Apéry's constant and other "geometric" numbers: towards understanding the motivic Galois group

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The story is based on the papers

- Yves André, Galois theory, motives and transcendental numbers, 2008
- Maxim Kontsevich and Don Zagier, Periods, 2001

$$\mathbb{N} = \Big\{1, 2, 3, \dots\Big\}$$

$$\mathbb{Z} = \Big\{\dots, -2, -1, 0, 1, 2, \dots\Big\}$$

$$\mathbb{Q} = \left\{\frac{p}{q} \;\middle|\; p \in \mathbb{Z},\ q \in \mathbb{N},\ \mathrm{g.c.d.}(p, q) = 1\right\}$$

$$\mathbb{R}$$

$$\mathbb{C} = \Big\{x + i \cdot y \;\Big|\; x, y \in \mathbb{R}\Big\}$$

A number $x \in \mathbb{C}$ is called *algebraic* if it satisfies a polynomial equation with rational coefficients:

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0 , \quad a_i \in \mathbb{Q}$$

Notation: $x \in \overline{\mathbb{Q}}$

Choose the equation of minimal possible degree. Its complex roots are then called the *conjugates* of x:

$$x_1 = x , \ x_2 , \ \ldots \ x_n$$

Example 1: $x^2 - x - 1 = 0, \quad x_{1,2} = \frac{1\pm\sqrt{5}}{2}$.

Example 2: $x = e^{\frac{2\pi i}{5}} = \cos(72^\circ) + i\sin(72^\circ)$

$$= \frac{\sqrt{5}-1}{4} + i\sqrt{\frac{5+\sqrt{5}}{8}}$$

$$x^5 = 1$$

$$x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1) = 0$$

$$x_{1,2} = \frac{\sqrt{5}-1}{4} \pm i\sqrt{\frac{5+\sqrt{5}}{8}}, \quad x_{3,4} = -\frac{\sqrt{5}+1}{4} \pm i\sqrt{\frac{5-\sqrt{5}}{8}}$$

Example 3: There are three sets of conjugates among 9th roots of 1.

$$x^9 - 1 = (x-1)(x^2+x+1)(x^6+x^3+1)$$

$$\begin{array}{ccccccc} \mathbb{N} & \subset & \mathbb{Z} & \subset & \mathbb{Q} & \subset & \overline{\mathbb{Q}} \\ & & & & \cap & & \cap \\ & & & & \mathbb{R} & \subset & \mathbb{C} \end{array}$$

Numbers which are not algebraic are called *transcendental*.

$$\pi = 3.14159265358979323846264338 3...$$

is transcendental (F. Lindeman, 1882)

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.71828182845904523536028747 1...$$

is transcendental (Ch. Hermite, 1873)

Basic question: Is there anything analogous to *conjugates* for (some) transcendental numbers?

Naive approach: look for a formal power series with rational coefficients as a substitute for the minimal polynomial.

E.g.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n} = 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 + \ldots = \frac{\sin(x)}{x}$$

vanishes at $x = \pi$, but also at

$$x = m\pi \quad \text{for all} \quad m \in \mathbb{Z}, m \neq 0 .$$

A.Hurwitz:
For any $\alpha \in \mathbb{C}$, there exists a power series with rational coefficients which defines an entire function of exponential growth, and vanishes at α.

However, it turns out that there are uncountably many such series. In fact, such a series can be found which vanishes not only at α, but also at any other fixed number β.

Naive approach fails.

Periods

A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

Examples: $\sqrt{2} = \frac{1}{2} \int_{0 \leq x^2 \leq 2} dx$, $\log(2) = \int_1^2 \frac{dx}{x}$.

All algebraic numbers are periods. Logarithms of algebraic numbers are periods. Periods form an algebra, i.e. the sum and the product of two periods is a period again.

$$\pi = \iint_{x^2+y^2 \leq 1} dx\, dy = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} \in \mathcal{P}$$

Many infinite sums of elementary expressions are periods. E.g. all values of the Riemann zeta function

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s}$$

at integer arguments $s \geq 2$ are periods. E.g.

$$\iiint\limits_{0<x<y<z<0} \frac{dxdydz}{(1-x)yz} \;=\; \int_0^1 \int_0^z \frac{1}{yz} \sum_{n=0}^{\infty} \int_0^y x^n \, dxdydz$$

$$= \int_0^1 \int_0^z \frac{1}{yz} \sum_{n=0}^{\infty} \frac{y^{n+1}}{n+1} dydz$$

$$= \int_0^1 \frac{1}{z} \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)^2} dz \;=\; \sum_{n=0}^{\infty} \frac{1}{(n+1)^3} \;=\; \zeta(3)$$

