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Dedicated to memory of V. A. Geyler, highly intelligent and clever Russian mathematician

Parametrization of Supersingular Perturbations in the Method of Rigged Hilbert Spaces

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Abstract. A classification of bounded below supersingular perturbations $\tilde{A}$ of a self-adjoint operator $A \geqslant 1$ is suggested. In the A-scale of Hilbert spaces $\mathcal{H}_{-k} \sqsupset \mathcal{H} \sqsupset \mathcal{H}_k = \operatorname{Dom} A^{k/2}$, $k > 0$, a parametrization of operators $\tilde{A}$ in terms of bounded mappings $S\colon \mathcal{H}_k \to \mathcal{H}_{-k}$ such that $\ker S$ is dense in $\mathcal{H}_{k/2}$ is obtained.

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1. INTRODUCTION

Let $A \geqslant 1$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$ with norm $\|\cdot\|$ and with inner product $(\cdot\,,\cdot)$. One can think that $\mathcal{H} = L_2(\mathbb{R}^d, dx)$, $d \geqslant 1$ and $A = H_0 + 1$, where $-H_0 = \Delta$ stands for the Laplacian. To the operator A we assign the A-scale of Hilbert spaces

$$\mathcal{H}_{-k} \sqsupset \mathcal{H}_0 \equiv \mathcal{H} \sqsupset \mathcal{H}_k \equiv \mathcal{H}_k(A), \quad k > 0. \tag{1.1}$$

Here $\mathcal{H}_k = \operatorname{Dom} A^{k/2}$ with respect to the norm $\|\varphi\|_k := \|A^{k/2}\varphi\|$, $\varphi \in \operatorname{Dom} A^{k/2}$, and $\mathcal{H}_{-k}$ is the completion of $\mathcal{H}$ with respect to the negative norm $\|h\|_{-k} := \|A^{-k/2}h\|$, $h \in \mathcal{H}$ (for details, see [12, 13, 3, 16, 2]). The symbol $\sqsupset$ stands for a dense and continuous embedding.

A simple but important fact is that the operator A (as well as the A-scale) can be reconstructed from a couple of spaces $\mathcal{H} \sqsupset \mathcal{H}_k$ (or the conjugate couple $\mathcal{H}_{-k} \sqsupset \mathcal{H}$) with an arbitrary fixed $k > 0$. Namely, $A = \sqrt[k]{D_k}$, where D_k stands for the restriction of the canonical unitary isomorphisms $D_{-k,k} : \mathcal{H}_k \to \mathcal{H}_{-k}$ on the set $\mathcal{D}_k \equiv \mathcal{H}_{2k} = \{\varphi \in \mathcal{H}_k \mid D_{-k,k}\varphi \in \mathcal{H}\}$ (for details, see [3]). Note that the relationship between A and D_k is based on the spectral theorem [12, 13], and therefore this relationship is not evident, except for the case $k = 2$. However, A is uniquely defined by D_k with any chosen $k > 0$. Let us use a similar relationship for the perturbed operators. Thus, let $\tilde{A} \neq A$ be a perturbed operator. Let $\tilde{A} \geqslant 1$. Then we associate with $\tilde{A}$ the new scale of Hilbert spaces,

$$\tilde{\mathcal{H}}_{-k} \sqsupset \mathcal{H}_0 \equiv \mathcal{H} \sqsupset \tilde{\mathcal{H}}_k, \quad \tilde{\mathcal{H}}_k = \operatorname{Dom} \tilde{A}^{k/2}, \qquad k > 0. \tag{1.2}$$

The idea of the method of rigged Hilbert spaces is that, for a given "singular perturbant" S of A (any order of singularity for S is admitted), we first construct one of the couples $\mathcal{H} \sqsupset \tilde{\mathcal{H}}_k$ or $\tilde{\mathcal{H}}_{-k} \sqsupset \mathcal{H}$ and then define the perturbed operator $\tilde{A}$ as the operator associated with the scale (1.2). In fact, the method generalizes and develops the well-known method of form-sums [5, 7, 10, 11], where $\tilde{A}$ is defined as the operator associated with the triplet $\tilde{\mathcal{H}}_{-1} \sqsupset \mathcal{H} \sqsupset \tilde{\mathcal{H}}_1$. Thus, to consider singular higher-order perturbations, cf. [9, 20] (i.e., to treat the situation with supersingular perturbants which are more singular than those in the so-called $\mathcal{H}_{-2}$-class), we must first construct one of the spaces $\tilde{\mathcal{H}}_k$ or $\tilde{\mathcal{H}}_{-k}$, $k > 2$. We say that an operator $\tilde{A}$ is a singular perturbation of A in the wide sense if there is a $k \geqslant 2$ such that the operators $\tilde{A}^{k/2}, A^{k/2}$ coincide on a set $\mathcal{M}_k \subset \tilde{\mathcal{H}}_k \bigcap \mathcal{H}_k$ dense in $\mathcal{H}$ (for the precise definition, see the next section). Clearly, in this case, the inner products on $\tilde{\mathcal{H}}_k, \mathcal{H}_k$ partially coincide, $(\varphi,\psi)_k = (\varphi,\psi)_k^{\sim}$ for $\varphi,\psi \in \mathcal{M}_k$. In particular, if $k > 2$, then the set $\mathcal{M}_k$ is possibly dense not only in $\mathcal{H}$ but also in $\mathcal{H}_2 = \operatorname{Dom} A$, and the symmetric operator $\dot{A} := A \restriction \mathcal{M}_k$ is essentially self-adjoint.

In the present paper we show that the difference between the (super)singular perturbation $\tilde{A}$ and A can be described in terms of operators S taking $\mathcal{H}_k$ to $\mathcal{H}_{-k}$, $k \geqslant 2$, which vanish on $\mathcal{M}_k$ (or in terms of the associated quadratic forms γ_S, which are singular on $\mathcal{H}$). Thus, one of the objectives of this paper is to give a parametrization of the bounded below perturbations ($\tilde{A} > \tilde{m} > -\infty$) that are singular in the wide sense in terms of mappings $S\colon \mathcal{H}_k \to \mathcal{H}_{-k}$, $k \geqslant 2$, with some additional properties.

