A VARIANT OF INVERSE NEGATIVE EIGENVALUES PROBLEM IN SINGULAR PERTURBATION THEORY

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Abstract. Let $A \geq 0$ be a self-adjoint unbounded operator in a Hilbert space $\mathcal{H}$. Let $\mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_{-1}$ be the rigged Hilbert space associated with A, where the norms in $\mathcal{H}_{\pm 1}$ are defined as $\|\cdot\|_{\pm 1} = ((A+1)^{\pm 1}\cdot, \cdot)^{1/2}$. Given $\psi_j \in \mathcal{H}_1 \setminus \mathcal{D}(A)$, a sequence of vectors orthonormal in $\mathcal{H}$, and $E_j \leq 0$, $j = 1, ..., N$, a sequence of non-positive numbers, we explicitly construct an operator T, acting from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$, such that the perturbed operator $\widetilde{A} = A \widetilde{+} T$, defined by the generalized operator sum, solves the negative eigenvalues problem, $\widetilde{A}\psi_j = E_j \psi_j$. We show that T is uniquely defined if $\operatorname{rank} T = N$. Besides we give a general description of all self-adjoint operators $T' : \mathcal{H}_1 \to \mathcal{H}_{-1}$ such that $\widetilde{A}' = A \widetilde{+} T'$ solve the same negative eigenvalues problem.

1. Introduction

Let $A \geq 0$ be a positive unbounded self-adjoint operator in a complex separable Hilbert space $\mathcal{H}$ with an inner product $(\cdot, \cdot)$ and the norm $\|\cdot\|$. We will assume that $\inf_{\|f\|=1}(Af, f) = 0$ and $(Af, f) > 0$, $f \neq 0$, i.e., that A is invertible.

Let $\{\mathcal{H}_k(A)\}_{k \in \mathbf{R}^1}$ be the $A-$scale of Hilbert spaces (for details see below Appendix). Here we use only the following part of this scale:

$$\mathcal{H}_{-1} \supset \mathcal{H}_0 \equiv \mathcal{H} \supset \mathcal{H}_1, \tag{1}$$

where $\mathcal{H}_1 \equiv \mathcal{H}_1(A)$ coincides with the domain $\mathcal{D}(A^{1/2})$ in the norm $\|\varphi\|_1 := \|(A + I)^{1/2}\varphi\|$, where I stands for identity, and $\mathcal{H}_{-1} \equiv \mathcal{H}_{-1}(A)$ is the dual space ($\mathcal{H}_{-1}$ is the completion of $\mathcal{H}$ in the norm $\|f\|_{-1} := \|(A + I)^{-1/2} f\|$). Obviously A is bounded as a map from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$, and therefore the expression $\langle \varphi, A\psi \rangle$ has sense for any $\varphi, \psi \in \mathcal{H}_1$, where $\langle \cdot, \cdot \rangle$ denotes the dual inner product between $\mathcal{H}_1$ and $\mathcal{H}_{-1}$.

Let $T : \mathcal{H}_1 \to \mathcal{H}_{-1}$ be a closed symmetric operator acting in the $A-$scale. We say that an operator $T : \mathcal{H}_1 \to \mathcal{H}_{-1}$ is $\mathcal{H}-$*singular* if the range $\mathcal{R}(T)$ contains elements which do not belong to $\mathcal{H}$. It may happens that $\mathcal{R}(T) \cap \mathcal{H} = \{0\}$ and moreover that T belongs to the $\mathcal{H}_{-1}$-class (see Appendix).

Given A and T we construct the singularly perturbed operator $\widetilde{A} = A \widetilde{+} T$ using the operation of generalized operator sum, which extends the well-known form-sum procedure. By definition (for details see Appendix and ref. [11,12,18,23,25], [28])

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the *generalized operator sum,* $\widetilde{A} = A\tilde{+}T$, is the "restriction" of the operator sum $\mathbf{A} + T : \mathcal{H}_1 \to \mathcal{H}_{-1}$ onto $\mathcal{H}$, where $\mathbf{A}$ stands for the closure of A as an operator from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$. Precisely,

$$\mathcal{D}(\widetilde{A}) = \{\varphi \in \mathcal{H}_1 \cap \mathcal{D}(T) : \mathbf{A}\varphi + T\varphi \in \mathcal{H}\}, \quad \widetilde{A}\varphi = \mathbf{A}\varphi + T\varphi. \tag{2}$$

In this paper we discuss the following variant of the inverse negative eigenvalues problem (c.f. with [3], [15]).

Let $\psi_j \in \mathcal{H}_1(A) \setminus \mathcal{D}(A)$, $j = 1, ..., N$, be an arbitrary sequence of vectors, which is orthonormal in $\mathcal{H}$, $(\psi_j, \psi_k) = \delta_{jk}$, and let $E_j \le 0$, $j = 1, ..., N$, be a sequence of non-positive numbers. We want to construct a singular operator $T : \mathcal{H}_1 \to \mathcal{H}_{-1}$ (possibly of the $\mathcal{H}_{-1}$-class) such that the perturbed operator $\widetilde{A} = A\tilde{+}T$ is self-adjoint in $\mathcal{H}$ and solves the negative eigenvalues problem,

$$\widetilde{A}\psi_j = E_j\psi_j, \quad j = 1, ..., N. \tag{3}$$

In this paper we construct the required operator T by the explicit inductive method using, at each step, a rank one singular perturbation.

We start with an observation that for any self-adjoint operator A, a regular rank one perturbation $A_1 = A + \alpha_1 (\cdot, \omega_1)\, \omega_1$ with $\omega_1 = (A - E_1)\psi_1$ and $\alpha_1 = -\frac{1}{(\psi_1, \omega_1)}$ solves the problem $A_1\psi_1 = E_1\psi_1$ for any beforehand given real number E_1 and any vector $\psi_1 \in \mathcal{D}(A)$, under the condition that $(A\psi_1,\ \psi_1) \ne E_1 \left\|\psi_1\right\|^2$ (in fact this condition might be omitted). We can repeat this simple construction for any real number E_2 and any other vector $\psi_2 \in \mathcal{D}(A)$, taking A_1 in place of A. Then the operator $A_2 = A_1 + \alpha_2 (\cdot, \omega_2)\, \omega_2$ with $\omega_2 = (A_1 - E_2)\psi_2$ and $\alpha_2 = -\frac{1}{(\psi_2, \omega_2)}$ solves the problem $A_2\psi_2 = E_2\psi_2$. If $\psi_2 \perp \psi_1$, then, at the second step, the previous eigenvalue pair, $E_1,\ \psi_1$, is preserved, i.e., the operator A_2 solves also the problem $A_2\psi_1 = E_1\psi_1$. Similarly by induction, at the Nth step, we obtain the operator $A_N = A_{N-1} + \alpha_N (\cdot, \omega_N)\, \omega_N$ which is a rank N perturbation of A and which solves the problem (3) with $\psi_j \in \mathcal{D}(A)$, $(\psi_j, \psi_k) = \delta_{jk}$, $j, k = 1, ..., N$, and any real numbers E_j.

