Nahm's conjecture about modularity of q-series

(joint work with S. Zwegers)

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Let $r \geq 1$ and

$$F_{A,B,C}(q) = \sum_{n \in (\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2}n^T A n + n^T B + C}}{(q)_{n_1} \dots (q)_{n_r}}, \quad |q| < 1$$

where

$$A \in M_r(\mathbb{Q}) \text{ positive definite, symmetric}$$

$$B \in \mathbb{Q}^n, \quad C \in \mathbb{Q}, \qquad (q)_n = \prod_{k=1}^{n} (1 - q^k)$$

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Problem (Werner Nahm): for which triples of parameters (A, B, C) the function $F_{A,B,C}$ is a modular form?

the case $r = 1$

Theorem(D. Zagier, M.Terhoeven, 2007) All triples (A, B, C) in $\mathbb{Q}_+ \times \mathbb{Q} \times \mathbb{Q}$ for which $F_{A,B,C}$ is modular are given in the following table.

A	B	C	$F_{A,B,C}(e^{2\pi iz})$
2	0	$-1/60$	$\theta_{5,1}(z)/\eta(z)$
	1	$11/60$	$\theta_{5,2}(z)/\eta(z)$
1	0	$-1/48$	$\eta(z)^2/\eta(\frac{z}{2})\eta(2z)$
	$1/2$	$1/24$	$\eta(2z)/\eta(z)$
	$-1/2$	$1/24$	$2\eta(2z)/\eta(z)$
$1/2$	0	$-1/40$	$\theta_{5,1}(\frac{z}{4})\eta(2z)/\eta(z)\eta(4z)$
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Here $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$, $\theta_{5,j}(z) = \sum_{n\equiv 2j-1(10)} (-1)^{[n/10]} q^{n^2/40}$.

Lemma. Let $F(q)$ be a modular form of weight w for a subgroup of finite index $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$. Then when $\varepsilon \searrow 0$

$$F(e^{-\varepsilon}) \sim b\,\varepsilon^{-w} e^{\frac{a}{\varepsilon}}\Big(1 + o(\varepsilon^N)\Big) \qquad \forall N \geq 0$$

for appropriate numbers $a, b \in \mathbb{C}$. Moreover, $a \in \pi^2\mathbb{Q}$ here.

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Strategy: compute the asymptotics of $F_{A,B,C}(e^{-\varepsilon})$ when $\varepsilon \searrow 0$.

Lemma. The system of equations

$$1 - Q_i = \prod_{j=1}^{r} Q_j^{A_{ij}}, \qquad i = 1, \dots, r$$

has a unique solution with $Q_i \in (0, 1)$ for all $1 \le i \le r$.

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Let $q \to 1$ and $n_i \to \infty$ so that $q^{n_i} \to Q_i$. Then

$$\frac{a_{n+e_i}(q)}{a_n(q)} = \frac{q^{n^TAe_i + \frac{1}{2}e_i^TAe_i + e_i^TB}}{1 - q^{n_i+1}} \to \frac{Q_1^{A_{i1}} \dots Q_r^{A_{ir}}}{1 - Q_i} = 1 .$$

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For fixed small ε terms $a_n(e^{-\varepsilon})$ are maximal around

$$n \approx \Big(-\frac{\log(Q_1)}{\varepsilon}, \dots, -\frac{-log(Q_r)}{\varepsilon}\Big).$$

Theorem. There is an asymptotic expansion

$$F_{A,B,C}(e^{-\varepsilon}) \sim \beta e^{\frac{\alpha}{\varepsilon} - \gamma\varepsilon}\Big(1 + \sum_{m=1}^{\infty} c_m \varepsilon^m\Big), \quad \varepsilon \searrow 0$$

with the coefficients given as follows:

$$\alpha = \sum_{i=1}^{r}\big(L(1) - L(Q_i)\big) > 0,$$

$$\beta = \det \widetilde{A}^{-1/2} \prod_i Q_i^{B_i}(1 - Q_i)^{-1/2}, \quad \gamma = C + \frac{1}{24}\sum \frac{1+Q_i}{1-Q_i},$$

where $L(x) = Li_2(x) + \frac{1}{2}\log(x)\log(1-x)$ is the Rogers dilog,

$$\widetilde{A} = A + \mathrm{diag}\{\xi_1, \dots, \xi_r\}, \quad \xi_i = \frac{Q_i}{1 - Q_i} > 0$$

and

$$c_m \; = \; \det \widetilde{A}^{1/2}(2\pi)^{-r/2} \int C_{2m}(B, \xi, t) e^{-\frac{1}{2} t^T \widetilde{A} t} dt$$

where C_{2m} are certain polynomials in $3r$ variables.