2. METHOD OF RIGGED HILBERT SPACES

Let us describe the idea of the method of rigged Hilbert spaces in more detail. Consider a part of the A-scale (1.1),

$$\mathcal{H}_{-2} \sqsupset \mathcal{H}_{-1} \sqsupset \mathcal{H}_0 \equiv \mathcal{H} \sqsupset \mathcal{H}_1 \sqsupset \mathcal{H}_2 \equiv \mathcal{H}_2(A), \tag{2.1}$$

where $\mathcal{H}_2 = \operatorname{Dom} A$ with respect to the norm $\|\varphi\|_2 := \|A\varphi\|$ and $\mathcal{H}_1 = \operatorname{Dom} A^{1/2}$ with respect to the norm $\|\varphi\|_1 := \|A^{1/2}\varphi\|$, and where $\mathcal{H}_{-2}$ and $\mathcal{H}_{-1}$ are dual spaces. Recall that there is a one-to-one correspondence between the operators $A = A^* \geqslant 1$ in $\mathcal{H}$ and the rigged Hilbert spaces of the form (2.1) or the entire scale of Hilbert spaces (1.1) (see, e.g., [3, Th. 2.1]). By construction, the linear functional $l_\omega(\varphi) := \langle \varphi, \omega \rangle_{1,-1}$ is well defined for each $\omega \in \mathcal{H}_{-1}$, and it is continuous on $\mathcal{H}_1$ (the symbol $\langle \cdot, \cdot \rangle_{k,-k}$, $k > 0$, stands for the dual inner product between $\mathcal{H}_k$ and $\mathcal{H}_{-k}$). Due to the Riesz theorem, we have $l_\omega(\varphi) = (\varphi, \psi)_1$, where $\psi = \psi(\omega) \in \mathcal{H}_1$, $\|\psi\|_1 = \|\omega\|_{-1}$. Let $D_{-1,1} : \mathcal{H}_1 \ni \psi \to \omega \in \mathcal{H}_{-1}$ be the canonical unitary isomorphism (see [12, 13, 3]), and let $D_1 := D_{-1,1} \upharpoonright \mathcal{D}_1$ and $\mathcal{D}_1 := \{\varphi \in \mathcal{H}_1 | D_{-1,1}\varphi \in \mathcal{H}\}$. In this case, one can readily see that $A = D_1$, and $\mathcal{D}_1 = \mathcal{H}_2$ with respect to the norm $\|\varphi\|_2 = \|A\varphi\|$. Similarly, repeating the above argument for $k = 2$, we can see that $A^2 = D_2$ with $\mathcal{D}_2 = \mathcal{H}_4$, and therefore $A = \sqrt{D_2}$. Thus, in the general case, we have $A = (D_k)^{1/k}$, $k \geqslant 2$.

Recall that the rigged Hilbert space $\mathcal{H}_{-k} \sqsupset \mathcal{H} \sqsupset \mathcal{H}_k$ (as well as the entire A-scale) can be reconstructed (see [12, 13]) from any couple of pre-rigged spaces, $\mathcal{H}_{-k} \sqsupset \mathcal{H}$ or $\mathcal{H} \sqsupset \mathcal{H}_k$. Thus, if we change the inner product in one of the spaces $\mathcal{H}_k$, $k > 0$ (or $\mathcal{H}_{-k}$), i.e., replace the product $(\cdot\,,\cdot)_k$ by $(\cdot\,,\cdot)^{\sim}_k$ (or $(\cdot\,,\cdot)_{-k}$ by $(\cdot\,,\cdot)^{\sim}_{-k}$), then, taking the couple $\mathcal{H} \sqsupset \tilde{\mathcal{H}}_k$ (or $\tilde{\mathcal{H}}_{-k} \sqsupset \mathcal{H}$), we can construct a new scale of spaces. In particular, we obtain a new rigged chain of the form

$$\tilde{\mathcal{H}}_{-2} \sqsupset \tilde{\mathcal{H}}_{-1} \sqsupset \mathcal{H} \sqsupset \tilde{\mathcal{H}}_1 \sqsupset \tilde{\mathcal{H}}_2. \tag{2.2}$$

By definition, the operator $\tilde{A}$ associated with (2.2) is a singular perturbation of A. If $k > 2$, then we obtain a singular perturbation $\tilde{A}$ of A in the wide sense, or a supersingular perturbation, in a similar way. Thus, in the general case $k \geqslant 2$, the operator $\tilde{A} \geqslant 1$ is defined as $\tilde{A} = \sqrt[k]{\tilde{D}_k}$, where $\tilde{D}_k$ stands for the restriction of the canonical unitary isomorphism $\tilde{D}_{-k,k} : \tilde{\mathcal{H}}_k \to \tilde{\mathcal{H}}_{-k}$ to the set $\tilde{\mathcal{D}}_k = \tilde{\mathcal{H}}_{2k} = \{\varphi \in \tilde{\mathcal{H}}_k \mid \tilde{D}_{-k,k}\varphi \in \mathcal{H}\}$. If the lower bound of the operator $\tilde{A}$ is less than one ($\tilde{A} \geqslant \tilde{m}$, $\tilde{m} < 1$), it is necessary to make additional changes in the scales (1.1) and (1.2). Namely, the norm $\|\cdot\|_{\pm k}$ must be replaced by $\|\cdot\|_{\pm k, c} := \|(A + c)\cdot\|_{\pm k}$, $c = 1 - \tilde{m}$.

We stress that, although the construction of perturbed operators $\tilde{A}$ is not explicit in the general case, new facts about singular perturbations can be observed. In particular, using Krein's formula for resolvents and the explicit representation for integral kernels $(1 - \Delta)^{-k/2}$ in terms of Bessel functions (see [1, 23]), we can obtain information on the additional point spectrum of $\tilde{A}$ and, moreover, if perturbations are given by the δ-potential $\delta(x)$, $x \in \mathbb{R}^d$, with an arbitrary dimension $d \geqslant 1$, then one can write out an explicit form of the generalized integral kernels for operators $\tilde{A}^{-k/2}$, $k > d/2$.

3. MAIN RESULT

Let an operator $A \geqslant 1$ be fixed. We assume that the lower bound of A is equal to 1, i.e., $\inf_{\|f\|=1}(Af, f) = 1$.

Definition 3.1. A self-adjoint operator $\tilde{A} \geqslant \tilde{m} > -\infty$ ($\tilde{m} := \inf_{\|f\|=1}(\tilde{A}f, f)$) is said to be a *singular perturbation of A in the wide sense* (we write $\tilde{A} \in \mathcal{P}_k(A)$) if the set

$$\mathcal{M}_k := \{\varphi \in \mathcal{H}_k | A_c^{k/2}\varphi = \tilde{A}_c^{k/2}\varphi\}, \qquad \mathcal{H}_k = \mathcal{M}_k \oplus \mathcal{N}_k,\ \mathcal{N}_k \neq \{0\} \tag{3.1}$$

is dense in $\mathcal{H}_{k/2}$ for some $k > 1$,

$$\mathcal{H}_{k/2} \sqsupset \mathcal{M}_k, \tag{3.2}$$

where $\tilde{A}_c := \tilde{A} + c$, $A_c := A + c$, $c \in [0, \infty)$ ($c = 0$ if $\tilde{m} \geqslant 1$ and $c = 1 - \tilde{m}$ if $\tilde{m} < 1$).

This definition enables us to introduce a classification of singular perturbations of A. Note that the numbers k and c are minimal possible in (3.1) and (3.2).