We will show in this paper that just described way admits an extension to the case where all $\psi_j \in \mathcal{H}_1(A) \setminus \mathcal{D}(A)$ and $E_j \le 0$. Moreover a similar result is true (see [17]) for an arbitrary self-adjoint operator A in the case where $\psi_j \in \mathcal{H}$, $\text{span}\{\psi_j\} \cap \mathcal{D}(A) = \{0\}$, and $E_j \in \mathbf{R}^1$.

We remark that in the case of a rank one perturbation, $\widetilde{A} = A\tilde{+}\alpha \langle \cdot, \omega \rangle\, \omega$, such that $\widetilde{A}$ solves the problem $\widetilde{A}\psi = E\psi$, there exists a one-to-one correspondence between the sets of pairs $\{E,\ \psi\}$ and $\{\alpha, \omega\}$. Moreover this fact allows us to prove the uniqueness theorem for the operator T in the representation $\widetilde{A} = A\tilde{+}T$ under the condition that $\text{rank}\, T = N$.

In general, the operator T is not unique. In section 5 we give a description of all set operators $T' : \mathcal{H}_1 \to \mathcal{H}_{-1}$ such that $\widetilde{A}' = A\tilde{+}T'$ solves the same negative eigenvalues problem.

2. Singular rank one perturbations

Here we show (by explicit construction) that for any $\psi \in \mathcal{H}_1$ and a real E there exists a unique (singular) rank one perturbation $\widetilde{A}$ of A, which solves the problem

$$\widetilde{A}\psi = E\psi.$$

We start with a brief account in the theory of weak singular rank one perturbation of A which might be considered now as a bounded from below (unbounded) operator (for more details see Appendix and [5,7,16,23,24,25,30]).

Let a vector $\omega \in \mathcal{H}_{-1} \setminus \mathcal{H}$, $\|\omega\|_{-1} = 1$ be fixed. We assign to ω the bounded operator $T = T_{\omega} = \langle \cdot, \omega \rangle \omega$ acting from $\mathcal{H}_1(A)$ to $\mathcal{H}_{-1}(A)$ as follows:

$$T_{\omega}\varphi = \langle \varphi, \omega \rangle \omega, \quad \varphi \in \mathcal{H}_1(A).$$

It is known that each T_{ω} produces a family of self-adjoint singular rank one perturbations $\widetilde{A} = A_{\alpha,\omega}$ with a parameter $\alpha \in (\mathbf{R}^1 \setminus \{0\}) \cup \infty$ which is called a coupling constant. We always write

$$A_{\alpha,\omega} = A \widetilde{+} \alpha T_{\omega} = A \widetilde{+} \alpha \langle \cdot, \omega \rangle \omega.$$

If $\alpha \neq \infty$, then $A_{\alpha,\omega}$ is defined as the generalized operator sum [18]. In the case $\alpha = \infty$, the operator $A_{\infty,\omega}$ is defined (cf. with [16]) as the Friedrichs extension of the densely defined symmetric operator

$$\overset{\circ}{A} := A \restriction \mathcal{D},$$

where the domain

$$\mathcal{D} = \{\varphi \in \mathcal{D}(A) : \langle \varphi, \omega \rangle = 0\}$$

is the maximal dense in $\mathcal{H}$ lineal on which all $A_{\alpha,\omega}$ coincide with A. In any case the resolvent of $A_{\alpha,\omega}$ has the representation by Krein's formula,

$$\begin{aligned} \widetilde{R}_z &= (A_{\alpha,\omega} - zI)^{-1} = (A - zI)^{-1} - \frac{1}{\alpha^{-1} + \langle \eta_z, \omega \rangle} (\cdot, \eta_{\overline{z}}) \eta_z, \\ &= R_z - b_{\alpha}^{-1}(z)(\cdot, \eta_{\overline{z}})\eta_z \end{aligned} \tag{4}$$

where

$$R_z = (A - zI)^{-1}, \quad \eta_z = \mathbf{R}_z \omega, \quad \mathbf{R}_z = (\mathbf{A} - zI)^{-1}, \quad b_{\alpha}(z) = \frac{1}{\alpha} + \langle \eta_z, \omega \rangle. \tag{5}$$

The domain $\mathcal{D}(A_{\alpha,\omega})$ belongs to $\mathcal{H}_1(A)$ and may be described as follows. For any z from resolvent set of $A_{\alpha,\omega}$,

$$\mathcal{D}(A_{\alpha,\omega}) = \left\{\psi \in \mathcal{H}_1(A) : \psi = \varphi - b_{\alpha}^{-1}(z) \langle \varphi, \omega \rangle \eta_z, \ \varphi \in \mathcal{D}(A)\right\},$$

and

$$(A_{\alpha,\omega} - zI)\psi = (A - zI)\varphi.$$

For $\alpha = \infty$, $b_{\infty}(z) = \langle \eta_z, \omega \rangle$ and we get the Friedrichs extension $A_{\infty,\omega}$ of $\overset{\circ}{A}$, which, in the case of a positive operator A, can be described also as follows:

$$\begin{gathered} \mathcal{D}(A_{\infty,\omega}) = \{\psi \in \mathcal{H}_1(A) : \psi = \varphi - \langle \varphi, \omega \rangle \eta, \ \varphi \in \mathcal{D}(A)\}, \\ A_{\infty,\omega}\psi = A\varphi + \langle \varphi, \omega \rangle \eta, \end{gathered}$$

where we put $\eta = (\mathbf{A} + I)^{-1}\omega$ and use the equality $\langle \eta, \omega \rangle = \|\omega\|^2_{-1} = 1$. Thus each $A_{\alpha,\omega}$ is a self-adjoint extension of $\overset{\circ}{A}$.

Let now A be a positive operator. It is known (for details, see [4,5,7,8,25,24]) that a pair $\psi \in \mathcal{D}(A_{\alpha,\omega}) \setminus \mathcal{D}(A)$, $E < 0$, solves the eigenvalue problem $A_{\alpha,\omega}\psi = E\psi$, if and only if the vector ψ has the form

$$\psi = (\mathbf{A} - E)^{-1}\omega$$

and E is a root of the equation

$$b_{\alpha}(E) = \frac{1}{\alpha} + \langle \eta_E, \omega \rangle = 0. \tag{6}$$

Does there exist a solution E of (6) for any α if $\omega \in \mathcal{H}_{-1} \setminus \mathcal{H}$ is fixed ? We give an answer to this question under the assumptions that $\alpha < 0$.