Values of the gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$$

are closely related to periods:

$$\Gamma\left(\frac{p}{q}\right)^q \in \mathcal{P} \qquad p, q \in \mathbb{N}$$

For instance,

$$\Gamma\left(\frac{1}{2}\right)^2 = \pi \,, \qquad \Gamma\left(\frac{1}{3}\right)^3 = 2^{\frac{4}{3}} \, 3^{\frac{1}{2}} \, \pi \int_0^1 \frac{dx}{\sqrt{1-x^3}} \,.$$

Identities between periods

(1) additivity (in the integrand and in the domain of integration)

$$\int_a^b (f(x)+g(x))\,dx \;=\; \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b f(x)\,dx \;=\; \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

(2) change of variables

$$\int_{f(a)}^{f(b)} F(y)\,dy \;=\; \int_a^b F(f(x))f'(x)\,dx$$

(3) Newton-Leibniz formula

$$\int_a^b f'(x)\,dx \;=\; f(b) - f(a)$$

Conjectural principle: if a period has two integral representations, then one can pass from one formula to another using only (multidimensional generalizations) of the rules (1)-(3).

As an example, let us proof the identity

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

The following proof is originally due to E.Calabi: we start with the integral

$$\int_0^1 \int_0^1 \frac{1}{1-xy} \frac{dxdy}{\sqrt{xy}} = \sum_{n=0}^{\infty} (n + \frac{1}{2})^{-2} = 3\zeta(2)$$

On the other hand, the change of variables

$$x = \xi^2 \frac{1+\eta^2}{1+\xi^2}\,, \qquad y = \eta^2 \frac{1+\xi^2}{1+\eta^2}$$

has the Jacobian

$$\left|\frac{\partial(x,y)}{\partial(\xi,\eta)}\right| \;=\; \frac{4\xi\eta(1-\xi^2\eta^2)}{(1+\xi^2)(1+\eta^2)} \;=\; 4\frac{(1-xy)\sqrt{xy}}{(1+\xi^2)(1+\eta^2)}$$

and therefore

$$\int_0^1\int_0^1 \frac{1}{1-xy}\frac{dx\,dy}{\sqrt{xy}} \;=\; 4\iint_{\xi,\eta>0,\xi\eta\le 1} \frac{d\xi}{(1+\xi^2)}\frac{d\eta}{(1+\eta^2)}$$

$$= \; 2\int_0^\infty \frac{d\xi}{(1+\xi^2)}\int_0^\infty \frac{d\eta}{(1+\eta^2)} \;=\; \frac{\pi^2}{2}$$

For a normal extension of fields $K \subset L$ the *Galois group* is defined as

$$Gal(L/K) \;=\; \{\text{ authomorphisms of } L \text{ that preserve } K\}\,.$$

For an algebraic number x with the conjugates $x_1 = x, x_2, \ldots, x_n$ one considers the field

$$\mathbb{Q}(x_1, \ldots, x_n)$$

and the group

$$G = Gal\Big(\mathbb{Q}(x_1, \ldots, x_n)/\mathbb{Q}\Big)\,.$$

Fundamental observations of Galois theory:

- Elements of G permute the numbers $x_1, \dots, x_n$.
- An element $y \in \mathbb{Q}(x_1, \dots, x_n)$ is preserved by all automorphisms $g \in G$ if and only if $y \in \mathbb{Q}$.

It follows that G is a subgroup of the group of permutations of $x_1, \dots, x_n$. Regarding $V = \mathbb{Q}(x_1, \dots, x_n)$ as a $\mathbb{Q}$-vector space, we then have that at the same time

$$G \subset S_n \ \text{ (the group of permutations of } n \text{ elements)}$$
$$G \subset GL(V) \ \text{ (the group of linear transformations of } V\text{)}$$

Finally, every algebraic number $x \in \overline{\mathbb{Q}}$ comes along with the following structure:

- the set of conjugates $x_1, \ldots, x_n$
- a finite dimensional $\mathbb{Q}$-vector space $V = \mathbb{Q}(x_1, \ldots, x_n)$
- a finite group G, which is a subgroup of permutations of the above set and acts in the above vector space by $\mathbb{Q}$-linear transformations:

$$G \subset S_n\,, \qquad G \subset GL(V)$$

$\mathcal{P}$ appears to be a natural set of numbers for which one could expect to generalize this structure.