In this paper, we study only operators $\tilde{A} \in \mathcal{P}_k(A)$ with $k \geqslant 2$. An operator $\tilde{A}$ is said to be a *supersingular perturbation* of A, if $k \geqslant 4$ (see [10, 11]). If $k = 2$, then $\tilde{A}$ is a *purely strong singular perturbation* of A (in this case, we say that $\tilde{A}$ *corresponds to $\mathcal{H}_{-2}$-class perturbations*, see [14, 15]).

Let an operator S act on the A-scale, $S\colon \mathcal{H}_k \to \mathcal{H}_{-k}$, $k > 1$, and let it be self-adjoint, which means that $\langle S\varphi, \psi\rangle_{k,-k} = \langle \varphi, S\psi\rangle_{-k,k} =: \gamma_S(\varphi, \psi)$, $\varphi, \psi \in \operatorname{Dom} S = \operatorname{Dom} S^*$.

Definition 3.2. We say that an operator $S : \mathcal{H}_k \to \mathcal{H}_{-k}$ *belongs to the class* $\mathcal{S}_{-k}(A)$, $k > 1$, if the subset $\ker S := \{\varphi \in \mathcal{H}_k| : \gamma_S[\varphi] = 0\}$ is dense in $\mathcal{H}_{k/2}$,

$$\mathcal{H}_{k/2} \sqsupset \ker S. \tag{3.3}$$

Recall that an operator $S \in \mathcal{S}_{-k}(A)$ with $k = 2$ is usually referred to as a *singular operator of $\mathcal{H}_{-2}$-class*. If $k \geqslant 4$, then the operators S (or the quadratic forms γ_S) are known as supersingular perturbations. In this case, the set $\ker S$ is dense in $\mathcal{H}_2 = \operatorname{Dom} A$, and the symmetric operator $\dot{A} := A \restriction \ker S$ is essentially self-adjoint. In this case, the problem to define the perturbed operator $\tilde{A}$ remains unsolved, except for some special constructions, see, e.g., [22].

The next theorem gives the simplest version of our result.

Theorem 3.3 ($k = 2$, $c = 0$). *There is a bijective correspondence between the operators $\tilde{A} \in \mathcal{P}_2(A)$, $\tilde{A} \geqslant 1$, $\inf_{\|f\|=1}(\tilde{A}f, f) = 1$ and the positive bounded operators $S \in \mathcal{S}_{-2}(A)$ such that*

$$\|\varphi\|_1^2 \leqslant \langle S\varphi, \varphi\rangle_{-2,2} \equiv \gamma_S[\varphi] \leqslant \|\varphi\|_2^2, \qquad \varphi \in \mathcal{N}_2 := \mathcal{H}_2 \ominus \ker S. \tag{3.4}$$

Proof. Let $\tilde{A} \geqslant 1$ belong to the class $\mathcal{P}_2(A)$. Then, according to (3.1), there is a set $\mathcal{M}_2$ in $\mathcal{H}_2$ which is dense in $\mathcal{H}_1$ and such that $\tilde{A}$ and A coincide on this set. Hence, the operators $\tilde{A}$ and A are distinct self-adjoint extensions of the symmetric operator $\dot{A} := \tilde{A} \restriction \mathcal{M}_2 = A \restriction \mathcal{M}_2$. Since $\mathcal{M}_2$ is dense in $\mathcal{H}_1$, it follows that the Friedrichs extension A_∞ of $\dot{A}$ coincides with A and, moreover, it is the maximal positive extension. Therefore, $\tilde{A} \leqslant A$. Consider the bounded positive operator

$$B := \tilde{A}^{-1} - A^{-1}. \tag{3.5}$$

Obviously, $B = 0$ on $\mathcal{M}_0 := A\mathcal{M}_2$. Since $\tilde{A}, A \geqslant 1$, the operator $b := B \restriction \mathcal{N}_0$, where $\mathcal{N}_0 := \mathcal{H} \ominus \mathcal{M}_0$, satisfies the inequalities

$$0 < b < 1, \qquad 0 < b \leqslant 1 - A^{-1}. \tag{3.6}$$

Thus, on $\mathcal{N}_0$, we have

$$0 < A^{-1} \leqslant s_0 < 1, \quad \text{where} \quad s_0 := 1 - b. \tag{3.7}$$

Now let us introduce an operator $S\colon \mathcal{H}_2 \to \mathcal{H}_{-2}$ which is equal to zero on $\mathcal{M}_2$ and coincides with $\mathbf{A}s_0A$ on $\mathcal{N}_2$, where $\mathbf{A}$ stands for the closure of $A\colon \mathcal{H} \to \mathcal{H}_{-2}$. It follows from (3.6) and (3.7) that $0 < (A^{-1}h, h) \leqslant (s_0h, h) < \|h\|^2$, where $h \in \mathcal{N}_0$ and $h \neq 0$, and

$$(A\varphi, \varphi) = \|\varphi\|_1^2 \leqslant (s_0A\varphi, A\varphi) = \langle S\varphi, \varphi\rangle_{-2,2} < \|\varphi\|_2^2 \quad \text{and} \quad A^{-1}h = \varphi \in \mathcal{N}_2,$$

which proves (3.3). Conversely, starting from a positive operator $S\colon \mathcal{H}_2 \to \mathcal{H}_{-2}$ of class $\mathcal{S}_{-2}(A)$ which satisfies condition (3.3), we define the operator

$$s_0 := \mathbf{A}^{-1}(S \restriction \mathcal{N}_0)A^{-1}, \quad \text{where} \quad \mathcal{N}_0 := (A \ker S)^\perp.$$

Due to (3.3), this operator satisfies inequalities (3.7). Next introduce the operator $b := 1 - s_0$ and denote by B the extension of b by zero to $\mathcal{M}_0$. It is clear now that the operator $\tilde{A}$ given by the relation $\tilde{A} := A^{-1} + B$ belongs to the class $\mathcal{P}_2(A)$. Indeed, by condition (3.3), the set $\mathcal{M}_2 := \ker S$ is dense in $\mathcal{H}_1$, and $\tilde{A} \geqslant 1$ by (3.6). $\square$

Now let us consider the case in which the lower bound $\tilde{m} = \inf_{\|f\|=1}(\tilde{A}f, f)$ of the operator $\tilde{A} \in \mathcal{P}_2(A)$ is less than one, $\tilde{m} < 1$. In this case, introduce the space $\mathcal{H}_{k,c}$, $c = 1 - \tilde{m} > 0$, which coincides with the domain $\operatorname{Dom} A^{k/2}$ with the norm $\|f\|_{k,c} := \|(A + c)^{k/2}f\|$, $f \in \operatorname{Dom} A^{k/2}$. Certainly, the last norm is equivalent to the norm $\|f\|_k$.