Let us consider the function

$$a_{\omega}(E) := \langle \eta_E, \omega \rangle = \left\langle (\mathbf{A} + E)^{-1}\omega, \omega \right\rangle = \int_0^{\infty} \frac{1}{\lambda - E}\, d\mu_{\omega}(\lambda), \quad E \le 0, \tag{7}$$

where $d\mu_{\omega}(\lambda) = d\,\langle \mathbf{E}_{\lambda}\omega, \omega\rangle$ stands for the spectral measure of A associated to ω and where $\mathbf{E}_{\lambda}$ is the resolution of identity for A. It is possible that $a_{\omega}(0) = \infty$. Evidently, $a_{\omega}(E)$ is a continuous, nonnegative, and non-decreasing function on $E \in (-\infty, 0]$ with

$$\lim_{E \to -\infty} a_{\omega}(E) = 0.$$

Due to (6), $A_{\alpha,\omega}\psi = E\psi$ for some vector $\psi \in \mathcal{D}(A_{\alpha,\omega}) \setminus \mathcal{D}(A)$ if and only if

$$\langle \omega, \mathbf{R}_E \omega \rangle = -\frac{1}{\alpha}. \tag{8}$$

Thus the answer to the above question depends of whether the value

$$a_{\omega}(0) := \lim_{E \to 0} a_{\omega}(E)$$

is finite or not. Since $a_{\omega}(E)$ grows monotonically from 0 to $a_{\omega}(0)$, when E runs over the interval $(-\infty, 0]$, a solution $E \le 0$ always exists for all negative α such that

$$-\frac{1}{\alpha} \le a_{\omega}(0).$$

Therefore if

$$a_{\omega}(0) = \int_0^{\infty} \frac{1}{\lambda} d\mu_{\omega}(\lambda) < \infty,$$

then the maximal value of the coupling constant α insuring the existence of a solution $E \le 0$ is

$$\alpha = -\frac{1}{a_{\omega}(0)}.$$

However if

$$\lim_{E \to 0} a_{\omega}(E) = a_{\omega}(0) = +\infty,$$

then $A_{\alpha,\omega}$ possesses a negative eigenvalue E for any fixed $\alpha < 0$. We emphasize that the solution E, if it exists, is unique since the function $a_{\omega}(E)$ grows monotonically when $E \to 0$.

We can give some characterization of vectors ω such that $A_{\alpha,\omega}$ possesses an eigenvalue $E<0$ for any fixed $\alpha<0$. We need some additional preparations before formulating of our result.

Introduce the "homogeneous" version of the positive space $\mathcal{H}_1(A)$. Let $H_1 \equiv H_1(A)$ denote the completion of $\mathcal{D}(A)$ under the norm $\|f\|_{H_1} := (Af,f)^{1/2}$. We recall that by assumption A is invertible. It is clear that $\|\cdot\|_{H_1} \leq \|\cdot\|_{\mathcal{H}_1}$ and therefore $\|\cdot\|_{H_{-1}} \geq \|\cdot\|_{\mathcal{H}_{-1}}$, where $H_{-1} \equiv H_{-1}(A)$ is the completion of the range $\mathcal{R}(A)$ under the inner product $(f,g)_{H_{-1}} := (A^{-1}f,g)$. Thus we have

$$\mathcal{H}_{-1}(A) \supseteq H_{-1}(A), \quad H_1(A) \supseteq \mathcal{H}_1(A). \tag{9}$$

We observe (see (7)) that

$$a_{\,\omega}(0) = \|\omega\|^2_{H_{-1}}.$$

Therefore $\omega \in H_{-1}$ iff $a_{\,\omega}(0) < \infty$. Thus if we assume that $\omega \in H_{-1}$, then $\lim_{E\to 0} a_{\,\omega}(E) = a_{\,\omega}(0) < \infty$, and the root E of equation (8) is absent, if the value of the coupling constant satisfies $-\frac{1}{a_{\,\omega}(0)} < \alpha < 0$. Indeed if E is a root of the equation $\alpha = -\left\langle \omega, (\mathbf{A}-E)^{-1}\omega \right\rangle^{-1}$ then the coupling constant α should satisfy the inequality

$$-\frac{1}{\alpha} \leq a_{\,\omega}(0).$$

Thus we have the following result (cf. with [4]).

Theorem 1. *For a fixed element $\omega \in \mathcal{H}_{-1} \setminus \mathcal{H}$, $\|\omega\|_{-1} = 1$, the operator $A_{\alpha,\omega} = A \widetilde{+} \alpha \langle \cdot, \omega \rangle \omega$ possesses exactly one negative eigenvalue $E<0$ for any $\alpha<0$, if and only if*

$$\omega \in \mathcal{H}_{-1} \setminus H_{-1}. \tag{10}$$

In such a case, the uniquely defined eigenvalue E is a root of the equation

$$a_{\,\omega}(E) + \alpha^{-1} = 0, \tag{11}$$

and the corresponding eigenvector has the form $\psi = (\mathbf{A}-E)^{-1}\omega$.

In the considered situation we have a one-to-one correspondence between pairs $\{\alpha,\omega\}$ and $\{E,\psi\}$. Now we show that such a correspondence exists in a more general situation.

Theorem 2. *For each bounded from below (unbounded) self-adjoint operator A in $\mathcal{H}$, any nonzero vector $\psi \in \mathcal{H}_1$, and a real number E, there exists a uniquely defined rank one (singular) perturbation $\widetilde{A} = A_{\alpha,\omega}$ with $\omega \in \mathcal{H}_{-1}$ and α given by*

$$\omega = (\mathbf{A}-E)\psi, \quad \alpha = -\frac{1}{\langle \psi, (\mathbf{A}-E)\psi \rangle}, \tag{12}$$

which solves the problem

$$\widetilde{A}\psi = E\psi. \tag{13}$$

In other words, the operator

$$\widetilde{A} = A_{\alpha,\omega} = A \widetilde{+} \alpha \langle \cdot, \omega\rangle \omega \tag{14}$$

with $\omega \in \mathcal{H}_{-1}$ and a coupling constant $\alpha \in (\mathbf{R}^1 \setminus \{0\}) \cup \infty$ defined by (12), for any beforehand given vector $0 \neq \psi \in \mathcal{H}_1$ and a real number E, solves the problem (13). We put $\widetilde{A} = A$, if $\omega = 0$, i.e., if $A\psi = E\psi$. Conversely if $\widetilde{A}$ is a rank one weak singular perturbation of A (see Appendix) and $\widetilde{A}$ solves the problem (13), then this operator admits the representation (14), where the coupling constant α and the element ω are uniquely connected (precisely ω is defined up to the factor $e^{i\theta}$) with ψ and E satisfying relations (12). Thus this relations establish a one-to-one correspondence between $\{E, \psi\}$ and $\{\alpha, \omega\}$ under the condition that $\widetilde{A}$ solves problem (13).

