Remarks on the inverse spectral theory for singularly perturbed operators

S. Albeverio[*], A. Konstantinov[†], and V. Koshmanenko[‡]

Abstract. Let A be an unbounded from above self-adjoint operator in a separable Hilbert space $\mathcal{H}$ and $E_A(\cdot)$ its spectral measure. We discuss the inverse spectral problem for singular perturbations $\tilde{A}$ of A ($\tilde{A}$ and A coincide on a dense set in $\mathcal{H}$). We show that for any $a \in \mathbb{R}$ there exists a singular perturbation $\tilde{A}$ of A such that $\tilde{A}$ and A coincide in the subspace $E_A((-\infty, a))\mathcal{H}$ and simultaneously $\tilde{A}$ has an additional spectral branch on $(-\infty, a)$ of an arbitrary type. In particular, $\tilde{A}$ may possess the prescribed spectral properties in the resolvent set of the operator A on the left from a point a. Moreover, for an arbitrary self-adjoint operator T in $\mathcal{H}$ there exists $\tilde{A}$ such that T is unitary equivalent to a part of $\tilde{A}$ acting in an appropriate invariant subspace.

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[*]Institut für Angewandte Mathematik, Universität Bonn, Wegelerstr. 6, D-53115 Bonn, Germany; SFB 611, Bonn, Germany; BiBoS, Bielefeld - Bonn, Germany; IZKS, Bonn, Germany, CERFIM, Locarno, Switzerland; Acc Arch., Mendrisio, Switzerland, e-mail: albeverio@uni-bonn.de

[†]Department of Mathematics, Kyiv University, 64 Volodymyrs'ka str, 01033, Kyiv, Ukraine, e-mail: konst@faust.kiev.ua.

[‡]Institute of Mathematics, National Academy of Sciences, 3 Tereshchenkivs'ka str, Kyiv, 01601, Ukraine, e-mail: kosh@imath.kiev.ua

1 Introduction

Let A be a self-adjoint unbounded operator defined on the domain $\mathfrak{D}(A) \equiv \mathrm{dom}(A)$ in a separable Hilbert space $\mathcal{H}$ with the inner product $(\cdot, \cdot)$. We shall say that an operator $\tilde{A} \neq A$ in $\mathcal{H}$ is a (pure) singular perturbation of A if the set

$$\mathfrak{D} := \{f \in \mathfrak{D}(A) \cap \mathfrak{D}(\tilde{A}) \,|\, Af = \tilde{A}f\}$$

is dense in $\mathcal{H}$. In this case one can define a densely defined symmetric operator $A_0 := A \restriction \mathfrak{D} = \tilde{A} \restriction \mathfrak{D}$. If in addition $\tilde{A}$ is self-adjoint then A and $\tilde{A}$ are different self-adjoint extensions of A_0.

We will denote by $\sigma(A)$, $\rho(A)$, and $E_A(\cdot)$ the spectrum, the resolvent set, and the spectral measure of A, respectively. The point, singular continuous, and absolutely continuous spectrum of a self-adjoint operator A are denoted by $\sigma_p(A)$, $\sigma_{sc}(A)$, and $\sigma_{ac}(A)$, respectively. For a Borel set $\Delta \subset \mathbb{R}$ we set $A_\Delta := A \restriction_{E_A(\Delta)\mathcal{H}}$. Clearly, A_Δ is as a self-adjoint operator in $\mathcal{H}_{A,\Delta} := \mathrm{Ran}(E_A(\Delta))$.

Assume that an open set $J \subset \mathbb{R}$ is a subset of $\rho(A)$. One can ask the question, whether there exists a singular perturbation $\tilde{A}$ having prescribed spectral properties in J. We show that the answer is positive if A is not semi-bounded from above and $J \subset (-\infty, a)$ for some $a \in \mathbb{R}$.

We remark that the first detailed investigation of the spectrum of self-adjoint extensions within a gap $J = (a, b)$ $(-\infty \leq a < b < +\infty)$ of a symmetric operator A_0 with finite deficiency indices (n, n) was carried out by M.G.Krein [15]. Namely, he proved that for any auxiliary self-adjoint operator T with the condition $\dim(\mathrm{Ran}(E_T(J))) \leq n$, there exist a self-adjoint extension $\tilde{A}$ such that

$$\tilde{A}_J \simeq T_J. \tag{1.1}$$

Here $A \simeq B$ means that A is unitary equivalent to B. For the operator T with an arbitrary pure point spectrum this result was generalized in [7] to the case of A_0 with infinite deficiency indices.