Theorem 3.4 ($c > 0$, $k = 2$). *There is a bijective correspondence between the operators $\tilde{A} \in \mathcal{P}_2(A)$, $\tilde{A} \geqslant \tilde{m}$, $\tilde{m} < 1$, and the bounded positive operators $S \in \mathcal{S}_{-2}(A)$ (S are singular in $\mathcal{H}$) which satisfy the inequalities*

$$\|\varphi\|_{1,c}^2 \leqslant \langle S\varphi, \varphi\rangle_{-2,2} \equiv \gamma_S[\varphi] \leqslant \|\varphi\|_{2,c}^2, \qquad \varphi \in \mathcal{N}_{2,c} := \mathcal{H}_{2,c} \ominus \mathcal{M}_{2,c}, \quad c = 1 - \tilde{m}, \tag{3.8}$$

and have the property

$$\inf_{\varphi \in \mathcal{N}_{2,c},\, \|\varphi\|_{1,c}=1} \gamma_S[\varphi] = 1. \tag{3.9}$$

Proof. According to (3.1), for any given operator $\tilde{A} \geqslant \tilde{m}$ in the class $\mathcal{P}_2(A)$ there is a subspace $\mathcal{M}_2$ in $\mathcal{H}_2$ dense in $\mathcal{H}_1$ and such that the operators $\tilde{A}_c = \tilde{A} + c$ and $A_c = A + c$ coincide on this subspace for $c = 1 - \tilde{m}$. Note that the lower bound of the operator $\tilde{A}_c$ is equal to one. Now we replace the norms $\|\cdot\|_k$ in the spaces $\mathcal{H}_k$, $k = 1, 2$, by the equivalent norms of the form $\|f\|_{k,c} := \|(A+c)^{k/2} f\|$, $f \in \operatorname{Dom} A^{k/2}$, i.e., we come to the space $\mathcal{H}_{k,c}$, which coincides with $\operatorname{Dom} A^{k/2}$ as a set. Thus, the mapping $A_c^{k/2} \colon \mathcal{H}_{k,c} \to \mathcal{H}$ becomes unitary. Further, we replace the notation $\mathcal{M}_2$ by $\mathcal{M}_{2,c}$. Thus, the operators $\tilde{A}_c$ and A_c are distinct self-adjoint extensions of the positive symmetric operator $\dot{A}_c := \tilde{A}_c \restriction \mathcal{M}_{2,c} = A_c \restriction \mathcal{M}_{2,c}$. Since the subspace $\mathcal{M}_{2,c}$ is dense in $\mathcal{H}_{1,c}$, the Friedrichs extension $A_{c,\infty}$ of the operator $\dot{A}_c$ coincides with A_c and, moreover, it is the maximal positive extension. Hence, $\tilde{A}_c \leqslant A_c$. Thus, we can introduce the bounded positive operator

$$B_c := \tilde{A}_c^{-1} - A_c^{-1} \tag{3.10}$$

on $\mathcal{H}$. Obviously $B_c = 0$ on $\mathcal{M}_0 := A_c \mathcal{M}_{2,c}$ and, on $\mathcal{N}_0 := \mathcal{H} \ominus \mathcal{M}_0$, the operator $b_c := B_c \restriction \mathcal{N}_0$ satisfies the inequalities

$$0 < b_c < 1, \qquad 0 < b_c \leqslant 1 - A_c^{-1}. \tag{3.11}$$

In other terms,

$$s := 1 - b_c, \qquad 0 < A_c^{-1} \leqslant s < 1. \tag{3.12}$$

Now let us define the operator $S : \mathcal{H}_2 \to \mathcal{H}_{-2}$. It is equal to zero on $\mathcal{M}_{2,c}$ and $S = \mathbf{A}_c s A_c$ on $\mathcal{N}_{2,c} = \mathcal{H}_{2,c} \ominus \mathcal{M}_{2,c}$, where $\mathbf{A}_c$ stands for the closure of $A_c \colon \mathcal{H} \to \mathcal{H}_{-2}$. It follows from (3.11), (3.12) that $(A_c^{-1} h, h) \leqslant (sh, h) < \|h\|^2$, $h \in \mathcal{N}_0$, and $(A_c \varphi, \varphi) \equiv \|\varphi\|_{1,c}^2 \leqslant (sA_c\varphi, A_c\varphi) = \langle S\varphi, \varphi\rangle_{-2,2} < \|\varphi\|_{2,c}^2$, $\varphi = A_c^{-1} h \in \mathcal{N}_{2,c}$, which proves (3.8). Further $\sup_{\|h\|=1} (\tilde{A}_c^{-1} h, h) = 1$ because the lower bound of the operator $\tilde{A}_c$ is equal to one. Therefore, $1 = \sup_{\|h\|=1}((A_c^{-1} + B)h, h) = \sup_{h\in\mathcal{N}_0, \|h\|=1}((A_c^{-1} + b)h, h)$. Replacing b_c by $1-s$ gives $1 = \sup_{h\in\mathcal{N}_0,\|h\|=1}(\|h\|_{-1,c}^2 + \|h\|^2 - (sh,h))$, and hence $\sup_{h\in\mathcal{N}_0,\|h\|=1}(\|h\|_{-1,c}^2 - (sh,h)) = 0$. Since the operator $A_c \colon \mathcal{H}_{2,c} \to \mathcal{H}$ is unitary, we obtain $\sup_{\varphi\in\mathcal{N}_{2,c},\|\varphi\|_{2,c}=1}(\|\varphi\|_{1,c}^2 - \gamma_S[\varphi]) = 0$, $\varphi = A_c^{-1} h$, which obviously coincides with (3.9). Conversely, let us begin with a positive (singular in $\mathcal{H}$) quadratic form γ_S or with the associated operator $S : \mathcal{H}_2 \to \mathcal{H}_{-2}$ in the class $\mathcal{S}_{-2}(A)$ (the set $\ker S$ is dense in $\mathcal{H}_1$). We certainly assume that conditions (3.8), (3.9) hold. Using S, we construct the operator $s := \mathbf{A}_c{}^{-1} S A_c^{-1} \restriction \mathcal{N}_0$, where $\mathcal{N}_0 := (A_c \ker S)^\perp$. Due to (3.8), the operator s satisfies inequality (3.12). Next introduce the positive operator $b_c := 1 - s$. Denote by B the extension of b_c to $\mathcal{M}_0$ by zero. It is clear now that the operator $\tilde{A}_c$ given by $\tilde{A}_c^{-1} := A_c^{-1} + B$ belongs to the class $\mathcal{P}_2(A)$ because the operators $\tilde{A}_c$ and A_c coincide on the subspace $\mathcal{M}_{2,c}$ dense in $\mathcal{H}_1$. Thus, the lower bound of $\tilde{A}_c$ is equal to one. Therefore, the lower bound of the operator $\tilde{A} = \tilde{A}_c - c$ is equal to $\tilde{m} = 1 - c < 1$. □

The next theorem gives the most general version of our result.