Proof. Let $\psi \in \mathcal{H}_1$ and $E \in \mathbf{R}^1$ be given. In the trivial case where $\psi \in \mathcal{D}(A)$ and $A\psi = E\psi$, we put $\widetilde{A} = A$, and (13) is satisfied. Otherwise $\omega \neq 0$ and $\alpha \neq 0$ too. Assume $\alpha \neq \infty$. By the direct checking we find that the operator

$$\widetilde{A} = A \widetilde{+} \left(-\frac{1}{\langle \psi, (\mathbf{A}-E)\psi\rangle}\right) \langle \cdot, (\mathbf{A}-E)\psi\rangle (\mathbf{A}-E)\psi, \tag{15}$$

solves the problem $\widetilde{A}\psi = E\psi$. Moreover for any other operator $\widetilde{A}' = A \widetilde{+} \alpha' \langle \cdot, \omega' \rangle \omega'$, $\omega' \neq 0$, $\alpha' \neq \infty$, which solves the same problem, the element ω' necessarily has the form $\omega' = e^{i\theta}\omega$, $\theta \in [0, 2\pi)$ and $\alpha' = \alpha$, where ω and α are given by (12). If ψ and E are such that $\langle \mathbf{A}\psi, \psi\rangle = E \left\|\psi\right\|^2$, $(\omega \neq 0)$, then $\alpha = \infty$, we define $\widetilde{A}$ by Krein's formula

$$\widetilde{R}_z = (A_{\infty,\omega} - zI)^{-1} = (A - zI)^{-1} - \frac{1}{\langle \eta_z, \omega\rangle} (\cdot, \eta_{\overline{z}})\eta_z, \tag{16}$$

with $\eta_z = \mathbf{R}_z\omega$ where $\omega = (\mathbf{A}-E)\psi$. We recall that now $\mathbf{A}\psi \neq E\psi$. By the direct checking we find that $\widetilde{R}_z\psi = \frac{1}{E-z}\psi$, i.e., $\widetilde{A} = A_{\infty,\omega}$ solves the problem (13). Conversely if $\widetilde{A}$ is the Friedrichs extension of a symmetric operator $\overset{\circ}{A}$ defined by some element $\omega \in \mathcal{H}_{-1} \setminus \mathcal{H}$, and $\widetilde{A}$ solves the problem (13), then the resolvent of $\widetilde{A}$ has the form (16) where necessarily $\omega = (\mathbf{A}-E)\psi$, and $\langle \mathbf{A}\psi, \psi\rangle = E \left\|\psi\right\|^2$, which corresponds to $\alpha = \infty$. □

Thus in the family of rank one perturbations $A_{\alpha,\omega}$ such that each $A_{\alpha,\omega}$ solves the problem $\widetilde{A}\psi = E\psi$ with a real E and some $\psi \in \mathcal{H}_1$, we have a one-to-one correspondence between pairs $\{E \in \mathbf{R}^1, \psi \in \mathcal{H}_1\}$ and $\{\alpha \in (\mathbf{R}^1 \setminus \{0\}) \cup \infty,\ \omega \in \mathcal{H}_{-1}\}$.

We remark also that a similar result is true (for details see [17]) for a strongly (pure) singular rank one perturbation $\widetilde{A} \in \mathcal{P}_{\rm ss}(A)$ (for notation see Appendix) with $\psi \in \mathcal{H}$ and $\omega \in \mathcal{H}_{-2}$. In this case, the operator $\widetilde{A}$ may be defined for any beforehand given $E \in \mathbf{R}^1$ and $\psi \in \mathcal{H}$ by Krein's formula

$$(\widetilde{A} - z)^{-1} = (A - z)^{-1} + b_z^{-1}(\cdot, \eta_{\overline{z}})\eta_z, \qquad \operatorname{Im} z \neq 0,$$

with

$$\eta_z = (A - E)(A - z)^{-1}\psi$$

and

$$b_z = (E - z)(\psi, \eta_{\bar{z}}).$$

The above described procedure of constructing the operator $\widetilde{A}$ which solves the problem (13) is called the *inverse eigenvalues method* in the singular perturbation theory. In the next section we will use this methods in the case of finite rank perturbations.

3. Singular finite rank perturbations

Let $A = A^* \geq 0$ be invertible. Given a sequence of vectors $\psi_j \in \mathcal{H}_1(A) \setminus \mathcal{D}(A)$ orthonormal in $\mathcal{H}$ and a sequence of non-positive numbers $E_j \leq 0$, $j = 1, ..., N$, introduce the sequence of operators $A_1, ..., A_N$ as follows. At the first step we put

$$A_1 = A \widetilde{+} \alpha_1 \langle \cdot, \omega_1 \rangle \omega_1, \quad \omega_1 \equiv \omega_1^0 = (\mathbf{A} - E_1)\psi_1$$

with

$$\alpha_1 = -\frac{1}{\langle \psi_1, (\mathbf{A} - E_1)\psi_1 \rangle} = -\frac{1}{\langle \psi_1, \mathbf{A}\psi_1 \rangle - E_1} = -\frac{1}{a_{11}^0 - E_1},$$

where

$$a_{11}^0 := \langle \psi_1, \mathbf{A}\psi_1 \rangle .$$

We remark that under starting assumptions $\infty \neq \alpha_1 < 0$ since $\langle \psi_1, \mathbf{A}\psi_1 \rangle > 0$ and $-E \geq 0$. According to constructions of the previous section, A_1 is a weak singular rank one perturbation of A, which solves the problem $A_1\psi_1 = E_1\psi_1$ and therefore it is a uniquely defined by beforehand given vector $\psi_1 \in \mathcal{H}_1(A) \setminus \mathcal{D}(A)$ and number $E_1 \leq 0$. We note that A_1 may be written also in the form

$$A_1 = E_1 P_{\psi_1} \oplus A_1^{\perp},$$

where P_{ψ_1} stands for the orthogonal projector onto the vector ψ_1 and $A_1^{\perp}$ is the part of A acting in the subspace orthogonal to ψ_1. It is clear that $A_1^{\perp} \geq 0$.