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$$\exp\Big[\big(B+\frac{1}{2}\frac{Q}{1-Q}\big)T\varepsilon^{1/2} - \sum_{m=3}^{\infty}\frac{1}{m!}B_m\big(\frac{T}{\varepsilon^{1/2}}\big)Li_{2-m}(Q)\,\varepsilon^{m-1}\Big] = 1+\sum_{m=1}^{\infty}D_m\Big(B,\frac{Q}{1-Q},T\Big)\,\varepsilon^{m/2}.$$

Then

$$C_m(B,\xi,t) = \sum_{m_1+\cdots+m_r=m}\prod_{i=1}^{r}D_{m_i}(B_i,\xi_i,t_i).$$

Corollary. If $F_{A,B,C}$ is modular then

- its weight $w = 0$
- $\alpha \in \pi^2\mathbb{Q} \iff \sum_{i=1}^{r} L(Q_i) \in \pi^2\mathbb{Q}$
- $e^{-\gamma\varepsilon}\left(1 + \sum_{p=1}^{\infty} c_p \varepsilon^p\right) = 1 \iff c_p = \frac{\gamma^p}{p!} \quad \forall p$

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Since

$$c_m \;=\; \det \widetilde{A}^{1/2} (2\pi)^{-r/2} \int C_{2m}(B,\xi,t) e^{-\frac{1}{2} t^T \widetilde{A} t} dt$$

are polynomials in the entries of B, ξ and $\widetilde{A}^{-1}$, we have infinitely many polynomial equations:

$$\left(c_m - \frac{1}{m!} c_1^m\right)\left(B, \xi, \widetilde{A}^{-1}\right) \;=\; 0\,, \quad m = 2, 3, \ldots$$

Back to the case $r = 1$: let

$$\widetilde{c}_m(B,\xi,A) = (A+\xi)^{3m}\left[c_m - \frac{1}{m!}c_1^m\right]\left(B,\xi,\frac{1}{A+\xi}\right), \quad m = 2,3,\dots$$

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Using *Magma* we have got the following decomposition of the radical of

$$I = \langle \widetilde{c}_2, \widetilde{c}_3, \widetilde{c}_4, \widetilde{c}_5 \rangle \subset \mathbb{Q}[B,\xi,A]$$

into prime ideals

$$\mathrm{Rad}(I) \;=\; \mathcal{P}_1 \cdot \ldots \cdot \mathcal{P}_{14}$$

where the generators of $\mathcal{P}_i$ are given below:

i	generators of $\mathcal{P}_i$		
1		ξ	
2		$\xi + 1$	
3	$B - 1/2,$	$\xi + 2,$	A
4	$B - 1,$	$\xi + 2,$	A
5	$B,$	$\xi + 2,$	A
6	$B + 1/2,$	$\xi^2 + 3\xi + 1,$	$A + 1$
7	$B - 1/2,$	$\xi^2 + 3\xi + 1,$	$A + 1$
8	$B + 1/2,$	$\xi - 1,$	$A - 1$
9	$B,$	$\xi - 1,$	$A - 1$
10	$B - 1/2,$	$\xi - 1,$	$A - 1$
11	$B - 1,$	$\xi^2 - \xi - 1,$	$A - 2$
12	$B,$	$\xi^2 - \xi - 1,$	$A - 2$
13	$B - 1/2,$	$\xi^2 + \xi - 1,$	$A - 1/2$
14	$B,$	$\xi^2 + \xi - 1,$	$A - 1/2$

A	B	C	$F_{A,B,C}(e^{2\pi i z})$
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Theorem. Consider $A = \begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix}$, $a > \frac{1}{2}$, $a \neq 1$. Then all pairs (B, C) such that $F_{A,B,C}$ is modular are:

B	C	$F_{A,B,C}(e^{2\pi i z})$
$\begin{pmatrix} b \\ -b \end{pmatrix}$	$\frac{b^2}{2a} - \frac{1}{24}$	$\frac{1}{\eta(z)} \sum_{n \in \mathbb{Z} + \frac{b}{a}} q^{an^2/2}$
$\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$	$\frac{1}{8a} - \frac{1}{24}$	$\frac{2}{\eta(z)} \sum_{n \in \mathbb{Z} + \frac{1}{2a}} q^{an^2/2}$
$\begin{pmatrix} 1-\frac{a}{2} \\ \frac{a}{2} \end{pmatrix}$ and $\begin{pmatrix} \frac{a}{2} \\ 1-\frac{a}{2} \end{pmatrix}$	$\frac{a}{8} - \frac{1}{24}$	$\frac{1}{2\eta(z)} \sum_{n \in \mathbb{Z} + \frac{1}{2}} q^{an^2/2}$

Theorem. Modular functions $F_{A,B,C}(z)$ with the matrix A being of the form $\begin{pmatrix} a & \frac{1}{2}-a \\ \frac{1}{2}-a & a \end{pmatrix}$ exist if and only if $a=1$, $a=3/4$ or $a=1/2$. Below is the list of all such modular functions.

A	B	C	$F_{A,B,C}(e^{2\pi iz})$
$\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$-\frac{1}{20}$	$(\theta_{5,\frac{3}{4}}(2z)+\theta_{5,\frac{13}{4}}(2z))\eta(z)/\eta(2z)\eta(z/2)$ $+\,2\theta_{5,2}(2z)\eta(2z)/\eta(z)^2$
	$\begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$	$\frac{1}{20}$	$2\theta_{5,1}(2z)\eta(2z)/\eta(z)^2$ $+\theta_{5,\frac{3}{2}}(z)\theta_{5,2}(2z)\eta(z)^3/\eta(z/2)^2\eta(2z)^2\eta(10z)$
$\begin{pmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \end{pmatrix}$ and $\begin{pmatrix} -\frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$	$-\frac{1}{80}$	$\theta_{5,1}(\frac{z}{8})\eta(z)/\eta(\frac{z}{2})\eta(2z)$
	$\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$	$\frac{1}{80}$	$\theta_{5,2}(\frac{z}{8})\eta(z)/\eta(\frac{z}{2})\eta(2z)$
$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$-\frac{1}{20}$	$(\theta_{5,1}(\frac{z}{4})\eta(2z)/\eta(z)\eta(4z))^2$
	$\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$	0	$\theta_{5,1}(\frac{z}{4})\theta_{5,2}(\frac{z}{4})(\eta(2z)/\eta(z)\eta(4z))^2$
	$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$	$\frac{1}{20}$	$(\theta_{5,2}(\frac{z}{4})\eta(2z)/\eta(z)\eta(4z))^2$

The Bloch group $B(K)$ of a field K is defined as the quotient of the kernel of the map

$$\begin{aligned} \mathbb{Z}[K] &\to \Lambda^2 K^* \\ [x] \mapsto x \wedge (1-x)\,, \quad [0],[1] &\mapsto 0 \end{aligned}$$

by a subgroup generated by the elements of the form

$$[x] + [y] + [1-xy] + \Big[\frac{1-x}{1-xy}\Big] + \Big[\frac{1-y}{1-xy}\Big]\,.$$

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If K is a number field then

$$B(K) \otimes_{\mathbb{Z}} \mathbb{Q} \cong K_3(K) \otimes_{\mathbb{Z}} \mathbb{Q}$$

and the regulator map is given explicitly on $B(K)$ by

$$\begin{aligned} B(K) &\to \mathbb{R}^{r_2} \\ [x] &\mapsto (D(\sigma_1(x)), \dots, D(\sigma_{r_2}(x))) \end{aligned}$$

where

$$D(x) \;=\; \Im\big(Li_2(x) + \log(1-x)\log|x|\big)$$

is the Bloch-Wigner dilogarithm function.

