equiv 0$ of system (4.10) is asymptotically stable if all elements of the matrices $-(f'_\pm(0))^{-1}$ are nonnegative. In the case $n = 2$, this condition reduces to the form

$$a_{11} + |b_1| < 0, \qquad a_{22} + |b_2| < 0, \qquad |b_1 a_{22} + b_2 a_{11}| < a_{11}a_{22} - a_{12}a_{21} + b_1 b_2.$$

In the general case, for the positive invertibility of the matrices $-\big(f'_\pm(0)\big)^{-1}$, it suffices that the following inequalities be true:

$$a_{ii} + |b_i| + r(M) < 0, \quad i = \overline{1,n},$$

where $r(M)$ is the spectral radius of the nonnegative matrix $M = A - A \odot I$. This statement is a corollary of the theorem on a two-sided estimate for a positive-invertible operator and the fact that $(\gamma I - M)^{-1} \geq 0 \Longleftrightarrow \gamma > r(M)$ [1].

Example 4.2. Consider the system

$$\dot{x} = (Ax + b) \odot c(x), \qquad x \in R^n, \quad t \geq 0, \tag{4.11}$$

where $c(x) = \big[c_1(x_1), \ldots, c_n(x_n)\big]^T$ is a continuous vector function with nonnegative components that is differentiable in the neighborhood of the isolated equilibrium state $\theta = -A^{-1}b$.

In the neighborhood of the point $x = \theta$, we have

$$f'(x) = \mathrm{diag}\{c(x)\}\, A + \mathrm{diag}\{Ax + b\}\, c'(x), \qquad f'(\theta) = \mathrm{diag}\{c(\theta)\}\, A.$$

It can be shown that system (4.11) is monotone with respect to the cone $\mathcal{K} = R^n_+$ if all off-diagonal elements of the matrix A are nonnegative. Moreover, by virtue of Theorem 4.1, the state $x \equiv \theta$ of this system is asymptotically stable if $c(\theta) \overset{\mathcal{K}}{>} 0$ and all elements of the matrix A^{-1} are nonpositive.

5. Pseudolinear Differential System

Consider the nonlinear differential system

$$\dot{x} = A(x,t)\, x, \qquad x \in R^n, \quad t \geq \tau \geq 0, \tag{5.1}$$

where $A(x,t)$ is a continuous matrix function. If $x \equiv \theta$ is an equilibrium state of system (5.1), then one of the following conditions is satisfied:

(a) $\theta = 0$;

(b) $\theta \neq 0$ and $A(\theta, t) \equiv 0$;

(c) $\theta \neq 0,\ A(\theta, t) \not\equiv 0,$ and $A(\theta, t)\theta \equiv 0.$

Lemma 5.1. *System (5.1) has the invariant set* $\mathcal{K}(\theta) = \left\{x\colon x \overset{\mathcal{K}}{\geq} \theta\right\}$, *where* $\mathcal{K} = R_+^n$ *is the cone of nonnegative vectors if*

$$a_{ij}(x,t) \geq 0, \quad i \neq j, \qquad A(x,t)\theta \overset{\mathcal{K}}{\geq} 0, \quad x \in \partial\mathcal{K}(\theta), \quad t \geq 0. \tag{5.2}$$

In particular, system (5.1) is positive with respect to the cone $\mathcal{K}$ *if*

$$a_{ij}(x,t) \geq 0, \quad i \neq j, \qquad x \in \partial\mathcal{K}, \quad t \geq 0. \tag{5.3}$$

Lemma 5.2. *System (5.1) is positive with respect to the ellipsoidal cone*

$$\mathcal{K}_t = \left\{x\colon x^T Q(t) x \geq 0,\ h^T(t) x \geq 0\right\}$$

$\big(h(t)$ *is an eigenvector of a symmetric differentiable matrix* $Q(t)$ *with inertia* $i(Q(t)) = \{1, n-1, 0\}\big)$ *if*

$$\dot{Q}(t) + \alpha(x,t)Q(t) + A^T(x,t)Q(t) + Q(t)A(x,t) \overset{\mathcal{K}}{\geq} 0, \quad x \in \partial\mathcal{K}_t, \quad t \geq 0, \tag{5.4}$$

where $\mathcal{K}$ *is the cone of symmetric nonnegative-definite matrices.*

Lemmas 5.1 and 5.2 follow from Theorem 3.1 (see Remark 3.1).

