p-adic Frobenius structure and monodromy

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Equivalence of differential systems

$$K \supset \mathbb{C}(t) \quad \text{differential field}, \qquad A, B \in \mathbb{C}(t)^{r\times r}$$

$$(1)\ \left\{\frac{dU}{dt} = AU\right\} \qquad\qquad (2)\ \left\{\frac{dV}{dt} = BV\right\}$$

Differential system (1) is equivalent to (2) over K if there exists an invertible matrix $H \in K^{r\times r}$ such that

$$\frac{dH}{dt} = BH - HA.$$

Observe:

- U is a solution to (1) $\Rightarrow$ $V = HU$ is a solution to (2)
- U, V fundamental solution matrices for (1) and (2) $\Rightarrow$ $H = V\Lambda U^{-1}$ where Λ is a constant marix:

$$\frac{d}{dt}(V^{-1}HU) = -(V^{-1}B)HU + V^{-1}(BH - HA)U + V^{-1}H(AU) = 0$$

Example: local study / regular singularities

take $A \in \mathbb{C}(t)^{r\times r}$ with no pole at $t = 0$
$K = \mathcal{O}[t^{-1}]$ field of germs of meromorphic functions

there is $\Gamma \in \mathbb{C}^{r\times r}$ (unique up to conjugation) such that

$$\left\{ t\frac{dU}{dt} = AU \right\} \sim^{K} \left\{ t\frac{dV}{dt} = \Gamma V \right\}$$

- eigenvalues of Γ are called local exponents at $t = 0$
- $V = t^{\Gamma}$ is a fundamental solution matrix for the second system; $M_0 = \exp(2\pi i\Gamma)$ its monodromy around $t = 0$
- differential systems are equivalent over K if and only if their local monodromies M_0 are conjugate

p-adic Frobenius structure

Definition. A differential system with $A \in \mathbb{Q}(t)^{r\times r}$ admits a Frobenius structure of period $h = 1, 2, \dots$ if

$$(1)\ \left\{ t\frac{dV}{dt} = p^h A(t^{p^h})V \right\} \quad \sim^{E_p} \quad (2)\ \left\{ t\frac{dU}{dt} = A(t)U \right\}$$

$E_p =$ field of p-adic analytic elements

is the completion of $\mathbb{C}_p(t)$ w.r.t. the Gauss norm

$$|\sum a_i t^i|_{Gauss} = \max_i |a_i|_p$$

$$t\frac{d\Phi(t)}{dt} = A(t)\Phi(t) - p^h\Phi(t)A(t^{p^h}), \quad \Phi \in E_p^{r\times r}$$

- U is a solution to (2) $\Rightarrow$ $V(t) = U(t^{p^h})$ is a solution to (1)
- let U be a fundamental solution matrix of (2)
  $\exists \Lambda \in \mathbb{C}_p^{r\times r}$ such that $\Phi(t) = U(t)\Lambda U(t^{p^h})^{-1} \in E_p^{r\times r}$

Example: $\frac{dU}{dt} = \frac{1}{2}\frac{1}{1-t}U, \quad U(t) = \frac{1}{\sqrt{1-t}}$

$$p \neq 2 \qquad \Phi(t) = U(t)U(t^p)^{-1} \in^? E_p$$

$$\sqrt{\frac{1-t^p}{1-t}} = (1-t)^{\frac{p-1}{2}}\left(\frac{1-t^p}{(1-t)^p}\right)^{1/2} \qquad g(t) = \tfrac{1-t^p-(1-t)^p}{p} \in \mathbb{Z}[t]$$

$$= (1-t)^{\frac{p-1}{2}}\left(1+p\tfrac{g(t)}{(1-t)^p}\right)^{1/2} = (1-t)^{\frac{p-1}{2}}\sum_{k\geqslant 0}\binom{1/2}{k}p^k\frac{g(t)^k}{(1-t)^{pk}}$$