Further this problem was intensively studied in a series of papers [2, 8, 9]. The complete solution of the above problem was recently obtained in [10]. It was shown that in the case $J = (a, b)$ and $n \leq \infty$ for any auxiliary self-adjoint operator T there exist a self-adjoint extension $\tilde{A}$ of A_0 satisfying (1.1). In particular it means that there exists a self-adjoint extension $\tilde{A}$ of A_0 having an arbitrary beforehand given structure and type of spectrum in the gap J.

On the other hand it is known that a similar result is not valid in the essentially more difficult case of a symmetric operator with several gaps.

This problem was studied in [1, 6, 11], where the spectral properties of self-adjoint extensions were described in terms of abstract boundary conditions and the corresponding Weyl functions. In particular, in [1] it was considered a symmetric operator A_0 of the special structure, namely,

$$A_0 = \bigoplus_{k=1}^{\infty} S_k,$$

where each S_k is unitary equivalent to a fixed densely defined closed symmetric operator S with equal positive deficiency indices. It was assumed that there exists a self-adjoint extension S^0 of S such that open set $J \subset \rho(S^0) \cap \mathbb{R}$. Then one can associate to the pair $\{S, S^0\}$ a boundary triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ (see, [11]) such that $S^0 = S^* \restriction \ker \Gamma_0$. Under the additional assumption that the Weyl function M (see, [1, 11]) corresponding to Π is monotone with respect to J it was shown that for any auxiliary self-adjoint operator T there exist a self-adjoint extension $\tilde{A}$ of A_0 satisfying (1.1).

In the present paper we consider the above problem from the point of view of singular perturbation theory. Instead of self-adjoint extensions of a fixed symmetric operator A_0 we consider singular perturbations $\tilde{A}$ of a fixed self-adjoint operator A. Therefore the corresponding symmetric operator A_0 is not unique. This gives the freedom of choice of a dense domain $\mathfrak{D} \equiv \mathfrak{D}(A_0) = \mathfrak{D}(A) \cap \mathfrak{D}(\tilde{A})$ and allows to involve in the consideration a wider class of operators $\tilde{A}$. In particular, we can consider instead of an interval (a, b) an arbitrary open set J which is upper semi-bounded (cf., [10]). We also show that for an an arbitrary self-adjoint operator T in $\mathcal{H}$ there exists a self-adjoint singular perturbation $\tilde{A}$ such that T is unitary equivalent to a certain part of $\tilde{A}$ acting in an appropriate invariant subspace.

The spectral inverse problem in such a setting (in the case of the point spectrum) have been investigated in [3, 13]. In particular, it was shown that for any unbounded self-adjoint operator A and a sequence $\{\lambda_k : k \geq 1\}$ of real numbers there exists a singular perturbation $\tilde{A}$ of A such that all λ_k are eigenvalues of $\tilde{A}$. Moreover in [14, 12, 3, 13, 4] the inverse eigenvalue problem of the form

$$\tilde{A}\psi_k = \lambda_k \psi_k, \ k = 1, 2, ...$$

was studied, for a given sequence $\{\lambda_k : k \geq 1\}$ of real numbers and an orthonormal system $\{\psi_k : k \geq 1\}$ satisfying the condition

$$\overline{\mathrm{span}\{\psi_k : k \geq 1\}} \cap \mathfrak{D}(A) = \{0\}.$$

Here $\overline{M}$ denotes the closure of the set M.

The aim of this note is to present new observations in the problem of construction of singular perturbations $\tilde{A}$ with the prescribed spectral properties, in particular, in the resolvent set of the operator A.

2 Two theorems

In the following we assume, without loss of generality, that an unbounded self-adjoint operator A in a separable Hilbert space $\mathcal{H}$ is not semi-bounded from above. The main results of this note are formulated in the following two theorems.

Theorem 2.1. *Let A be an unbounded (at least from above) self-adjoint operator in a separable Hilbert space $\mathcal{H}$. Then for any fixed $a \in \mathbb{R}$ and an auxiliary self-adjoint operator T in $\mathcal{H}$ there exists a self-adjoint singular perturbation $\tilde{A}$ of A of the form*

$$\tilde{A} = A_{(-\infty,a)} \oplus A', \tag{2.1}$$

where the self-adjoint operator A' in $\mathcal{H}_{[a,\infty)} = E_A([a,\infty))\mathcal{H}$ is such that

$$A'_{(-\infty,a)} \simeq T_{(-\infty,a)}. \tag{2.2}$$

In particular, for an arbitrary open set $J \subset \rho(A) \cap (-\infty, a)$ there exists a self-adjoint singular perturbation $\tilde{A}$ of the form (2.1) such that

$$\tilde{A}_J = A'_J \simeq T_J. \tag{2.3}$$

Moreover, we will show that for an arbitrary self-adjoint operator T in $\mathcal{H}$ there exists a self-adjoint singular perturbation $\tilde{A}$ such that T is unitary equivalent to an appropriate part of $\tilde{A}$.