Theorem 3.5 ($c \geqslant 0,\ k \geqslant 2$). *There is a bijective correspondence between the family of singular perturbed operators $\tilde{A} \in \mathcal{P}_k(A)$, $k \geqslant 2$, that are bounded below, $\tilde{A} \geqslant \tilde{m} > -\infty$, and the family of bounded positive operators $S \colon \mathcal{H}_k \to \mathcal{H}_{-k}$ of class $\mathcal{S}_{-k}(A)$ which satisfy the inequalities*

$$\|\varphi\|_{k/2,c}^2 \leqslant \gamma_S[\varphi] \equiv \langle S\varphi, \varphi\rangle_{-k,k} < \|\varphi\|_{k,c}^2, \qquad \varphi \in \mathcal{N}_{k,c} := \mathcal{H}_{k,c} \ominus \ker S, \tag{3.13}$$

and are such that

$$\inf_{\varphi\in\mathcal{N}_{k,c},\ \|\varphi\|_{k/2,c}=1} \gamma_S[\varphi] = 1, \tag{3.14}$$

where $c = 0$ if $\tilde{m} \geqslant 1$ and $c = 1 - \tilde{m} > 0$ if $\tilde{m} < 1$. The correspondence can be given by the relation

$$(\tilde{A} + c)^{-k/2} = A_c^{-k/2} + B_{c,k}, \qquad B_{c,k} = 1 - \mathbf{A}_c^{-k/2} S A_c^{-k/2}. \tag{3.15}$$

Proof. We must consider the case $c > 0$, $k > 2$ only. Let an operator $\tilde{A} \geqslant \tilde{m} > -\infty$, $\tilde{m} < 1$, belong to the class $\mathcal{P}_k(A)$, $k > 2$. According to (3.1), there is a subspace $\mathcal{M}_k$ in $\mathcal{H}_k$ dense in $\mathcal{H}_{k/2}$ on which the operators $\tilde{A}_c^{k/2}$ and $A_c^{k/2}$ coincide, where $c = 1 - \tilde{m}$. Denote this subspace by $\mathcal{M}_{c,k}$. Simultaneously, let us replace the norm in $\mathcal{H}_k$ by the equivalent norm $\|\cdot\|_{c,k}$. Hence, the operators $\tilde{A}_c^{k/2}$ and $A_c^{k/2}$ are distinct self-adjoint extensions of the positive symmetric densely defined operator $\dot{A}_{(k,c)} := \tilde{A}_c^{k/2} \restriction \mathcal{M}_{c,k} = A_c^{k/2} \restriction \mathcal{M}_{c,k}$. Since the subspace $\mathcal{M}_{c,k}$ is dense in $\mathcal{H}_{c,k/2}$, it follows that the Friedrichs extension of the operator $\dot{A}_{(k,c)}$ coincides with $A_c^{k/2}$. Thus, this is the maximal positive extension. Hence, $\tilde{A}_c^{k/2} \leqslant A_c^{k/2}$. Introduce the bounded positive operator

$$B_{ck} := \tilde{A}_c^{-k/2} - A_c^{-k/2}. \tag{3.16}$$

Clearly, $B_{ck} = 0$ on $\mathcal{M}_0 := A_c^{k/2}\mathcal{M}_{c,k}$, and the operator $b_{ck} := B_{ck} \restriction \mathcal{N}_0$ satisfies the inequalities

$$0 < b_{ck} < 1, \qquad 0 < b_{ck} \leqslant 1 - A_c^{-k/2} \tag{3.17}$$

on the subspace $\mathcal{N}_0 := \mathcal{H} \ominus \mathcal{M}_0$. Thus, for $s_{ck} := 1 - b_{ck}$ in $\mathcal{N}_0$, we have

$$0 < A_c^{-k/2} \leqslant s_{ck} < 1. \tag{3.18}$$

The operator S is zero on $\mathcal{M}_{c,k}$. On $\mathcal{N}_{c,k}$, the operator S is $S = \mathbf{A}_c^{k/2} s_{ck} A_c^{k/2}$. It follows from (3.17), (3.18) that $(A_c^{-k/2}h, h) \leqslant (s_{ck}h, h) < \|h\|^2$, $h \in \mathcal{N}_0$, or $\|\varphi\|_{k/2,c}^2 = (A_c^{k/2}\varphi, \varphi) \leqslant (s_{ck}A_c^{k/2}\varphi, A_c^{k/2}\varphi) = \langle S\varphi, \varphi\rangle_{-k,k} < \|\varphi\|_{k,c}^2 = (A_c^{k/2}\varphi, A_c^{k/2}\varphi)$, $\varphi = A_c^{-k/2}h \in \mathcal{N}_{c,k}$, which proves (3.13). Further, $\sup_{\|h\|=1}(\tilde{A}_c^{-k/2}h, h) = 1$ because the lower bound of the operator $\tilde{A}_c$ is equal to 1. Therefore,

$$1 = \sup_{\|h\|=1}((A_c^{-k/2} + B_{ck})h, h) = \sup_{h\in\mathcal{N}_0, \|h\|=1}((A_c^{-k/2} + b_{ck})h, h).$$

Replacing b_{ck} on $1 - s_{ck}$, we obtain $1 = \sup_{h\in\mathcal{N}_0,\|h\|=1}(\|h\|_{-k/2,c}^2 + \|h\|^2 - (s_{ck}h, h)$. This means that

$$\sup_{h\in\mathcal{N}_0,\|h\|=1}(\|h\|_{-k/2,c}^2 - (s_{ck}h, h)) = 0 \quad \text{and} \quad \sup_{\varphi\in\mathcal{N}_{k,c},\|\varphi\|_{k,c}=1}(\|\varphi\|_{k/2,c}^2 - \gamma_S[\varphi]) = 0$$

(which is obviously equivalent to (3.14)) for $h = A_c^{k/2}\varphi$, $\varphi \in \mathcal{N}_{k,c}$ because the operator $A_c^{k/2}\colon \mathcal{H}_{k,c} \to \mathcal{H}$ is unitary. Conversely, beginning with the operator $S : \mathcal{H}_k \to \mathcal{H}_{-k}$ belonging to the class $\mathcal{S}_{-k}(A)$, $k > 2$ (it is supposed that S satisfies conditions (3.13), (3.14)), we construct the operator $s_{ck} := \mathbf{A}_c^{-k/2} S A_c^{-k/2} \restriction \mathcal{N}_0$ on the subspace $\mathcal{N}_0 := (A_c^{k/2} \ker S)^{\perp}$. It obviously satisfies inequality (3.18). Then we define the operator $b_{ck} := 1 - s_{ck}$, and denote by B_{ck} the extension of b_{ck} by zero to $\mathcal{M}_{0ck}$. It is clear now that the operator $\tilde{A} = \tilde{A}_c - c$ given by the rule $\tilde{A}_c^{-k/2} := A_c^{-k/2} + B_{ck}$ belongs to the class $\mathcal{P}_k(A)$. That is, the lower bound of $\tilde{A}$ satisfies $1 > \tilde{m} = 1 - c > -\infty$. □