It can be established that the Gâteaux derivative of the vector function $f(x,t) = A(x,t)\,x$ can be represented in the form

$$f'(x,t) = A(x,t) + \left[\frac{\partial A(x,t)}{\partial x_1}\,x, \ldots, \frac{\partial A(x,t)}{\partial x_n}\,x\right], \tag{5.5}$$

where $\dfrac{\partial A(x,t)}{\partial x_k}$ are the matrices composed of the Gâteaux derivatives of the functions $a_{ij}(x,t)$ with respect to $x_k,\ k = 1, \ldots, n.$ Indeed,

$$\frac{1}{\varepsilon}\left[f(x+\varepsilon h, t) - f(x,t)\right] = A(x+\varepsilon h, t)h + \frac{1}{\varepsilon}\left[A(x+\varepsilon h, t) - A(x,t)\right]x$$

$$\underset{\varepsilon\to 0}{\to} A(x,t)h + \sum_j \left(\sum_k \frac{\partial a_{*j}(x,t)}{\partial x_k} h_k\right) x_j$$

$$= A(x,t)h + \sum_k \left(\sum_j \frac{\partial a_{*j}(x,t)}{\partial x_k} x_j\right) h_k = f'(x,t)h,$$

where $a_{*j}(x,t)$ is the jth column of the matrix $A(x,t)$. Moreover, if h may be an arbitrary element of a certain cone $\mathcal{K} \subset R^n$, then (5.5) is the Gâteaux derivative with respect to $\mathcal{K} \subset R^n$ of the vector function $f(x,t)$. In particular, for the cones $\pm\mathcal{K} = R^n_{\pm}$, the following relations can be used:

$$f'_{\pm}(x,t) = A(x,t) + \left[\frac{\partial A(x,t)}{\partial x_{1\pm}}\, x, \ldots, \frac{\partial A(x,t)}{\partial x_{n\pm}}\, x\right],$$

where $\dfrac{\partial A(x,t)}{\partial x_{k\pm}}$ are, respectively, the right and the left derivatives of the matrix $A(x,t)$ with respect to x_k, $k = 1, \ldots, n$.

We select a subclass of systems (5.1) with diagonal matrices $A(x,t)$:

$$\dot{x} = \operatorname{diag}\{a(x,t)\}x \equiv a(x,t) \odot x, \qquad x \in R^n, \quad t \geq 0, \tag{5.6}$$

where $a(x,t)$ is a continuous vector function and $\odot$ denotes the elementwise product of vectors. System (5.6) is a generalized Kolmogorov model of the dynamics of growth and interaction of n populations [2].

Conditions (5.3) are satisfied, and system (5.6) has the invariant cone $\mathcal{K} = R^n_+$. For system (5.6) to be monotone in the cone $\mathcal{K}$, it is necessary and sufficient that $a \in \mathcal{F}_2^+$.

The linear systems (4.8) have the form

$$\dot{h} = A_{\pm}(x,t)h, \qquad A_{\pm}(x,t) = \operatorname{diag}\{x\}\, a'_{\pm}(x,t) + \operatorname{diag}\{a(x,t)\}, \tag{5.7}$$

where $a'_{\pm}(x,t)$ are the derivatives of the vector function $a(x,t)$ with respect to the cones $\pm\mathcal{K}$. Systems (5.7) are positive with respect to $\mathcal{K}$ if all off-diagonal elements of the matrices $a'_{\pm}(x,t)$, $x \in \mathcal{K}$, are nonnegative. If the vector function $a(x,t)$ is differentiable with respect to x (in the ordinary sense) at every point $x \in \mathcal{K}$, then system (5.6) is monotone in $\mathcal{K}$ under the conditions

$$\frac{\partial a_i(x,t)}{\partial x_j} \geq 0, \qquad i \neq j, \quad x \geq 0.$$

This means, in particular, that the growth of each population leads to an increase in the growth rate of any other population.