Theorem 2.2. *Let A be an unbounded self-adjoint operator in a separable Hilbert space $\mathcal{H}$ and T be an arbitrary auxiliary self-adjoint operator in $\mathcal{H}$. Then there exists a self-adjoint singular perturbation $\tilde{A}$ of A of the form*

$$\tilde{A} = A' \oplus A'', \tag{2.4}$$

where A' is similar to T,

$$A' \simeq T. \tag{2.5}$$

3 Proofs

Proof of Theorem 2.1. Fix $a \in \mathbb{R}$ and consider the orthogonal decomposition $A = A_{(-\infty,a)} \oplus A_{[a,\infty)}$ where the self-adjoint operators $A_{(-\infty,a)}$ and $A_{[a,\infty)}$ act in the Hilbert spaces $\mathcal{H}_{(-\infty,a)}$ and $\mathcal{H}_{[a,\infty)}$, resp. Let $\dot{A}$ be an arbitrary densely defined symmetric restriction of $A_{[a,\infty)}$ with infinite deficiency indices. Then $\dot{A} \geq a$ and according to [10] for any auxiliary self-adjoint operator T there exists a self-adjoint extension A' of $\dot{A}$ (acting in $\mathcal{H}_{[a,\infty)}$) such that $A'_{(-\infty,a)} \simeq T_{(-\infty,a)}$. Define the singular perturbation $\tilde{A}$ of A by

$$\tilde{A} := A_{(-\infty,a)} \oplus A'. \tag{3.1}$$

Clearly $\tilde{A}$ satisfies (2.2). In particular, for any open subset $J \subset \rho(A) \cap (-\infty, a)$ one can take T_J instead of $T_{(-\infty,a)}$, and get in the same way a self-adjoint extension A' of $\dot{A}$ such that

$$A'_{(-\infty,a)} = A'_J \simeq T_J. \tag{3.2}$$

By (3.1) and (3.2)

$$\tilde{A}_{(-\infty,a)} = A_{(-\infty,a)} \oplus A'_{(-\infty,a)} \simeq A_{(-\infty,a)} \oplus T_J. \tag{3.3}$$

Note that $A_{(-\infty,a)} = A_{(-\infty,a)\setminus J}$ since $J \subset \rho(A)$. Therefore (see, (3.3)) $\tilde{A}$ satisfies (4.1). $\square$

Proof of Theorem 2.2. First suppose in addition that the operator A is not semi-bounded from below (recall that we assume throughout the paper that A is not semi-bounded from above). Denote $\mathbb{R}_+ := [0, \infty)$, $\mathbb{R}_- := (-\infty, 0)$. In this case the positive and negative parts $A^{\pm} := A_{\mathbb{R}_\pm}$ of A are unbounded self-adjoint operators in $\mathcal{H}_\pm := \mathcal{H}_{\mathbb{R}_\pm}$. So we can apply Theorem 2.1 separately to A^+ and A^-. Let T be an arbitrary self-adjoint operator in $\mathcal{H}$. Then by Theorem 2.1 there exist self-adjoint singular perturbations $\widetilde{A^{\pm}}$ of $A^{\pm}$ in $\mathcal{H}_\pm$ such that

$$\widetilde{A^+}_{(-\infty,0)} \simeq T^-, \text{ and } \widetilde{A^-}_{[0,\infty)} \simeq T^+.$$

Define the operator

$$\tilde{A} := \widetilde{A^+} \oplus \widetilde{A^-} = \widetilde{A^+}_{(-\infty,0)} \oplus \widetilde{A^+}_{[0,\infty)} \oplus \widetilde{A^-}_{(-\infty,0)} \oplus \widetilde{A^-}_{[0,\infty)}.$$

It has the form (2.4) with

$$A' := \widetilde{A^+}_{(-\infty,0)} \oplus \widetilde{A^-}_{[0,\infty)}, \text{ and } A'' := \widetilde{A^+}_{[0,\infty)} \oplus \widetilde{A^-}_{[0,\infty)}.$$

Clearly, $\tilde{A}$ is a singular perturbation of A such that its part A' satisfies the condition (2.5).