$$\in \mathbb{Z}[t,(1-t)^{-1}]\widehat{\ } \subset E_p$$

For a ring R such that $\underset{s\geqslant 1}{\cap} p^sR = 0$ we denote its p-adic completion by $\widehat{R} = \varprojlim R/p^sR$

$$\sqrt{\frac{1-t^5}{1-t}} = 1+\tfrac{1}{2}t+\tfrac{3}{8}t^2+\tfrac{5}{16}t^3+\tfrac{35}{128}t^4-\tfrac{65}{256}t^5+\ldots \equiv 1+3t+t^2 \pmod 5$$

$$\equiv \tfrac{1+8t+11t^2+20t^3+5t^4+14t^5+17t^6+24t^7}{(1-t)^5} \pmod{5^2}$$

$$\equiv \tfrac{1+57t+28t^2+9t^3+10t^4+9t^5+28t^6+57t^7+t^8}{(1-t)^6} \pmod{5^3}$$

$$(*) \qquad t(1-t)\frac{d^2}{dt^2} + (c-(a+b+1)t)\frac{d}{dt} - ab$$

$$a, b, c-a, c-b \notin \mathbb{Z} \quad \left(\Leftrightarrow \text{ irreducible monodromy on } \mathbb{P}^1\backslash\{0,1,\infty\}\right)$$

Theorem (Dwork) Differential equation (*) with $a, b, c \in \mathbb{Q} \cap \mathbb{Z}_p$ has a Frobenius structure of period

$$h = \min\{m \geqslant 1 : (p^m-1)a, (p^m-1)b, (p^m-1)c \in \mathbb{Z}\}.$$

Remark: multiplying (1) by t we obtain the operator

$$\theta(\theta+c-1) - t(\theta+a)(\theta+b), \quad \theta = t\frac{d}{dt},$$

so the local exponents are

0	∞	1
0	a	0
$1-c$	b	$c-a-b$

and h in the theorem is such that the eigenvalues of local monodromies M_0, M_1 and M_∞ are (p^h-1)st roots of unity.

Remark (continued): a Frobenius structure is a solution to the 1st order differential system

$$t\frac{d\Phi(t)}{dt} = A(t)\Phi(t) - p^h\Phi(t)A(t^{p^h}).$$

If there is a meromorphic solution $\Phi \in K^{r\times r}$, $K = \mathcal{O}[t^{-1}]$ then

$$M_0 \sim M_0^{p^h} \Rightarrow \text{ eigenvalues are } (p^h - 1)\text{st roots of } 1$$

It is often the case that $\Phi \in (E_p \cap K)^{r\times r}$.
Similar considerations apply at other singular points.

Example: $\theta^2 - t(\theta + \frac{1}{3})(\theta + \frac{2}{3}) \qquad A(t) = \begin{pmatrix} 0 & 1 \\ \frac{t}{1-t} & -\frac{2}{9}\frac{t}{1-t} \end{pmatrix}$

$$u_0(t) = {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}, 1; t) = 1 + \tfrac{2}{3^2}t + \tfrac{10}{3^4}t^2 + \tfrac{560}{3^8}t^3 + \ldots \in \mathbb{Z}[[3^{-3}t]]$$
$$u_1(t) = \log(t)u_0(t) + \tfrac{5}{9}t + \tfrac{19}{54}t^2 + \ldots = \log(t)u_0(t) + u_1^{an}(t)$$

$$U(t) = \begin{pmatrix} u_0 & u_1 \\ \theta u_0 & \theta u_1 \end{pmatrix} = \begin{pmatrix} u_0 & u_1^{an} \\ \theta u_0 & \theta u_1^{an} + u_0 \end{pmatrix} \begin{pmatrix} 1 & \log(t) \\ 0 & 1 \end{pmatrix}, \ \theta(U) = AU$$
$$U^{an}(t) := \begin{pmatrix} u_0 & u_1^{an} \\ \theta u_0 & \theta u_1^{an} + u_0 \end{pmatrix} = U\Big|_{\text{``}\log(t)=0\text{''}} \in \mathbb{Q}[[t]]^{2\times 2}, \ U^{an}(0) = Id$$