At the second step we construct a weak singular rank one perturbation of A_1,

$$A_2 = A_1 \widetilde{+} \alpha_2 \left\langle \cdot, \omega_2^1 \right\rangle \omega_2^1, \quad \omega_2^1 = (\mathbf{A}_1 - E_2)\psi_2,$$

where

$$\omega_2^1 = (\mathbf{A}_1 - E_2)\psi_2 \in \mathcal{H}_{-1}(A_1),$$

since $\psi_2 \in \mathcal{H}_1(A)$ and norms in $\mathcal{H}_1(A)$ and $\mathcal{H}_1(A_1)$ are equivalent, and where

$$\alpha_2 = -\frac{1}{\langle \psi_2, (\mathbf{A}_1 - E_2)\psi_2 \rangle} = -\frac{1}{a_{22}^1 - E_2}.$$

We find that

$$a_{22}^1 := \langle \psi_2, \mathbf{A}_1\psi_2 \rangle = \langle \psi_2, \mathbf{A}\psi_2 \rangle + \alpha_1 \left| \langle \psi_2, \omega_1 \rangle \right|^2 = a_{22}^0 - \frac{1}{a_{11}^0 - E_1} |a_{21}^0|^2,$$

$$a_{21}^0 := \langle \psi_2, A\psi_1 \rangle ,$$

and therefore

$$\alpha_2 = -\frac{1}{a^0_{22} - E_2 - \dfrac{|a^0_{21}|^2}{a^0_{11} - E_1}}.$$

By the construction, A_2 solves the problem $A_2\psi_2 = E_2\psi_2$, and moreover it solves the problem $A_2\psi_1 = E_1\psi_1$ too since $\psi_1 \perp \psi_2$. Thus by our construction and due to Theorem 2, the operator A_2 is uniquely defined by the vectors $\psi_j \in \mathcal{H}_1(A)$ and the numbers $E_j \leq 0$, $j = 1, 2$, singular rank two perturbation of A, which solves the eigenvalues problem $A_2\psi_j = E_j\psi_j$, $j = 1, 2$. We note that A_2 may be written in the form

$$A_2 = E_1 P_{\psi_1} \oplus E_2 P_{\psi_2} \oplus A_2^{\perp},$$

where P_{ψ_j}, $j = 1, 2$, stand for the orthogonal projectors onto vector ψ_j and $A_2^{\perp}$ is the part of A_2 acting in the subspace orthogonal to the vectors ψ_1, ψ_2. Obviously A_2, as a rank two perturbation of A, has the representation

$$A_2 = A \widetilde{+} T_2$$

with a singular rank two operator $T_2 : \mathcal{H}_1(A) \to \mathcal{H}_{-1}(A)$,

$$T_2 = \alpha_1 \left\langle \cdot, \omega^0_1 \right\rangle \omega^0_1 + \alpha_2 \left\langle \cdot, \omega^1_2 \right\rangle \omega^1_2,$$

which can also be written in the form

$$T_2 = \sum_{j,k=1}^{2} t_{jk} \left\langle \cdot, \omega_j \right\rangle \omega_k, \ \omega_j := (\mathbf{A} - E_j)\psi_j,$$

where

$$\begin{aligned} t_{11} &= \alpha_1 + \alpha_2(\alpha_1)^2 \left|a^0_{21}\right|^2, \\ t_{12} &= \alpha_1\alpha_2 a^0_{21}, \\ t_{21} &= \alpha_1\alpha_2 a^0_{12}, \\ t_{22} &= \alpha_2. \end{aligned}$$

Thus

$$A_2 = A \widetilde{+} T_2 = A \widetilde{+} \left(\sum_{j,k=1}^{2} t_{jk} \left\langle \cdot, \omega_j \right\rangle \omega_k \right).$$

We can continue our constructions to any finite step up to $n = N$. At the nth step we construct a weak singular rank one perturbation of the operator A_{n-1},

$$A_n = A_{n-1} \widetilde{+} \alpha_n \left\langle \cdot, \omega^{n-1}_n \right\rangle \omega^{n-1}_n,$$

where

$$\omega^{n-1}_n = (\mathbf{A}_{n-1} - E_n)\psi_n \in \mathcal{H}_{-1}(A_{n-1})$$

since the norms in $\mathcal{H}_1(A)$ and $\mathcal{H}_1(A_{n-1})$ are equivalent and $\psi_n \in \mathcal{H}_1(A)$, and where

$$\alpha_n = -\frac{1}{\left\langle \psi_n, \omega_n^{n-1} \right\rangle} = -\frac{1}{a_{nn}^{n-1} - E_n},$$
$$a_{nn}^{n-1} := \langle \psi_n, \mathbf{A}_{n-1}\psi_n \rangle$$
$$= \langle \psi_n, \mathbf{A}_{n-2}\psi_n \rangle + \alpha_{n-1} | \left\langle \psi_n, \omega_{n-1}^{n-2} \right\rangle |^2 \equiv a_{nn}^{n-2} + \alpha_{n-1} |a_{n,n-1}^{n-2}|^2,$$

with

$$a_{n,n-1}^{n-2} = \langle \psi_n, \mathbf{A}_{n-2}\psi_{n-1} \rangle .$$

Thus

$$\alpha_n = -\cfrac{1}{a_{nn}^{n-2} - E_n - \cfrac{|a_{n,n-1}^{n-2}|^2}{a_{n-1,n-1}^{n-3} - E_{n-1} - \cdots - \cfrac{|a_{21}^0|^2}{a_{11}^0 - E_1}}}.$$

We set

$$A_N = A_n = A \tilde{+} \alpha_1 \langle \cdot, \omega_1 \rangle \omega_1 \tilde{+} \cdots \tilde{+} \alpha_n \left\langle \cdot, \omega_n^{n-1} \right\rangle \omega_n^{n-1}, \quad N = n. \tag{17}$$

By Theorem 2 the operator A_N is uniquely defined by a vector $\psi_N \in \mathcal{H}_1(A)$ and a number $E_N \leq 0$, weak singular rank one perturbation of A_{n-1} which solves the eigenvalues problem $A_N \psi_N = E_N \psi_N$. Moreover by induction, A_N is uniquely defined by the vectors $\psi_j \in \mathcal{H}_1(A)$ and the numbers $E_j \leq 0$, $j = 1, ...N$, weak singular rank N perturbation of A, which solves the eigenvalues problem $A_N \psi_j = E_j \psi_j$ for all $E_j \leq 0$.

We note also that A_N can be written in the form

$$A_N = E_1 P_{\psi_1} \oplus \cdots \oplus E_N P_{\psi_N} \oplus A_N^{\perp},$$

where P_{ψ_j}, $j = 1, ..., N$, stands for the orthogonal projector onto the vector ψ_j, and $A_N^{\perp}$ is the part of A_N acting in the subspace orthogonal to the vectors ψ_j.