$$\begin{aligned}\mathcal{P} &= \Big\{ \text{integrals of rational functions with algebraic coefficients} \\ &\quad \text{over domains given by polynomial inequalities} \\ &\quad \text{with rational coefficients} \Big\} \\ &= \Big\{ \text{integrals of rational differential forms } \omega \\ &\quad \text{on smooth algebraic varieties } X \text{ defined over } \mathbb{Q} \\ &\quad \text{integrated over relative topological chains } \sigma \\ &\quad \text{with the boundary on a subvariety } D \subset X \text{ of codimension } 1 \Big\}\end{aligned}$$

$$2\pi i = \oint \frac{dx}{x}$$

$$X = \mathbb{C}^{\times} \cong \{(x,y) \in \mathbb{C}^2 \mid xy = 1\}$$

$$\omega = \frac{dx}{x}$$

$\sigma =$ a counterclockwise loop around the puncture

$$D = \emptyset$$

$$\iint\limits_{\substack{v^2(x^3-3x^2+2x)\le 1\\ 1\le x\le 2}} dx\,dv \;=\; 2\int_1^2 \frac{dx}{\sqrt{x^3-3x^2+2x}} \;=\; \int_\sigma \omega$$

$$X \;=\; \{(x,y)\in\mathbb{C}^2 \mid y^2 = x^3-3x^2+2x\}$$

$$\omega \;=\; \frac{dx}{y}$$

$\sigma \;=\;$ a loop through the points $(1,0)$ and $(2,0)$

$$D \;=\; \emptyset$$

Homology and Cohomology

X a smooth manifold of dimension n

k-chains in X: formal linear combinations with rational coefficients of smooth embeddings of the k-dimensional simplex Δ_k into X
Notation: $C_k(X)$

The *boundary* map: $\partial : C_k(X) \to C_{k-1}(X)$.

A simple computation shows that $\partial \circ \partial = 0$.

Homology and Cohomology (continuation)

The k-th *homology*

$$H_k(X) \;=\; \frac{\text{Kernel}(\partial : C_k(X) \to C_{k-1}(X))}{\text{Image}(\partial : C_{k+}(X) \to C_k(X))} \;=\; \frac{k-\text{cycles}}{k-\text{boundaries}}$$

is a finite-dimensional (!) vector space over $\mathbb{Q}$.

Its dual vector space is called the k-th *cohomology*:

$$H^k(X) \;=\; H_k(X)^* \;=\; \{\text{linear functionals on } H_k(X)\}\,.$$

$$\beta_k(X) \;=\; \dim H^k(X) \qquad \text{the } \textit{Betti numbers} \text{ of } X$$

$$X = \mathbb{C}^*$$
$$\beta_0 = \beta_1 = 1\,,\ \beta_2 = 0$$

$$X = \text{compactification of } \{(x,y) \in \mathbb{C}^2 \mid y^2 = x^3 - 3x^2 + 2x\}$$
$$\equiv \text{2-dimensional torus}$$
$$\beta_0 = 1\,,\ \beta_1 = 2\,,\ \beta_2 = 1$$

With a period

$$w = \int_\sigma \omega\,, \qquad \sigma \in H_k(X)$$

we associate a finite-dimensional $\mathbb{Q}$-vector space

$$V = H_\bullet(X) = \oplus_{r=0}^n H_r(X)$$

and a subgroup of the group of linear transformations of this space

$$G = Gal_{mot}(X) = \Big\{\text{linear transformations of } V \text{ which preserve all elements in the tensor algebra } \bigotimes_{m=0}^{\infty} V^{\otimes m} \text{ which correspond to algebraic cycles in multiple products } X \times \cdots \times X \Big\} \subset \mathrm{GL}(V)$$

Künneth formula:

$$H_r(X \times Y) = \bigoplus_{i+j=r} H_i(X) \otimes H_j(Y)$$

Algebraic subvariety $Z \subset X$ of dimension k can be *triangulated* into a chain $\sigma_Z \in C_{2k}(X)$ without a boundary, i.e. $\partial(\sigma_Z) = 0$, and its class in the homology group $[Z] \in H_{2k}(X)$ is independent of the triangulation.

A k-dimensional algebraic subvariety $Z \subset \underbrace{X \times \cdots \times X}_{m}$ then defines a class

$$[Z] \in \bigoplus_{i_1+\cdots+i_m=2k} H_{i_1}(X) \otimes \cdots \otimes H_{i_m}(X) \subset H_\bullet^{\otimes m}.$$

The *motivic Galois group* of an algebraic variety X is

$$Gal_{mot}(X) \;=\; \Big\{ \text{linear transformations of } H_\bullet \text{ which preserve all classes of algebraic cycles in the tensor algebra } \bigotimes_{m=0}^{\infty} H_\bullet^{\otimes m} \Big\} \subset GL(H_\bullet(X))$$

The *conjugates of a period* $w = \int_\sigma \omega$ are then all periods

$$w^g \;=\; \int_{g\,\sigma} \omega \,, \qquad g \in Gal_{mot}(X)\,.$$

For example, for an *elliptic curve*

$$
\begin{aligned}
X : y^2 &= x^3 + ax^2 + bx + c \\
&= (x-\alpha_1)(x-\alpha_2)(x-\alpha_3) \quad \alpha_i \neq \alpha_j
\end{aligned}
$$

we consider

$$
H_\bullet(X) = H_0(X) \oplus H_1(X) \oplus H_2(X) \cong \mathbb{Q} \oplus \mathbb{Q}^2 \oplus \mathbb{Q}\,.
$$