Consider now the case of a semi-bounded operator A. Suppose that $A \geq a$, $a \in \mathbb{R}$. Then one can decompose $[a, \infty)$ into a union of mutually disjoint Borel sets $\Delta_k \subset [a+k, \infty)$:

$$[a, \infty) = \bigcup_{k=0}^{\infty} \Delta_k,$$

in such a way that each $A^{(k)} := A_{\Delta_k}$ is an unbounded operator in the subspace $\mathcal{H}_k := \mathcal{H}_{\Delta_k}$. Note that

$$A = \bigoplus_{k=0}^{\infty} A^{(k)}.$$

Let T be an arbitrary self-adjoint operator in $\mathcal{H}$. Set $T^{(0)} := T_{(-\infty,a)}$, $T^{(k)} =: T_{[a+k-1,a+k)}$, $k \geq 1$. Then

$$T = \bigoplus_{k=0}^{\infty} T^{(k)}.$$

Applying Theorem 2.1 to $A^{(k)}$ we obtain that there exists a self-adjoint singular perturbation $\widetilde{A^{(k)}}$ of $A^{(k)}$ in $\mathcal{H}_k$ such that

$$\widetilde{A^{(k)}}_{(-\infty,a+k)} \simeq T^{(k)}. \tag{3.4}$$

Define the singular perturbation $\tilde{A}$ of A by

$$\tilde{A} := \bigoplus_{k=0}^{\infty} \widetilde{A^{(k)}}.$$

Clearly, $\tilde{A} = A' \oplus A''$, where

$$A' := \bigoplus_{k=0}^{\infty} \widetilde{A^{(k)}}_{(-\infty,a+k)}, \quad \text{and} \quad A'' := \bigoplus_{k=0}^{\infty} \widetilde{A^{(k)}}_{[a+k,\infty)}.$$

By (3.4) we have that

$$A' \simeq T.$$

□

4 Discussion

We emphasize that the singular perturbations $\tilde{A}$ in Theorems 2.1 and 2.2 are not uniquely defined since in our considerations the symmetric restrictions of $A_{[a,\infty)}$ and $A^{(k)}$ are arbitrary.

Theorem 2.1 shows that the spectral properties of $\tilde{A}$ and T in $J \subset \rho(A) \cap (-\infty, a)$ are the same. In particular

$$\sigma_\sharp(\tilde{A}) \cap J = \sigma_\sharp(T) \cap J \text{ for } \sharp = \text{ac, sc, p}.$$

Besides, on $E_A((-\infty,a) \setminus J)\mathcal{H}$ the operators A and $\tilde{A}$ coincide. If an unbounded self-adjoint operator A is not semi-bounded from below then one can replace in Theorem 2.1 $(-\infty, a)$ by (a, ∞).

Note also that Theorem 2.2 shows that for an arbitrary Borel set $\Delta \subset \mathbb{R}$ there exists a self-adjoint singular perturbation $\tilde{A}$ of the form (2.4) such that

$$A'_\Delta \simeq T_\Delta.$$

In particular, one can construct $\tilde{A}$ such that $\sigma_{ac}(\tilde{A}) = \sigma_{sc}(\tilde{A}) = \overline{\sigma_p(\tilde{A})} = \mathbb{R}$.

We remark that Theorem 2.1 shows that for any fixed $a \in \mathbb{R}$ there exists a singular perturbation $\tilde{A}$ which coincides with A on the subspace $E_A((-\infty,a))\mathcal{H}$ and has any before given additional kind of spectra on the left of the point a. Theorem 2.1 can be in some sense improved using the paper [16] and combining it with results from [10]. The following theorem holds.

Theorem 4.1. *Let A be an unbounded (at least from above) self-adjoint operator in a separable Hilbert space $\mathcal{H}$. Then for any fixed $a \in \mathbb{R}$ and an auxiliary self-adjoint operator T in $\mathcal{H}$ there exists a self-adjoint singular perturbation $\tilde{A}$ of A such that*

$$\tilde{A}_{(-\infty,a)} \simeq T_{(-\infty,a)}. \tag{4.1}$$

However this variant of our main result does not ensure that $\tilde{A}$ coincides with A on the subspace $E_A((-\infty,a))\mathcal{H}$.

Further, taking into account the paper [17] Theorem 2.2 takes the following stronger form:

Theorem 4.2. *Let A be an unbounded (at least from above) self-adjoint operator in a separable Hilbert space $\mathcal{H}$. Then for any auxiliary self-adjoint operator T in $\mathcal{H}$, which is unbounded from above, there exists a self-adjoint singular perturbation $\tilde{A}$ of A such that $\tilde{A} \simeq T$.*

One of the aim of our short paper is to show how recent results of usual spectral theory of self-adjoint extensions imply the corresponding results for singular perturbations. So Theorem 2.2 can be considered in particular as a simple proof of a weak version of the corresponding result from [17].

By the way Theorem 4.2 shows that the only condition for the existence of a singular perturbation obeying $\tilde{A} \simeq T$ is that both operators A and T together either semi-bounded from above or semi-bounded from below. However, if this condition is satisfied, then one can produce by singular perturbation any self-adjoint operator up to unitary equivalence.

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