4. CONSTRUCTION OF $\tilde{A}$-SCALE

It is clear from the previous section that, from any couple of spaces (either $\tilde{\mathcal{H}}_{-k} \sqsupset \mathcal{H}_0$ or $\mathcal{H}_0 \sqsupset \tilde{\mathcal{H}}_k$), we can reconstruct the entire A-scale. In particular,

$$\tilde{\mathcal{H}}_{-2} \sqsupset \tilde{\mathcal{H}}_{-1} \sqsupset \mathcal{H} \sqsupset \tilde{\mathcal{H}}_1 \sqsupset \tilde{\mathcal{H}}_2. \tag{4.1}$$

Moreover, we then can obtain the operator $\tilde{A}$ as an operator associated with this scale. In this section, we discuss the construction of a new rigged Hilbert space of the form (4.1) and its relation to a quadratic form $\gamma_S(\varphi, \psi) = \langle S\varphi, \psi\rangle_{-2,2}$, where $S \in S_{-2}(A)$ plays the role of a perturbation, in more detail. Let us show how to consider a strongly singular perturbation belonging to the class $\mathcal{H}_{-2}(A)$ by using the approach of rigged Hilbert spaces. We begin with the rigged triplet (1.1) associated with the free operator $A = A^* \geqslant 1$ on $\mathcal{H}$ and then come to the consideration of the chain of five spaces,

$$\mathcal{H}_- \equiv \mathcal{H}_{-2} \sqsupset \mathcal{H}_{-1} \sqsupset \mathcal{H} \sqsupset \mathcal{H}_1 \sqsupset \mathcal{H}_2 \equiv \mathcal{H}_+ (= \operatorname{Dom} A). \tag{4.2}$$

Using a positive quadratic form $\gamma \in \mathcal{H}_{-2}$, we define a new inner product on $\mathcal{H}_0$,

$$(h_1, h_2)^{\sim}_{-1} := (A^{-1}h_1, h_2) + \gamma(A^{-1}h_1, A^{-1}h_2), \qquad h_1, h_2 \in \mathcal{H}. \tag{4.3}$$

If the quadratic form γ satisfies the condition

$$-\|f\|_1^2 \leqslant \gamma[f] \leqslant \|f\|_2^2 - \|f\|_1^2, \qquad f \in \mathcal{H}_2 = \operatorname{Dom} A, \tag{4.4}$$

then $\tilde{A} \in \mathcal{P}_2(A)$. The following theorem shows that each singular perturbation corresponding to $\gamma \in \mathcal{H}_{-2}$ admits a construction by the modified form-sum method.

Theorem 4.1 [3]. *For each operator $\tilde{A} \in \mathcal{P}_2(A)$, $\tilde{A} \geqslant 1$ (i.e., $\tilde{A}$ corresponds to a class $\mathcal{H}_{-2}(A)$ perturbation), the inner product $(\cdot\,,\cdot)^{\sim}_{-2}$ in the space $\tilde{\mathcal{H}}_{-2} = \operatorname{Dom}\tilde{A}$ can be represented as the singular form-sum perturbation of the inner product in $\mathcal{H}_{-2}$,*

$$(\cdot\,,\cdot)^{\sim}_{-2} = (\cdot\,,\cdot)_{-2} + \tau(\cdot\,,\cdot), \tag{4.5}$$

where the quadratic form τ defined on $\mathcal{H}$ is singular on $\mathcal{H}_{-2}$, which means that $\ker\gamma$ *is dense in $\mathcal{H}_{-2}$.*

Proof. Let us sketch the proof. By Krein's formula (3.5), $(h_1, h_2)^{\sim}_{-2} = (\tilde{A}^{-1}h_1, \tilde{A}^{-1}h_2) = (A^{-1}h_1, A^{-1}h_2) + \tau(h_1, h_2)$, where

$$\tau(\cdot\,,\cdot) := (A^{-1}\cdot, B\cdot) + (B\cdot, A^{-1}\cdot) + (B\cdot, B\cdot). \tag{4.6}$$

Obviously, the form τ is Hermitian but not positive. By construction, we have $\ker\tau = \ker B \sqsubset \mathcal{H}_{-1}$ since the operator $\tilde{A}$ belongs to the class $\tilde{A} \in \mathcal{P}_2(A)$ (see Definition 3.1). Therefore, γ is singular on $\mathcal{H}_{-1}$, and in $\mathcal{H}_{-2}$ as well. □

The converse assertion is also true, but it requires additional constructions.

Let us show now that, for rank one singular perturbation $\tilde{A} \in \mathcal{P}_2(A)$, $\tilde{A} \geqslant 1$, the inner product in $\tilde{\mathcal{H}}_1$ also admits an interpretation as the generalized form-sum perturbation of the inner product in $\mathcal{H}_1$. Let $\tilde{A}$ be the rank-one singular perturbation corresponding to an element $\omega \in \mathcal{H}_{-2}\backslash\mathcal{H}_{-1}$. Consider a new inner product on $\mathcal{H}_2$, $\chi(\cdot,\cdot) = (\cdot,\cdot)_1 + \gamma(\cdot,\cdot)$, $\gamma(\cdot,\cdot) := \beta(A\cdot,\eta)(\eta,A\cdot)$, $\eta := \mathbf{A}^{-1}\omega$. Note that $\eta \in \mathcal{H}_0\backslash\mathcal{H}_1$, and therefore $\ker\gamma \sqsubset \mathcal{H}_1$. Obviously, $\gamma(\cdot,\cdot) \geqslant 0$. Further, if we take a constant β such that

$$0 < \gamma[\varphi] \leqslant \|\varphi\|_2^2 - \|\varphi\|_1^2, \qquad \varphi \in \mathcal{H}_2, \tag{4.7}$$

then the space $\tilde{\mathcal{H}}_1$ (given below) is densely embedded in $\mathcal{H}$. Note that, due to $\ker\gamma \sqsubset \mathcal{H}_1$, we have the orthogonal decomposition $\mathcal{H}_\chi = \mathcal{H}_1 \oplus \mathcal{H}_\gamma$, where $\mathcal{H}_\gamma$ is the Hilbert space obtained from $\operatorname{Dom} A$ by completing with respect to the inner product given by $\gamma(\cdot,\cdot)$. One can readily see that $\mathcal{H}_\gamma = \{c\eta_+\}$, $\eta_+ := A^{-1}\eta$, $c \in \mathbb{C}$, where we have used an explicit form of γ. Therefore, the mapping $\acute{A} := A|\mathcal{H}_\gamma \to \hat{\mathcal{N}}_0 = \{c\eta\}$, $c \in \mathbb{C}$, is unitary. Our theorem on the structure of the space $\tilde{\mathcal{H}}_1$ ($\tilde{\mathcal{H}}_1 = \mathcal{H}_1 \oplus \hat{\mathcal{N}}_0 \iff \tilde{A} \in \mathcal{P}_2(A)$) implies that the operator $T := \mathbf{1} \oplus \acute{A}$ is also unitary from the Hilbert space $\mathcal{H}_\chi$ into $\tilde{\mathcal{H}}_1$. Thus, we obtained $\tilde{\mathcal{H}}_1$, and hence $\tilde{A}$.