We now formulate a corollary of Theorem 4.1 for the stationary system

$$\dot{x} = A(x)\, x, \qquad x \in R^n, \quad t \geq \tau \geq 0. \tag{5.8}$$

Theorem 5.1. *Suppose that system (5.8) belongs to the class $\mathcal{M}_1(\theta)$ with respect to the cone $\mathcal{K} = R^n_+$ and there exist the Gâteaux derivatives of the vector function $f(x) = A(x)\, x$ with respect to $\pm\mathcal{K}$ at the point $x = \theta$:*

$$f'_{\pm}(\theta) = A(\theta) + B_{\pm}(\theta), \qquad B_{\pm}(\theta) = \left[\frac{\partial A(\theta)}{\partial \theta_{1\pm}}\, \theta, \ldots, \frac{\partial A(\theta)}{\partial \theta_{n\pm}}\, \theta\right].$$

The state $x \equiv \theta$ of system (5.8) is Lyapunov asymptotically stable if the system of inequalities

$$x_- \overset{\mathcal{K}}{\leq} \theta \overset{\mathcal{K}}{\leq} x_+, \qquad f'_+(\theta)x_+ \overset{\mathcal{K}}{<} B_+(\theta)\theta, \qquad f'_-(\theta)x_- \overset{\mathcal{K}}{>} B_-(\theta)\theta \tag{5.9}$$

is consistent with respect to x_+ and x_-.

In using Theorem 5.1, it is necessary to establish that system (5.8) belongs to the class $\mathcal{M}_1(\theta)$ and to solve the system of cone inequalities (5.9). In the analysis of the stability of the zero state of system (5.8) that does not belong to the class $\mathcal{M}_1(\theta)$ and in the analysis of the more general system (5.1), the following statement can be used:

Theorem 5.2. *Suppose that, for a certain matrix* $X(t) \equiv X^T(t)$, *the following relations are true:*

$$\alpha I \overset{\mathcal{K}}{\leq} X(t), \qquad Y(x,t) \overset{\mathcal{K}}{\leq} 0, \quad x \in S_0, \quad t \geq 0, \tag{5.10}$$

where $\alpha > 0$, $Y(x,t) \triangleq \dot{X}(t) + A^T(x,t)X(t) + X(t)A(x,t)$, $\mathcal{K}$ *is the cone of symmetric nonnegative-definite matrices, and* S_0 *is a certain neighborhood of the point* $x = 0$. *Then the isolated equilibrium state* $x \equiv 0$ *of system (5.1) is Lyapunov stable.*

If

$$\alpha I \overset{\mathcal{K}}{\leq} X(t) \overset{\mathcal{K}}{\leq} \beta I, \qquad Y(x,t) \overset{\mathcal{K}}{\leq} \gamma I, \quad x \in S_0, \quad t \geq 0, \tag{5.11}$$

where $\alpha > 0$, $\beta > 0$, *and* $\gamma < 0$, *then the state* $x \equiv 0$ *of system (5.1) is uniformly asymptotically stable.*

This statement is established on the basis of the first and the second Lyapunov theorems with the use of the relations

$$\alpha \|x\|^2 \leq \lambda_{\min}(X(t))\|x\|^2 \leq v(x,t) \leq \lambda_{\max}(X(t))\|x\|^2 \leq \beta\|x\|^2,$$

$$\dot{v}(x,t) = x^T Y(x,t) x \leq \lambda_{\max}(Y(x,t))\|x\|^2 \leq \gamma\|x\|^2,$$

where $v(x,t) = x^T X(t) x$ and $\lambda_{\max}(\cdot)$ $(\lambda_{\min}(\cdot))$ is the maximal (minimal) eigenvalue of a symmetric matrix. In this case, $v(x,t)$ is the Lyapunov function of the first or the second kind, respectively, and $\dot{v}(x,t)$ is its derivative along solutions of system (5.1).