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If $F_{A,B,C}$ is modular for some B, C then we must have

$$L(Q_1) + \cdots + L(Q_r) \in \pi^2 \mathbb{Q}$$

for the distinguished solution with all $Q_i \in (0, 1)$.

Conjecture.(Werner Nahm) For a positive definite symmetric $r \times r$ matrix with rational coefficients A the following are equivalent:

- There exist $B \in \mathbb{Q}^r$ and $C \in \mathbb{Q}$ such that $F_{A,B,C}$ is a modular function.
- The element $[Q_1] + \cdots + [Q_r]$ is torsion in the corresponding Bloch group for *every* solution of "Nahm's equation".

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True when $r = 1$: one can prove that all solutions of

$$1 - Q = Q^A$$

are totally real if and only if $A \in \{\frac{1}{2}, 1, 2\}$, and for the element $[Q]$ the condition of being torsion is equivalent to being totally real since

$$D(z) = 0 \iff z \in \mathbb{R}\,.$$

For $A = \begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix}$ all solutions are zero in the Bloch group:

$$\begin{cases} 1 - Q_1 &= Q_1^a \, Q_2^{1-a}, \\ 1 - Q_2 &= Q_1^{1-a} Q_2^a, \end{cases}$$

hence

$$\begin{aligned} \frac{1-Q_1}{Q_2} &= \left(\frac{Q_1}{Q_2}\right)^a = \frac{Q_1}{1-Q_2}, \\ (1-Q_1)(1-Q_2) &= Q_1 Q_2, \\ Q_1 + Q_2 &= 1 \quad \Rightarrow \quad [Q_1] + [Q_2] = 0 \text{ in } B(\mathbb{C}). \end{aligned}$$

A counterexample to the conjecture:

$$F_{\begin{pmatrix} 3/4 & -1/4 \\ -1/4 & 3/4 \end{pmatrix}, \begin{pmatrix} 1/4 \\ -1/4 \end{pmatrix}, -\frac{1}{80}} = \frac{\theta_{5,1}(\frac{z}{8})\eta(z)}{\eta(\frac{z}{2})\eta(2z)}$$

but there are non-torsion solutions to

$$\begin{cases} 1 - Q_1 = Q_1^{3/4} \, Q_2^{-1/4}, \\ 1 - Q_2 = Q_1^{-1/4} Q_2^{3/4}. \end{cases}$$

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Let $t = Q_1^{1/4} Q_2^{-1/4}$, then

$$\frac{1-Q_1}{Q_2^{1/2}} = t^3 \qquad \Rightarrow \qquad Q_2^{1/2} = t^{-3}(1-Q_1),$$
$$\frac{1-Q_2}{Q_1^{1/2}} = t^{-3} \qquad \Rightarrow \qquad Q_1^{1/2} = t^{3}(1-Q_2).$$

We substitute these equalities into $Q_1^{1/2} = t^2 Q_2^{1/2}$ and get

$$\begin{aligned} t^3(1-Q_2) &= t^2 t^{-3}(1-Q_1)\,, \\ t^4(1-Q_2) &= 1 - Q_1 = 1 - t^4 Q_2\,, \\ t^4 &= 1\,. \end{aligned}$$

Therefore all solutions are $(Q_1, Q_2) = (x, x)$ where

$$1 - x = t x^{1/2}\,, t^4 = 1$$

or

$$(1-x)^4 = x^2 \quad \Leftrightarrow \quad (x^2 - 3x + 1)(x^2 - x + 1) = 0\,.$$

Hence $(Q_1, Q_2) = \left(\frac{1+\sqrt{-3}}{2}, \frac{1+\sqrt{-3}}{2}\right)$ is a solution, and $2\left[\frac{1+\sqrt{-3}}{2}\right]$ is not torsion because

$$D\left(\frac{1+\sqrt{-3}}{2}\right) = 1.01494...$$

Counterexample with integer entries:

$$A = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad B = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \quad C = \frac{1}{15},$$

$$F_{A,B,C}(q) = \frac{\eta(2z)^2\theta_{5,1}(z)}{\eta(z)^3}.$$

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- The element $[Q_1] + \cdots + [Q_r]$ is torsion in the corresponding Bloch group for *EVERY* (?) solution of

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