A similar result holds for an arbitrary k. Choose a $k > 2$. Let A be an operator associated with

$$\mathcal{H}_{-k} \sqsupset \mathcal{H}_{-k/2} \sqsupset \mathcal{H}_0 \sqsupset \mathcal{H}_{k/2} \sqsupset \mathcal{H}_k = \operatorname{Dom} A^{k/2}. \tag{4.8}$$

Then each positive quadratic form $\gamma \in \mathcal{S}_{-k}$ defines a new negative inner product $(h_1,h_2)^{\sim}_{-k/2} := (A^{-k/2}h_1,h_2)_0 + \gamma(A^{-k/2}h_1, A^{-k/2}h_2)$, $h_1,h_2 \in \mathcal{H}$. If this form satisfies the condition $-\|f\|^2_{k/2} \leqslant \gamma[f] \leqslant \|f\|_k^2 - \|f\|^2_{k/2}$, $f \in \mathcal{H}_k$, then we can introduce a chain of new Hilbert spaces

$$\tilde{\mathcal{H}}_{-k} \sqsupset \tilde{\mathcal{H}}_{-k/2} \sqsupset \mathcal{H} \sqsupset \tilde{\mathcal{H}}_{k/2} \sqsupset \tilde{\mathcal{H}}_k \tag{4.9}$$

and construct the associated operator $\tilde{A} \in \mathcal{P}_k(A)$. In a similar way, we can also construct the positive space $\tilde{\mathcal{H}}_{k/2}$ for an arbitrary rank-one supersingular perturbation. Namely, let $\chi(\cdot,\cdot) = (\cdot,\cdot)_{k/2} + \gamma(\cdot,\cdot)$ be a new positive inner product on $\mathcal{H}_k$, where $\gamma(\cdot,\cdot) := \beta(A^{k/2}\cdot,\eta)_0(\eta,A^{k/2}\cdot)_0$, $\eta \in \mathcal{H}_0\backslash\mathcal{H}_{k/2}$, and the constant β is such that $0 < \gamma[\varphi] \leqslant \|\varphi\|_k^2 - \|\varphi\|^2_{k/2}$, $\varphi \in \mathcal{H}_k$. Then, similarly to the case of $k = 2$, using the operator $T = \mathbf{1} + \acute{A}^{k/2}$, we pass from $\mathcal{H}_\chi$ to $\tilde{\mathcal{H}}_{k/2}$, where $\acute{A} := A|\mathcal{H}_\gamma$ ($\mathcal{H}_\gamma$ stands for the Hilbert space obtained from $\operatorname{Dom} A^{k/2}$ by completing with respect to the inner product $\gamma(\cdot,\cdot)$).

In the above sense, every supersingular perturbation of class $\mathcal{H}_{-k}$ admits the definition using the modified form-sum method. Thus, we can formulate a general assertion in this direction.

Theorem 4.2. *For each $\tilde{A} \in \mathcal{P}_k(A)$, $\tilde{A} \geqslant 1$, the inner product $(\cdot,\cdot)^{\sim}_{-k}$ on the space $\tilde{\mathcal{H}}_{-k}$ is a singular form-sum perturbation of the inner product in $\mathcal{H}_{-k}$, $(\cdot,\cdot)^{\sim}_{-k} = (\cdot,\cdot)_{-k} + \tau_k(\cdot,\cdot)$, where the Hermitian quadratic form $\tau_k(\cdot,\cdot) := (A^{-k/2}\cdot, B\cdot) + (B\cdot, A^{-k/2}\cdot) + (B\cdot,B\cdot)$ is singular in $\mathcal{H}_{-k}$.*

The proof is essentially the same as for Theorem 4.1. Note that τ_k is Hermitian but not positive.

Thus, the results discussed above show that one can construct supersingular perturbations of any order (using the method of rigged Hilbert spaces) by changing the inner products in each of the spaces in (4.2) or in (4.8).

5. EXAMPLES

In this section, we show, in particular, that (among singular perturbations $\tilde{A} \in \mathcal{P}_k(A)$, $k > 2$) there are new kinds of operators which cannot be obtained as singular perturbations of order $k \leqslant 2$.

Example 5.1 (Rank one supersingular perturbations of order $k > 2$).

Consider the perturbation $\tilde{A}$ of A given by the quadratic form $\gamma_\omega[\varphi] = \langle\varphi,\omega\rangle_{k,-k}\langle\omega,\varphi\rangle_{-k,k}$, $\omega \in \mathcal{H}_{-k}\backslash\mathcal{H}_{-k/2}$, $k > 2$. The operator $S_\omega :\ \mathcal{H}_k \ni \varphi \to \langle\varphi,\omega\rangle_{k,-k}\omega \in \mathcal{H}_{-k}$ associated with γ_ω belongs to $\mathcal{S}_{-k}$ (see Definition 3.2) because the set $\ker S_\omega = \ker\gamma_\omega = \{\varphi \in \mathcal{H}_k | \langle\varphi,\omega\rangle_{k,-k} = 0\}$ is dense in $\mathcal{H}_{k/2}$ since $\omega \notin \mathcal{H}_{-k/2}$ (see [9, Th. A.1]). In particular, if $k \geqslant 4$, then $\ker S_\omega$ is dense in $\mathcal{H}_2 = \operatorname{Dom} A$, and one cannot use any of the usual methods to construct the perturbed operator $\tilde{A}$. Here we suggest the idea of regarding γ_ω or S_ω as a perturbation of $A^{k/2} \colon \mathcal{H}_k \to \mathcal{H}$. In other words, we define $\tilde{A}$ as $(A^{-k/2} + B_\omega)^{2/k}$, where B_ω is a rank-one operator in $\mathcal{H}$ of the form $B_\omega = \beta_\omega(\cdot,\eta_0)\eta_0$,

$\eta_0 := \mathbf{A}^{-k/2}\omega$, with an appropriate constant β_ω. Thus, if we set $\beta_\omega = 1 - \sigma_\omega$ (this corresponds to the representation $b = 1 - s$, see Section 3), then the constant σ_ω must satisfy the inequality: $\|\eta_0\|^2_{-1} \leqslant \sigma_\omega < 1$. Only in this case we obtain $\tilde{A} \geqslant 1$ (see [3, Example 3.1]).