Theorem 5.2 gives general methods for the analysis of the stability of zero equilibrium states of the class of systems (5.1), in particular (5.6) and (5.8), on the basis of solutions of the systems of functional matrix inequalities (5.10) and (5.11).

6. Positive Differential Systems with Delay

Consider the class of nonlinear differential systems with delay

$$\dot{X}(t) + L(t)X(t) = G(X(t-s), t), \qquad G(0,t) \equiv 0, \quad t \geq \tau \geq 0, \tag{6.1}$$

where $s > 0$ is a constant delay, $L(t) \colon \mathcal{E} \to \mathcal{E}$ is a linear bounded operator in the Banach space $\mathcal{E}$, and $G(X,t)$ is a continuous operator function that satisfies conditions for the existence of a unique solution $X(t) \in \mathcal{E}$ under the initial conditions

$$X(\xi) = \Psi(\xi), \quad \tau - s \leq \xi \leq \tau. \tag{6.2}$$

System (6.1) is called positive with respect to the cone $\mathcal{K}_t \subset \mathcal{E}$ if, for any $\tau \geq 0$, the fact that $\Psi(\xi) \overset{\mathcal{K}_\xi}{\geq} 0$ for all $\xi \in [\tau - s, \tau]$ implies that $X(t) \overset{\mathcal{K}_t}{\geq} 0$ for $t > \tau$.

Theorem 6.1. *The differential system (6.1) is positive with respect to the constant cone* $\mathcal{K}$ *if and only if, for any* $\tau \geq 0$ *and* $t \geq \tau$, *the following inclusions are true:*

$$W(t,\tau)\mathcal{K} \subseteq \mathcal{K}, \qquad G(\mathcal{K},t) \subseteq \mathcal{K}, \tag{6.3}$$

where $W(t,\tau)$ *is the evolution operator of the linear system*

$$\dot{X}(t) + L(t)X(t) = 0, \quad t \geq \tau \geq 0. \tag{6.4}$$

Proof. We define the sequence $t_k = \tau + ks$, $k = 0, 1, \ldots$. Then a solution of system (6.1) under conditions (6.2) satisfies the following integral relations on each segment $[t_k, t_{k+1}]$:

$$X(t) = W(t,t_k)X(t_k) + \int\limits_{t_k}^{t} W(t,\xi)G(X(\xi - s),\xi)d\xi, \quad t_k \leq t \leq t_{k+1}. \tag{6.5}$$

These relations can be directly established by performing differentiation and using the definition of evolution operator as a solution of the Cauchy problem

$$\frac{d}{dt}W(t,\xi) + L(t)W(t,\xi) = 0, \qquad W(\xi,\xi) = E, \quad t \geq \xi, \tag{6.6}$$

where E is the identity operator. Therefore, if $\Psi(\xi) \in \mathcal{K}$ in (6.2), then $X(t) \in \mathcal{K}$ for $t_k \leq t \leq t_{k+1}$, $k = 0, 1, \ldots$.

Let us show that inclusions (6.3) are necessary for the positivity of system (6.1). If

$$X(\xi) = \begin{cases} 0, & t_{k-1} \leq \xi \leq t_k - \varepsilon, \\ \Psi(\xi), & t_k - \varepsilon \leq \xi \leq t_k, \end{cases}$$

where $\varepsilon > 0$ and $\Psi(\xi) \in \mathcal{K}$, then, according to (6.5),

$$X(t) = W(t,t_k)X(t_k) \in \mathcal{K}, \quad t_k \leq t \leq t_{k+1} - \varepsilon.$$

By virtue of the closedness of the cone $\mathcal{K}$, as $\varepsilon \to 0$ we obtain the first inclusion in (6.3) on the interval $t_k \leq t \leq t_{k+1}$. Its validity for any $t \geq \tau$ is a corollary of the multiplicative representation of the operator

$$W(t,t_0) = W(t,t_k)W(t_k,t_{k-1})\ldots W(t_1,t_0), \quad t_k \leq t \leq t_{k+1},$$

where k is a certain natural number.