The above result we formulate as

Theorem 3. *Let $\widetilde{A}$ be a rank N weak singular perturbation of a self-adjoint operator $A \geq 0$, i.e.,*

$$\widetilde{A} \equiv A_N = A \tilde{+} T_N,$$

where $T_N : \mathcal{H}_1(A) \to \mathcal{H}_{-1}(A)$ is a $\mathcal{H}-$singular rank N operator (see Appendix). Let a set of vectors $\psi_j \in \mathcal{H}_1(A) \setminus \mathcal{D}(A)$, orthonormal in $\mathcal{H}$, and numbers $E_j \leq 0$, $j = 1, ..., N$, be arbitrary and fixed. Then $\widetilde{A}$ solves the eigenvalues problem

$$\widetilde{A}\psi_j = E_j \psi_j, \quad E_j \leq 0, \quad j = 1, ..., N,$$

if and only if the operator T_N admits the representation in the form

$$T_N = \sum_{j=1}^{N} \alpha_j \left\langle \cdot, \omega_j^{j-1} \right\rangle \omega_j^{j-1}, \tag{18}$$

where the coupling constants α_j *and the elements* ω_j^{j-1} *are uniquely defined by the vectors* ψ_j *and the numbers* E_j *according to the inductive formulae*

$$\omega_j^{j-1} := (\mathbf{A}_{j-1} - E_j)\psi_j \tag{19}$$

$$\alpha_j := -\frac{1}{\left\langle \psi_j, \omega_j^{j-1} \right\rangle}. \tag{20}$$

Thus we have a one-to-one correspondence between the set of any beforehand given vectors $\psi_j \in \mathcal{H}_1(A) \setminus \mathcal{D}(A)$ orthonormal in $\mathcal{H}$ and numbers $E_j \leq 0$, $j = 1, ..., N$, and a set of weak singular rank N perturbations $\widetilde{A}$ (see Appendix) which solve the eigenvalues problem (3); this correspondence is fixed by (18), (19), and (20).

Proof is inductive and uses Theorem 2. In one direction, the theorem is proved by the above described explicit construction of the operators A_N (see (17)). In the opposite direction theorem is proved in the case $N = 1$ (Theorem 2). In the general case, $N > 1$, we use the following reasoning. Assume $\widetilde{A}$ is a rank N weak singular perturbation of $A \geq 0$, i.e., $\widetilde{A} = A\tilde{+}T$, with some $\mathcal{H}-$singular rank N closed symmetric operator $T : \mathcal{H}_1(A) \to \mathcal{H}_{-1}(A)$ (in fact T is self-adjoint). Assume $\widetilde{A}$ solves the eigenvalues problem (3). Then

$$(A_N - \widetilde{A})\psi_j = 0, \quad j = 1, ..., N,$$

where A_N is defined by (17), and therefore

$$T_N\psi_j = T\psi_j = -\omega_j^{j-1}, \quad j = 1, ..., N.$$

This means that $T_N = T$ since all the vectors ω_j^{j-1} are linearly independent by construction, and both T_N, T are rank N self-adjoint operators. Thus $\widetilde{A} = A_N$. □

We remark that the required operators T of rank N can be constructed in another way as follows [29]. Put

$$T = \sum_{i,j=1}^{N} t_{ij} \left\langle \cdot, \omega_i \right\rangle \omega_j, \quad \omega_j := (\mathbf{A} - E_j)\psi_j.$$

We assert that $\widetilde{A} = A\tilde{+}T$ solves the problem (3), if $(t_{ij})_{i,j=1}^N$ is, up to sign, the matrix inverse to

$$\mathbf{a} = (a_{kj})_{k,j=1}^N, \quad a_{kj} := \left\langle \psi_k, (\mathbf{A} - E_j)\psi_j \right\rangle.$$

Indeed, the equation $\widetilde{A}\psi_k = E_k\psi_k$ can be equivalently rewritten in the form

$$A\psi_k + \sum_{i,j=1}^{N} t_{ij} \left\langle \psi_k, \omega_i \right\rangle \omega_j = E_k\psi_k,$$

or

$$\sum_{i,j=1}^{N} t_{ij} a_{ki} \omega_j = \sum_{j=1}^{N} \alpha_{jk} \omega_j = -(\mathbf{A} - E_k)\psi_k = -\omega_k,$$

where $\alpha_{jk} = \sum_{i=1}^{N} t_{ij} a_{ki}$. Because the vectors ω_j, $j = 1, ..., N$ are linearly independent, we conclude that $\widetilde{A}\psi_k = E_k \psi_k$, $k = 1, ..., N$, holds if and only if $\alpha_{jk} = -\delta_{jk}$. This means that $t_{ij} = -(\mathbf{a}^{-1})_{ij}$, where $\mathbf{a}^{-1}$ denote the matrix inverse to $\mathbf{a}$. Thus if we put

$$T = \sum_{i,j=1}^{N} (-1)(\mathbf{a}^{-1})_{ij} \langle \cdot, \omega_i \rangle \omega_j, \tag{21}$$

then $\widetilde{A} = A \tilde{+} T$ solves the problem (3).

4. Sequential method

Let an operator $A \geq 0$ be as above, $\psi_j \in \mathcal{H}_1(A) \setminus \mathcal{D}(A)$ be a sequences of vectors orthonormal in $\mathcal{H}$, and $E_j < 0$, $j = 1, ..., N$, be a sequence of negative numbers.

In this section we develop another sequential method of an explicit construction of the operator $T : \mathcal{H}_1 \to \mathcal{H}_{-1}$ such that $\widetilde{A} = A \tilde{+} T$ is self-adjoint and solves the negative eigenvalues problem,

$$\widetilde{A}\psi_j = E_j \psi_j, \quad E_j < 0, \quad j = 1, ..., N. \tag{22}$$

Similar to the previous section, in the construction given below we consecutively use only (singular) rank one perturbations of the form $\alpha \langle \cdot, \omega \rangle \omega : \mathcal{H}_1 \to \mathcal{H}_{-1}$, $0 \neq \alpha \in \mathbf{R}^1$, where the vectors

$$\omega \in \mathcal{N}_{-1} := \operatorname{span} \{(\mathbf{A} - E_k)\psi_k\}_{k=1}^{N} \subset \mathcal{H}_{-1}.$$

