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Algebraic hypergeometric functions II

Eisenstein theorem was proved last time.

Consider $x(t) = \sum_{n \geq 0} u_n t^n$

$$u_n = \prod_{m>0} (mn)!^{\gamma_m} \qquad \gamma = (\gamma_1, \gamma_2, \dots) \in \mathbb{Z}^\infty, \ \neq (0,0,0,\dots)$$

almost all $= 0$
(finite sequence)

Assume that $\sum_{m>0} m \gamma_m = 0$ ("regularity")

Today we will prove that if $x(t)$ is algebraic then

- all $u_n \in \mathbb{Z}$
- $d := -\sum_{m>0} \gamma_m = 1$.

<u>Lemma 1</u> $\exists$ minimal $r \geq 1$
$\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r \in \mathbb{Q} \cap (0;1]$
such that

$$x(t) = \sum_{n=0}^{\infty} \prod_{k=1}^{r} \frac{(\alpha_k)_n}{(\beta_k)_n} \lambda^n t^n,$$

where $\lambda = \prod_{m \geq 1} m^{m\gamma_m}$.

<u>Proof</u> $(mn)! = \prod_{k=1}^{m} \left(\frac{k}{m}\right)_n \cdot m^n$

$$u_n = \prod_{m>0} \prod_{k=1}^{m} \left[\left(\frac{k}{m}\right)_n \cdot m^n \right]^{\gamma_m} = \lambda^n \prod_{m>0} \prod_{k=1}^{m} \Big(\overset{\frac{k}{m}\ !!}{\alpha_{k,m}}\Big)_n^{\gamma_m}$$

after cancellation, which happens when $\frac{\tilde{k}}{\tilde{m}} = \frac{k}{m}$ and $\gamma_m \gamma_{\tilde{m}} < 0$,

we clearly have

$$u_n = \lambda^n \frac{(\alpha_1)_n \ldots (\alpha_r)_n}{(\beta_1)_n \ldots (\beta_r)_n} \quad \boxtimes$$

for some r, since $\sum m\gamma_m = 0$

Question: why can't it happen that $r=0$, that is all $\alpha_{k,m}$ cancel?

We will see soon why this is not possible. So far we have $r \geq 0$ in the above Lemma.

$$\prod_{m>0} (t^m - 1)^{\gamma_m} = \prod_{m>0} \prod_{k=1}^{m} \left(t - \exp(2\pi i \alpha_{k,m})\right)^{\gamma_m}$$

$$= \prod_{k=1}^{r} \frac{(t - \exp(2\pi i \alpha_k))}{(t - \exp(2\pi i \beta_k))} =: \frac{A(t)}{B(t)}$$

Observation

Since cyclotomic polynomials are irreducible, we have the following

Corollary If $\frac{p}{q} \in \{\alpha_1, \ldots, \alpha_r\}$, $(p,q) = 1$ then every $\frac{\hat{p}}{q} \in \{\alpha_1, \ldots, \alpha_r\}$ with the same multiplicity as $\frac{p}{q}$, where $\hat{p}$ is any other number $1 \leq \hat{p} \leq q$ with $(\hat{p}, q) = 1$.

The same is true for $\{\beta_1, \ldots, \beta_r\}$.

$$\nu_p(n!) = \sum_{j>0} \left\lfloor \frac{n}{p^j} \right\rfloor$$

$\uparrow$
p-adic valuation

$$\Rightarrow \quad \nu_p(u_n) = \sum_{j,m>0} \gamma_m \left\lfloor \frac{mn}{p^j} \right\rfloor$$

Let us introduce the <u>Landau function</u> of the sequence $\gamma = (\gamma_1, \gamma_2, \dots)$:

$$L(x) := \sum_{m\geq 1} \gamma_m \lfloor x\cdot m \rfloor = -\sum_{m\geq 1} \gamma_m \{x\cdot m\}$$

"regularity" condition $\sum m\gamma_m = 0$

<u>Lemma</u> ① $L(x)$ is 1-periodic and right-continuous

② for $0 \leq x \leq 1$ we have

$$L(x) = \#\{\alpha_k : \alpha_k \leq x\} - \#\{k : \beta_k \leq x\}$$

③ $L(x) + L(1-x) = d$ away from discontinuity points

<u>Proof</u> ① is obvious, it follows from the properties of $\{\cdot\}$-function

② $0 \leq x \leq 1$

$$L(x) = \sum_{m\geq 1} \gamma_m \lfloor x m \rfloor = \sum_{m\geq 1} \gamma_m \cdot \#\{1 \leq k \leq m : \underbrace{\tfrac{k}{m}}_{\alpha_{k,m}} \leq x\}$$

= (same cancellation as in the previous lemma takes place) $= \#\{1 \leq k \leq r : \alpha_k \leq x\}$ $- \#\{1 \leq k \leq r : \beta_k \leq x\}$.

③ By the above corollary, it is clear that $L(x) + L(1-x)$ is constant (away from discontinuity points): if $L(x)$ makes a jump at some $x = \frac{p}{q}$, then $L(1-x)$ makes a jump of the same size but in the opposite direction, because $x = \frac{q-p}{q}$ belongs to the set of α_s or β_s with the same multiplicity.

So, $L(x) + L(1-x) = \text{const}$

when $x \notin \{\alpha_1, .., \alpha_r, \beta_1, .., \beta_r\}$.