Set $\omega = \sigma_\omega(\mathbf{A}^{k/2}\psi - \lambda\psi)$, $\psi \in \mathcal{H}\backslash\mathcal{H}_{k/2}$, $\|\psi\| = 1$, $\lambda \in \mathbb{R}$. Then, using Krein's resolvent formula

$$(\tilde{A}^{k/2} - z)^{-1} = (A^{k/2} - z)^{-1} + B_\omega(z),$$
$$\text{where}\quad B_\omega(z) = \beta_{\omega,z}(\cdot\,,\eta_{\bar{z}})\eta_z, \quad \eta_z = (A^{k/2} - \lambda)(A^{k/2} - z)^{-1}\psi, \quad \beta_{\omega,z} = 1/\big((\lambda - z)(\psi, \eta_{\bar{z}})\big),$$

we solve the eigenvalue problem for $\tilde{A}^{k/2}$: $\tilde{A}^{k/2}\psi = \lambda\psi$ (for details, see [4, 6]). Therefore, the operator $\tilde{A}$ solves the eigenvalue problem $\tilde{A}\psi = \lambda^{2/k}\psi$ with the same eigenvector ψ.

The vector ψ can be taken in the space $\mathcal{H}_1$ (since $k > 2$ now). This is of importance and proves that one cannot solve the above eigenvalue problem in the framework of singular perturbations of class $\mathcal{P}_k(A)$ with $k \leqslant 2$ (see [6]).

Example 5.2 (Perturbations of Bessel potentials by the $\delta(x)$-function, $x \in \mathbb{R}^d,\ d > 3$).

Let Δ be the Laplace operator on $L_2(\mathbb{R}^d, dx)$. As is well known [1, 25], the inverse operator of $A = (1 - \Delta)^{k/2}$, $k > 0$, is the integral operator with Bessel kernel $G_k = \mathfrak{F}^{-1}((1 + |\xi|^2)^{-k/2})$, where $\mathfrak{F}^{-1}$ stands for the inverse Fourier transform. Recall that the representation by the Bessel potential is $\varphi = G_k * h = (1 - \Delta)^{-k/2}h$. In particular, if h ranges over $L_2(\mathbb{R}^d, dx)$, then $\varphi \in \operatorname{Dom}(1 - \Delta)^{k/2}$, which is a Sobolev space, $W_2^k = \mathcal{H}_k := \{\varphi = G_k * h,\ h \in L_2\}$. Recall the explicit presentation for G_k,

$$G_k(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{e^{ix\xi}}{(1 + |\xi|^2)^{k/2}}d\xi = (2\pi)^{-d/2}|x|^{-(d-2)/2}\int_0^\infty\frac{t^{d/2}}{(1 + t^2)^{k/2}}\mathfrak{J}_{(d-2)/2}(|x|t)dt,$$

where $\mathfrak{J}_\nu$ stands for the Bessel function of order ν. For more details on the explicit representation for G_k, see [1, 25].

Choose a $d > 3$ and take $k > d/2$. By the Sobolev embedding theorem, the above functions φ are continuous. Thus, we have an embedding $W_2^k \subset C(\mathbb{R}^d)$, which enables us to introduce the generalized Bessel kernel $\tilde{G}_k = G_k + \beta(\cdot, \eta_\delta)\eta_\delta$, $\beta > 0$, where $\eta_\delta = G_k * \delta$ stands for the convolution G_k with the delta function $\delta(x) \in W_2^{-k}$. If $d > k > d/2$, then $\delta \notin W_2^{-k/2}$. This ensures that the generalized Bessel potentials $\tilde{\varphi} = \tilde{G}_k * h$, $h \in L_2(\mathbb{R}^d, dx)$, form a new Hilbert space $\tilde{W}_2^k$ such that $\tilde{W}_2^k \sqsubset L_2$ and which coincides with the ordinary Sobolev space on a set $\mathcal{M}_k = \{\varphi \in W_k^2 \mid \varphi(0) = 0\}$ dense in L_2. The canonical unitary isomorphism $\tilde{D}_k\colon\ \tilde{W}_k^2 \to L_2(\mathbb{R}^d, dx)$, after the restriction to the space $\tilde{W}_{2k}^2 := \{\tilde{\varphi} \in \tilde{W}_k^2 \mid \tilde{D}_k\varphi \in L_2(\mathbb{R}^d, dx)\}$, defines a self-adjoint operator $\tilde{A}^{-k/2}$ in L_2, which corresponds to the heuristic expression "$(1 - \Delta)^{k/2} + \delta$." Finally, the operator $\tilde{A}$ (formally given as "$-((1 - \Delta)^{k/2} + \delta)^{2/k} + 1$") can be regarded as a uniquely defined version for the formal expression $-\Delta + \delta(x)$ in the case of $x \in \mathbb{R}^d,\ d > 3$.

Example 5.3 (Perturbations of Bessel potentials by fractal measures).

Let $\Gamma \subset \mathbb{R}^d$ be a compact support of a chosen fractal measure μ on $\mathbb{R}^d$ (for more details and definitions, see [24, 8, 21]; see also the related papers [19, 17, 18]). Assume that the $C_{k/2}$-capacity (in particular, the Lebesgue measure) of Γ is zero, $C_{k/2}(\Gamma) = 0,\ m(\Gamma) = 0$, but $C_k(\Gamma) > 0$. (For the definition of C_l-capacity and its properties, see [1, 23].) Recall only that the C_l-capacity of a set Γ can be defined as follows:

$$C_l(\Gamma) = \inf_{\nu(\Gamma)=1}\iint G_l(x - y)d\nu(x)d\nu(y),$$

where G_l stands for the Bessel kernel and the infimum is taken over all Borel measures such that $\Gamma \subset \operatorname{supp}\nu$. Further, assume that the quadratic form $\gamma_\mu[\varphi] := \int|\varphi|^2d\mu$, $\varphi \in W_2^k$, is bounded on W_2^k. Then there exists an associated operator $S_\mu\colon\ W_2^k\ \to\ W_2^{-k}$ acting as the multiplication by μ and belonging to the class $\mathcal{S}_{-k}$ (just due to $C_{k/2}(\Gamma) = 0$). Therefore, we can define a new version of the generalized Bessel kernel, $\tilde{G}_{k,\mu} = G_k + B_\mu$, where B_μ is the integral operator given by the kernel

$$B_\mu(x, y) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G_k(x - z)G_k(y - z')\mu(dz)\mu(dz').$$

Similarly to the previous example, one can introduce a new version of generalized Bessel potentials $\tilde{\varphi} = \tilde{G}_{k,\mu} * h$, $h \in L_2(\mathbb{R}^d, dx)$, to construct a new Hilbert space $\tilde{W}_{k,\mu} \sqsubset L_2$ and to define the singular perturbation corresponding to the heuristically given expression "$-\Delta + \mu$." This is a uniquely defined self-adjoint operator on L_2 formally given by the expression $(\tilde{G}_{k,\mu})^{2/k}$.

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