Let us introduce the following notations. Put

$$A_0^k \equiv A, \quad k = 1, ..., N,$$

and define the operators

$$A_1^k := A_0^k \tilde{+} t_{1k} \langle \cdot, \omega_1^k \rangle \omega_1^k, \quad t_{1k} = -\frac{1}{\langle \psi_1, \omega_1^k \rangle},$$

where

$$\omega_1^k = (\mathbf{A}_0^k - N E_k \delta_{1k})\psi_1 \in \mathcal{H}_{-1}$$

(here we used the same notation as in the previous section for different vectors $\omega \in \mathcal{N}_{-1}$). Similarly we define the operators

$$A_j^k := A_{j-1}^k \tilde{+} t_{1k} \langle \cdot, \omega_j^k \rangle \omega_j^k, \quad t_{1k} = -\frac{1}{\langle \psi_j, \omega_j^k \rangle}, \quad j \geq 1, \tag{23}$$

where

$$\omega_j^k = (\mathbf{A}_{j-1}^k - N E_k \delta_{jk})\psi_j \in \mathcal{H}_{-1}, \tag{24}$$

and $\mathbf{A}^k_{j-1}$ is the closure of A^k_{j-1} as a map from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$. We note that each operator A^k_j can be represented in the form $A^k_j = A\widetilde{+}T^k_j$, where T^k_j is a rank j self-adjoint operator acting from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$ (one can easily find its explicit form). By definition we put

$$A^{(k)} := A^k_{j=N} = A\widetilde{+}T^{(k)}, \tag{25}$$

where

$$T^{(k)} := T^k_{j=N} = \sum_{j=1}^{N} t_{jk} \left\langle \cdot, \omega^k_j \right\rangle \omega^k_j.$$

Define also

$$\widetilde{A} := \frac{1}{N}\sum_{k=1}^{N} A^{(k)}, \tag{26}$$

and $T : \mathcal{H}_1 \to \mathcal{H}_{-1}$ as follows:

$$T = \frac{1}{N}\sum_{k=1}^{N} T^{(k)} = \frac{1}{N}\sum_{k,j=1}^{N} t_{kj} \left\langle \cdot, \omega^k_j \right\rangle \omega^k_j, \tag{27}$$

where the vectors $\omega^k_j \in \mathcal{H}_{-1}$ are given by (24) and where all the coefficients

$$t_{jk} = -\frac{1}{\left\langle \psi_j, \omega^k_j \right\rangle} = -\frac{1}{\left\langle \psi_j, (\mathbf{A}^k_{j-1} - NE_k\delta_{jk})\psi_j \right\rangle} < \infty. \tag{28}$$

Our aim in this section is to prove

Theorem 4. *Given a sequence of vectors $\psi_j \in \mathcal{H}_1(A) \setminus \mathcal{D}(A)$ orthonormal in $\mathcal{H}$ and a sequence of negative numbers $E_j < 0$, $j = 1, ..., N$, let $\widetilde{A}$ be the operator constructed according to (23)–(28). Then $\widetilde{A}$ is self-adjoint and solves the negative eigenvalues problem (22) in the exact sense, i.e., $\widetilde{A}\psi = E\psi$, $E < 0$, $E \neq E_j$, implies $\psi \equiv 0$.*

To prove this, we start with

Proposition 1. *The operator $\widetilde{A}$ defined by (26) coincides with the generalized operator sum,*

$$\widetilde{A} = A\widetilde{+}T. \tag{29}$$

Proof. Using above notations and definitions (see (23)–(28)), we have

$$\begin{aligned} A^{(1)} &\equiv A^1_{j=N} = A^1_{j=N-1}\widetilde{+}t_{N1} \left\langle \cdot, \omega^1_N \right\rangle \omega^1_N \\ &= A^1_{N-2}\widetilde{+}t_{N-1,1} \left\langle \cdot, \omega^1_{N-1} \right\rangle \omega^1_{N-1}\widetilde{+}t_{N1} \left\langle \cdot, \omega^1_N \right\rangle \omega^1_N \\ &= A\widetilde{+}\sum_{j=1}^{N} t_{j1} \left\langle \cdot, \omega^1_j \right\rangle \omega^1_j. \end{aligned}$$

Similarly for any $k \geq 1$,

$$A^{(k)} \equiv A^k_{j=N} = A^k_{j=N-1} \widetilde{+} t_{Nk} \left\langle \cdot, \omega^k_N \right\rangle \omega^k_N$$
$$= A \widetilde{+} \sum_{j=1}^{N} t_{jk} \left\langle \cdot, \omega^k_j \right\rangle \omega^k_j \equiv A \widetilde{+} T^{(k)}.$$

Therefore,

$$\widetilde{A} = \frac{1}{N} \sum_{k=1}^{N} A^{(k)} = A \widetilde{+} \frac{1}{N} \sum_{k=1}^{N} T^{(k)} = A \widetilde{+} T.$$

□

Let

$$\mathbf{A}^k_0 : \mathcal{H}_1 \to \mathcal{H}_{-1},$$
$$\mathbf{A}^k_j : \mathcal{H}_1 \to \mathcal{H}_{-1}, \quad k, j = 1, ..., N,$$

denote closures of the operators A and A^k_j as maps from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$.

Proposition 2. *All the operators $\mathbf{A}^k_0$, $\mathbf{A}^k_j$ are self-adjoint in the sense of a pair of spaces.*

Proof. Recall that $A^k_0 \equiv A$, and $A^k_j = A^k_{j-1} \widetilde{+} t_{jk} \left\langle \cdot, \omega^k_j \right\rangle \omega^k_j$. By the construction, all these operators, as maps from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$, are densely defined symmetric and bounded. Therefore their closures are self-adjoint. □

Proposition 3.

$$A^k_j \psi_l = 0, \quad \textit{if} \quad l \leq j < k.$$

Proof. By the construction, for $j = 1$ and any $k > 1$ we have

$$A^k_1 \psi_1 = \mathbf{A}^k_1 \psi_1 = \mathbf{A} \psi_1 - \frac{1}{\langle \psi_1, \mathbf{A} \psi_1 \rangle} \langle \psi_1, \mathbf{A} \psi_1 \rangle \mathbf{A} \psi_1 = 0. \tag{30}$$