If $X(\tau) = 0$, then, for a certain $\xi \in (\tau, t)$, we have

$$X(t) = (t - \tau)W(t,\xi)G(X(\xi - s),\xi), \quad t > s,$$

$$X(\xi - s) \in \mathcal{K} \implies W(t,\xi)G(X(\xi - s),\xi) \in \mathcal{K}.$$

Moreover, if $t \to \tau$, then $\xi \to \tau$ and $W(t, \xi) \to E$. By virtue of the closedness of the cone and the continuous dependence of W and G on their arguments, $G(X, \tau)$ belongs to $\mathcal{K}$, provided that $X = X(\tau - s) \in \mathcal{K}$. Therefore, the second inclusion (6.3) is also necessary for the positivity of system (6.1).

The theorem is proved.

Theorem 6.1 is a generalization of the known criterion for the positivity of the class of systems (6.1) with respect to the cone of nonnegative vectors R^n_+ [13].

Consider a subclass of autonomous differential systems with delay, namely,

$$\dot{X}(t) + LX(t) = G(X(t-s)), \qquad G(0) = 0, \quad t \geq \tau \geq 0. \tag{6.7}$$

If $\Psi(\xi) \equiv 0$ in (6.2), then $X \equiv 0$ is the trivial solution of system (6.7).

The trivial solution $X \equiv 0$ of system (6.7) is Lyapunov stable if, for any $\varepsilon > 0$ and $\tau \geq 0$, there exists $\delta = \delta(\varepsilon, \tau) > 0$ such that the inequality $\|X(\tau)\|_s < \delta$ implies that $\|X(t)\| < \varepsilon$ for all $t > \tau$, where

$$\|X(\tau)\|_s = \max_{\tau - s \leq \xi \leq \tau} \|X(\xi)\|.$$

The solution $X \equiv 0$ is called asymptotically stable if it is Lyapunov stable and, for any $\tau \geq 0$, there exists $\delta = \delta(\tau) > 0$ such that, for every solution $X(t)$ of system (6.7), the relation $\|X(\tau)\|_s < \delta$ implies that

$$\lim_{t \to \infty} \|X(t)\| = 0.$$

The solution $X \equiv 0$ is absolutely stable if it is asymptotically stable for any $s \geq 0$.

Theorem 6.2. *Suppose that*

$$e^{Lt} X \overset{\mathcal{K}}{\geq} 0, \qquad 0 \overset{\mathcal{K}}{\leq} G(X) \overset{\mathcal{K}}{\leq} PX, \qquad X \in \mathcal{K}, \quad t \geq 0, \tag{6.8}$$

where P is a linear positive operator, and there exist linear uniformly positive functionals $\varphi, \psi \in \mathcal{K}^*$ that satisfy the equation

$$M^* \varphi = \psi, \qquad M \triangleq L - P. \tag{6.9}$$

Then the solution $X \equiv 0$ of system (6.7) is absolutely stable.

Proof. According to Theorem 6.1, system (6.7) is positive under conditions (6.8). We construct the Lyapunov–Krasovskii functional

$$V(X_t) = \varphi(X(t)) + \int_{-s}^{0} \varphi\left(G(X(t+\xi))\right) d\xi, \tag{6.10}$$

where $X_t(\xi) = x(t+\xi)$ and $\varphi \in \mathcal{K}^*$. Expression (6.10) is a generalization of the functional used in [13] in the case of the cone R^n_+.

Note that, for any uniformly positive functional, the following estimate is true [1]:

$$\gamma_- \|X\| \leq \varphi(X) \leq \gamma_+ \|X\|, \quad X \in \mathcal{K},$$

where $\gamma_\pm > 0$ are certain constants independent of X. Therefore, $V(\Psi) \geq \varphi(\Psi(0)) \geq \gamma_- \|\Psi(0)\|$.