Both $H_0(X) = \mathbb{Q} \cdot [pt]$ and $H_2(X) = \mathbb{Q} \cdot [X]$ are spanned by

algebraic classes $[pt]$ and $[X]$ correspondingly. For a *generic* elliptic curve there are no nontrivial algebraic cycles in $X \times \cdots \times X$, and therefore

$$
Gal_{mot}(X) = GL(H_1(X)) \cong GL_2(\mathbb{Q})\,.
$$

The period

$$w_1 = \int_{\alpha_1}^{\alpha_2} \frac{dx}{\sqrt{x^3 + ax^2 + bx + c}}$$

has a conjugate

$$w_2 = \int_{\alpha_2}^{\alpha_3} \frac{dx}{\sqrt{x^3 + ax^2 + bx + c}},$$

and the whole set of its Galois conjugates is given by

$$\left\{\alpha_1 w_1 + \alpha_2 w_2 \mid \alpha_1, \alpha_2 \in \mathbb{Q}, \text{ not both zero}\right\}$$

It remains to consider also "nongeneric" elliptic curves. For any curve one can show that $w_1/w_2 \in \mathbb{C} \setminus \mathbb{R}$. In particular, the ratio of two periods w_1/w_2 is never rational. "Nongeneric" curves are those for which w_1/w_2 satisfies a quadratic equation with rational coefficients, so called *curves with complex multiplication*. These have extra algebraic cycles in $X \times X$, which the motivic Galois group must preserve.

Consider the field $K = \mathbb{Q}(w_1/w_2)$. It is a quadratic extension of $\mathbb{Q}$ and $Gal_{mot}(X)$ in this case is the normalizer N_K of a Cartan subgroup of $GL(H_1(X)) \cong GL_2(\mathbb{Q})$ isomorphic to the multiplicative group $K^\times = K \setminus \{0\}$ (vieved as a 2-dimensional torus over $\mathbb{Q}$). The answer for the set of conjugates of a period in this case is exactly the same.

Motives

$Var(\mathbb{Q})$ the category of algebraic varieties defined over $\mathbb{Q}$
One expects existence of an *abelian* category $MM = MM_{\mathbb{Q}}(\mathbb{Q})$ of *mixed motives* over $\mathbb{Q}$ with rational coefficients, and of a functor

$$h : Var(\mathbb{Q}) \to MM$$

which plays a role of universal cohomology theory. Its full subcategory NM (*pure* or *numerical* motives) has a simple description in terms of enumerative projective geometry: up to inessential technical modifications (idempotent completion and inversion of the reduced motive $\mathbb{Q}(-1)$ of the projective line), its objects are smooth projective varieties and morphisms are given by algebraic correspondences up to numerical equivalence.

Motivic Galois group

Cartesian product on $Var(\mathbb{Q})$ corresponds via h to a certain tensor product $\otimes$ on MM, which makes MM into a *tannakian category*. There is a $\otimes$-functor

$$H : MM \to Vec_{\mathbb{Q}}$$

such that $H(h(X)) = H^{\bullet}(X)$. For any motive M one denotes by $\langle M \rangle$ the tannakian subcategory of MM generated by a motive M: its objects are given by algebraic construction on M (sums, subquotients, duals, tensor products). The *motivic Galois group* is the group-scheme

$$Gal_{mot}(M) \;=\; Aut^{\otimes} H\Big|_{\langle M \rangle}$$

of automorphisms of the restriction of the $\otimes$-functor H to $\langle M \rangle$.

$2\pi i$ is a period of so-called *Lefschetz motive* $\mathbb{Q}(-1) = H^1(\mathbb{P}^1)$. $Gal_{mot}(\mathbb{Q}(-1)) = \mathbb{Q}^\times$ and the conjugates are all nonzero rational multiples of $2\pi i$.

$\log q$ for $q \in \mathbb{Q} \setminus \{-1, 0, 1\}$ is a period of so-called Kummer 1-motive M_q. Grothendieck's conjecture for M would imply that $\log q$ and π are algebraically independent. If so, the conjugates are $\log q + \mathbb{Q}\pi i$.

$\zeta(s)$ for an odd integers $s > 1$ is a period of so-called mixed Tate motive over $\mathbb{Z}$. Grothendieck's conjecture would imply that $\zeta(3), \zeta(5), \dots$ are algebraically independent and the conjugates are $\zeta(s) + \mathbb{Q}(\pi i)^s$.