To find this constant we $x \to 1$: since $L(1) = L(0) = 0$, for x close to 1 but < 1 we must have

$$L(x) = \#\{k : \beta_k = 1\} - \#\{k : \alpha_k = 1\}$$

$$= -\sum \gamma_m = d. \qquad \boxtimes$$

Observation: If in the first Lemma we get $r = 0$, so all $\alpha_{k,m}$ cancelled out, then $L(x) \equiv 0$ and

$\nu_p(u_n) = 0 \quad \forall n, \forall p \quad \Rightarrow \quad u_n = 1, \ \forall n$

$\Rightarrow \lambda = \prod m^{m \cdot \gamma_m} = 1 \quad \Rightarrow \quad \gamma_m = 0 \ \forall m$. Contradiction. So $r \geq 1$.

Lemma $\forall n \ u_n \in \mathbb{Z} \iff \forall x \ L(x) \geq 0$

Proof ($\Leftarrow$) follows from $v_p(u_n) = \sum_{j>0} L\left(\frac{x}{p^j}\right)$

($\Rightarrow$) Suppose $L(x_0) < 0$

Choose δ s.t. $L(x) \equiv 0$ on $[0, \delta]$

and x_1 s.t. $L(x) < 0$ for $x \in [x_0, x_1]$.

Then for any $p \gg 0$ we have

$p\delta > 1$ and $\exists\, k = k_p$ s.t. $\frac{k}{p} \in [x_0, x_1]$.

For this k

$$v_p(u_k) = \sum_{j>0} L\left(\frac{k}{p^j}\right) = L\left(\frac{k}{p}\right) < 0.$$

because $\frac{k}{p^2} < \frac{1}{p} < \delta$.

$\boxtimes$

$$x(t) = \sum_{n \geq 0} u_n t^n$$

$$\lambda = \prod_{m \geq 1} m^{m\gamma_m}$$

$$x\left(\frac{t}{\lambda}\right) = \sum_{n=0}^{\infty} \prod_{k=1}^{r} \frac{(\alpha_k)_n}{(\beta_k)_n} t^n$$ satisfies

the hypergeometric diff. equation:

$$(*) \quad t(D+\alpha_1)\dots(D+\alpha_r) - (D+\beta_1-1)\dots(D+\beta_r-1)$$

annihilates $x\left(\frac{t}{\lambda}\right)$ $\qquad D = t\frac{d}{dt}$

Take $t_0 \in \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ as a base point and let

V = $\mathbb{C}$-vector space of solutions to $(*)$ near t_0.

$\dim_{\mathbb{C}} V = r$

Consider the monodromy representation:

$$G := \pi_1\left(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\} \right)$$

$$\rho : G \longrightarrow GL(V)$$

$$M := \rho(G) \subset GL(V)$$

is called the monodromy group of $(*)$.

Note that the action of M in V is irreducible. This follows from the crijterion of irreducibility for hypergeometric differential operators:

$$\alpha_i \neq \beta_j \mod \mathbb{Z} \quad \forall i, j$$

which is clear in our case (see Lemma 1).

<u>Lemma</u> If $x(t)$ is algebraic then $\# M < \infty$.

<u>Proof</u> $F(x(t), t) = 0$ for some $F \in \mathbb{C}[X, Y]$, polynomial.

$$F(X, t) = \sum_{i=0}^{N} a_i(t) X^i$$

have no monodromy

Therefore $Mx(t) \subset$ finite set of roots of $\sum_{i=0}^{N} a_i(t) X^i$,

$$\# Mx(t) < \infty.$$

As M acts irreducibly, then $\mathrm{Span}_{\mathbb{C}}(Mx(t)) = V$.

Let $m_1 x(t), \dots, m_r x(t)$ be a basis of V.

$$e_i := m_i x(t)$$

Since Me_i is a finite set, then there are finitely many possibilities for the columns of matrices in M written in this basis, so $\# M < \infty$. $\boxtimes$

Theorem Let $x(t) = \sum_{n \geq 1} u_n t^n$

with $u_n = \prod_{m \geq 1} (mn)!^{\gamma_m}$

is an algebraic function.

Then $u_n \in \mathbb{Z}$ $\forall n$ and $d = 1$.

Proof Suppose not all $u_n \in \mathbb{Z}$. Then $\mathcal{L}(x_0) < 0$ for some x_0, and therefore for all $p >> 0$ there exists n s.t. $\nu_p(u_n) < 0$. But this contradicts Eisenstein's theorem.

It remains to prove that $d = 1$. Since $\mathcal{L}(x) \geq 0$, we have $d > 0$. Suppose that $d > 1$. Recall that

$$d = \# \{ 1 \leq k \leq r : \beta_k = 1 \}.$$

From Levelt's Theorem we know that

$$M_0 \sim \begin{pmatrix} 0 & & & b_0 \\ 1 & 0 & & \vdots \\ & 1 & \ddots & \vdots \\ & & \ddots & b_{r-1} \end{pmatrix} \qquad (*)$$

where $B(t) = \prod_{k=1}^{r} (t - \exp(2\pi i \beta_k)) = t^r + \sum_{i=0}^{r-1} b_i t^i$.

$\Rightarrow$ 1 is a rooot of $B(t)$ of mutliplicity d.

But $(*)$ has $\mathrm{rk}(M_0 - Id) = r-1$ $\Rightarrow$ geometric multiplicity of eigenvalue 1 is 1 $\Rightarrow$

$$\dim \operatorname{Ker}(M_0 - \mathrm{Id}) = 1,$$

so there is one Jordan block of size d corresponding to this eigenvalue:

$$\underbrace{\begin{pmatrix} 1 & 1 & & 0 \\ & 1 & 1 & \\ & & \ddots & 1 \\ 0 & & & 1 \end{pmatrix}}_{d}$$

If $d > 1$, then no power of this Jordan block is Id, which can not happen for an element of a finite group. Thus $d = 1$.

Out theorem is proved. $\boxtimes$