The derivative of functional (6.10) along the solutions of system (6.7), with regard for (6.8) and (6.9), satisfies the relations

$$\dot V(X_t) = \varphi(-LX(t) + G(X(t))) \le -\varphi(MX(t)) = -\psi(X(t)) \le -\gamma\|X(t)\|,$$

where $\gamma > 0$. Therefore, the solution $X \equiv 0$ of the positive system (6.7) is asymptotically stable for any $s \ge 0$ [14].

The theorem is proved.

Theorem 6.2 is a generalization of a criterion for the absolute stability of differential systems of the type (6.11) that are positive with respect to the cone R^n_+ [13].

Let $\mathcal{K}$ and $\mathcal{K}^*$ be solid cones in the spaces $\mathcal{E}$ and $\mathcal{E}^*$, respectively. Under the conditions of Theorem 6.2, the operator M^* must be positive invertible and such that $(M^*)^{-1} = (M^{-1})^*$. Therefore, in the construction of criteria for the absolute stability of the linear system

$$\dot X(t) + LX(t) = PX(t-s), \quad t \ge \tau \ge 0, \tag{6.11}$$

instead of Eq. (6.9) one can consider an analogous equation with operator M.

Theorem 6.3. *For the positive system (6.11), the following assertions are equivalent:*

(i) *the solution* $X \equiv 0$ *of system (6.11) is absolutely stable;*

(ii) *the operator* $M = L - P$ *is positive invertible;*

(iii) *there exist* $X, Y \in \operatorname{int}\mathcal{K}$ *that satisfy the equation* $MX = Y$;

(iv) $\operatorname{Re}\lambda \le \alpha < 0 \quad \forall\lambda \in \sigma(M)$.

Remark 6.1. If $P\mathcal{K} \subseteq \mathcal{K} \subseteq L\mathcal{K}$, then each of assertions (i)–(iv) in Theorem 6.3 is equivalent to the following assertion (see, e.g., [8]):

(iv) $\rho(T) < 1$, where $\rho(T)$ is the spectral radius of the pencil of operators $T(\lambda) = P - \lambda L$.

Example 6.1. The linear differential system

$$\dot x(t) = A(t)x(t) + B(t)x(t-s) \tag{6.12}$$

is positive with respect to the cone of nonnegative vectors $\mathcal{K} = R^n_+$ if and only if the off-diagonal elements of the matrix $A(t)$ and all elements of the matrix $B(t)$ are nonnegative for $t \ge 0$. Moreover, if the matrices A and B are constant, then the absolute stability of the solution $x \equiv 0$ of system (6.12) is equivalent to the consistency of the system of inequalities $(A+B)x \overset{\mathcal{K}}{<} 0 \overset{\mathcal{K}}{<} x$.

Example 6.2. The nonlinear matrix system

$$\dot X(t) + A(t)X(t) + X(t)A^*(t) = B(t)X(t-s)B^*(t) + X(t-s)C(t)X(t-s) \tag{6.13}$$

is positive with respect to the cone of Hermitian nonnegative-definite matrices $\mathcal{K}$ if $C(t) \overset{\mathcal{K}}{\geq} 0$ for $t \geq 0$. In the case considered, system (6.4) is described by the Lyapunov operator $L(t)X = A(t)X + XA^*(t)$, and the operators

$$W(t,\tau)X = \Delta(t,\tau)X\Delta^*(t,\tau), \qquad G(X,t) = B(t)XB^*(t) + XC(t)X,$$

where $\Delta(t,\tau)$ is the matrizant of the system $\dot{x}+A(t)x = 0$, are positive with respect to the cone $\mathcal{K}$ for $t \geq \tau \geq 0$. In the case of constant matrices A, B, and $C = 0$, the zero solution of the positive system (6.13) is absolutely stable if, for a certain positive-definite matrix $Y = Y^* > 0$, the matrix algebraic equation

$$AX + XA^* - BXB^* = Y$$

has a positive-definite solution $X = X^* > 0$.

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