Look for Λ such that $\Phi(t) = U(t)\Lambda U(t^p)^{-1} \in E_p^{2\times 2}$. Observe:

$$\begin{pmatrix} 1 & \log(t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \begin{pmatrix} 1 & -p\log(t) \\ 0 & 1 \end{pmatrix} = \text{const} \quad \Leftrightarrow \quad \begin{cases} \lambda_{21} = 0 \\ \lambda_{22} = p\lambda_{11} \end{cases}$$

$\Rightarrow$ analytic at $t = 0$ solutions to $\theta(\Phi) = A(t)\Phi(t) - p\Phi(t)A(t^p)$ are

$$\Phi(t) = U(t)\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ 0 & p\lambda_{11} \end{pmatrix} U(t^p)^{-1} = U^{an}(t)\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ 0 & p\lambda_{11} \end{pmatrix} U^{an}(t^p)^{-1}$$

Claim: for $p \neq 2, 3$, there exists $\alpha \in \mathbb{Z}_p$ such that

$$\Phi(t) = U^{an}(t) \begin{pmatrix} 1 & \alpha \\ 0 & p \end{pmatrix} U^{an}(t^p)^{-1} \in \mathbb{Z}[t, \widehat{(1-t})^{-1}]^{2\times 2}.$$

- α can be found experimentally: e.g. $p = 5$

  $\Phi(t) = \Phi_0(t) + \alpha\Phi_1(t), \qquad \Phi_i \in (\mathbb{Q} \cap \mathbb{Z}_p)[[t]]^{2\times 2}$

  $$\Phi_0(t) \equiv \begin{pmatrix} 1+3t & 3t \\ 0 & 0 \end{pmatrix} \mod 5, \quad \deg(\Phi_1(t) \mod 5) \approx \infty$$

  $\deg(\Phi_0(t) \mod 5^2) \approx \infty, \quad$ but

  $$\deg\left(\Phi_0(t) + i \cdot 5 \cdot \Phi_1(t) \mod 5^2\right) = \begin{cases} 3, i = 2 \\ \infty, i = 0, 1, 3, 4 \end{cases}$$

  $\Rightarrow \alpha = 2 \cdot 5 + O(5^2)$

$$\deg\left(\Phi_0(t) + 2\cdot 5\cdot \Phi_1(t) \mod 5^2\right) = 3$$

$$\deg\left(\Phi_0(t) + (2\cdot 5 + i\cdot 5^2)\cdot \Phi_1(t) \mod 5^3\right) \approx \infty,\ i = 0,1,2,3,4$$

$$\deg\left((\Phi_0(t) + (2\cdot 5 + i\cdot 5^2)\cdot \Phi_1(t))(1-t)^5 \mod 5^3\right) = \begin{cases} 8, i = 4 \\ \infty, i = 0,1,2,3 \end{cases}$$

$$\alpha = 2\cdot 5 + 4\cdot 5^2 + O(5^3)$$

$$\dots$$

$$\deg\left((\Phi_0(t) + \alpha\cdot \Phi_1(t))(1-t)^{5(m-2)} \mod 5^m\right) = 5(m-2) + 3$$

$$\begin{aligned}\alpha &= 2\cdot 5 + 4\cdot 5^2 + 3\cdot 5^3 + 4\cdot 5^4 + 3\cdot 5^5 + 2\cdot 5^6 + 3\cdot 5^7 + 4\cdot 5^8 + 5^9 + \dots \\ &= -12\cdot \log_5(3)\end{aligned}$$

For all p we tried, this experiment gives a unique α

$$\Phi(0) = \begin{pmatrix} 1 & -3(p-1)\log_p(3) \\ 0 & p \end{pmatrix}$$