Similarly, for $j = 1$ and any $k > 2$,

$$A^k_2 \psi_2 = \mathbf{A}^k_2 \psi_2 = \mathbf{A}^k_1 \psi_2 - \frac{1}{\left\langle \psi_2, \mathbf{A}^k_1 \psi_2 \right\rangle} \left\langle \psi_2, \mathbf{A}^k_1 \psi_2 \right\rangle \mathbf{A}^k_1 \psi_2 = 0,$$

and also

$$A^k_2 \psi_1 = \mathbf{A}^k_2 \psi_1 = \mathbf{A}^k_1 \psi_1 - \frac{1}{\left\langle \psi_2, \mathbf{A}^k_1 \psi_2 \right\rangle} \left\langle \psi_1, \mathbf{A}^k_1 \psi_2 \right\rangle \mathbf{A}^k_1 \psi_2 = 0,$$

since $\mathbf{A}^k_1 \psi_1 = 0$, and $\left\langle \psi_1, \mathbf{A}^k_1 \psi_2 \right\rangle = \left\langle \mathbf{A}^k_1 \psi_1, \psi_2 \right\rangle = 0$ due to (30), where we used the self-adjointness of the operator $\mathbf{A}^k_1$ in the $A-$scale (see the previous Proposition). By induction for an arbitrary $l \leq j < k$ we have

$$A^k_j \psi_l = \mathbf{A}^k_j \psi_l = \mathbf{A}^k_{j-1} \psi_l - \frac{1}{\left\langle \psi_j, \mathbf{A}^k_{j-1} \psi_j \right\rangle} \left\langle \psi_l, \mathbf{A}^k_{j-1} \psi_j \right\rangle \mathbf{A}^k_{j-1} \psi_j = 0,$$

since $\mathbf{A}^k_{j-1} \psi_l = 0$ if $l \leq j-1 < k$, and $\left\langle \psi_l, \mathbf{A}^k_{j-1} \psi_j \right\rangle = \left\langle \mathbf{A}^k_{j-1} \psi_l, \psi_j \right\rangle = 0$ too, where we again used that operators $\mathbf{A}^k_{j-1}$ are self-adjoint in the $A-$scale. □

Proposition 4.
$$A_j^k \psi_l = \delta_{jk} N E_k \psi_l, \quad \textit{if} \quad l \le j = k.$$

Proof. Let $l < j = k$. Then similarly to the previous arguments, we have

$$\begin{aligned} A_j^k \psi_l &= \mathbf{A}_j^k \psi_l \\ &= \mathbf{A}_{j-1}^k \psi_l - \frac{1}{\left\langle \psi_{k-1}, \mathbf{A}_{k-1}^k \psi_{k-1} \right\rangle} \left\langle \psi_l, \mathbf{A}_{k-1}^k \psi_{k-1} \right\rangle \mathbf{A}_{k-1}^k \psi_{k-1} = 0, \end{aligned}$$

since
$$\left\langle \psi_l, \mathbf{A}_{k-1}^k \psi_{k-1} \right\rangle = \left\langle \mathbf{A}_{k-1}^k \psi_l, \psi_{k-1} \right\rangle$$
and $\mathbf{A}_{k-1}^k \psi_l = 0$, $l \le j = k-1$. In the case where $l = j = k$ we have

$$\begin{aligned} & A_k^k \psi_k = \mathbf{A}_k^k \psi_k \\ & = A_{k-1}^k \psi - \frac{1}{\left\langle \psi_k, (\mathbf{A}_{k-1}^k - N E_k) \psi_k \right\rangle} \left\langle \psi_k, (\mathbf{A}_{k-1}^k - N E_k) \psi_k \right\rangle (\mathbf{A}_{k-1}^k - N E_k) \psi_k \\ & = \mathbf{A}_{k-1}^k \psi_k - (\mathbf{A}_{k-1}^k - N E_k) \psi_k = N E_k \psi_k. \end{aligned}$$

□

Proposition 5. *Let $k < j$. Then*

$$A_j^k \psi_k = N E_k \psi_k, \quad \textit{if} \quad k < j, \tag{31}$$

and

$$A_j^k \psi_l = 0, \quad \textit{if} \quad l \le j, \quad l \ne k. \tag{32}$$

Proof. By the definition

$$\begin{aligned} A_j^k \psi_k = \mathbf{A}_j^k \psi_k = A_{j-1}^k \psi_k &- \frac{1}{\left\langle \psi_j, (\mathbf{A}_{j-1}^k - N E_k \delta_{j-1,k}) \psi_j \right\rangle} \\ &\times \left\langle \psi_k, (\mathbf{A}_{j-1}^k - N E_k \delta_{j-1,k}) \psi_j \right\rangle (\mathbf{A}_{j-1}^k - N E_k \delta_{j-1,k}) \psi_j. \end{aligned}$$

Assume $k = j-1$, then obviously $A_{j-1}^k \psi_k = N E_k \psi_k$. Besides,

$$\left\langle \psi_k, \mathbf{A}_{j-1}^k \psi_j \right\rangle = \left\langle \mathbf{A}_{j-1}^k \psi_k, \psi_j \right\rangle = E_k \left\langle \psi_k, \psi_j \right\rangle = 0,$$

since $\psi_k \perp \psi_j, k \ne j$. Thus $A_j^k \psi_k = N E_k \psi_k$, if $j = k+1$. By induction, we have a similar relation for any $j > k$, i.e., (31) is proved. To prove (32) we have to consider only the case $l \le j$, $l > k$, since the case $l < k$ is already proved in the previous proposition.

At the first step, put $j = k+1 = l$. Then we have

$$\begin{aligned} A_{k+1}^k \psi_k &= \mathbf{A}_{k+1}^k \psi_k \\ &= A_k^k \psi_k - \frac{1}{\left\langle \psi_{k+1}, \mathbf{A}_k^k \psi_{k+1} \right\rangle} \left\langle \psi_{k+1}, \mathbf{A}_k^k \psi_{k+1} \right\rangle \mathbf{A}_k^k \psi_{k+1} = 0. \end{aligned}$$

At the second step we take $j = k+2$, and consider $l = k+1, k+2$. By a direct computation we have

$$\begin{aligned} A_{k+2}^{k}\psi_{k+1} &= \mathbf{A}_{k+2}^{k}\psi_{k+1} \\ &= A_{k+1}^{k}\psi_{k+1} - \frac{1}{\left\langle \psi_{k+2}, \mathbf{A}_{k+1}^{k}\psi_{k+2}\right\rangle} \left\langle \psi_{k+1}, \mathbf{A}_{k+1}^{k}\psi_{k+2}\right\rangle \mathbf{A}_{k+1}^{k}\psi_{k+2} = 0, \end{aligned}$$

since

$$\left\langle \psi_{k+1}, \mathbf{A}_{k+1}^{k}\psi_{k+2}\right\rangle = \left\langle \mathbf{A}_{k+1}^{k}\psi_{k+1}, \psi_{k+2}\right\rangle = 0$$

and $A_{k+1}^{k}\psi_{k+1} = 0$ similarly to the previous arguments. In the case $l = k+2$, we have

$$\begin{aligned} A_{k+2}^{k}\psi_{k+2} &= \mathbf{A}_{k+2}^{k}\psi_{k+2} \\ &= A_{k+1}^{k}\psi_{k+2} - \frac{1}{\left\langle \psi_{k+2}, \mathbf{A}_{k+1}^{k}\psi_{k+2}\right\rangle} \left\langle \psi_{k+2}, \mathbf{A}_{k+1}^{k}\psi_{k+2}\right\rangle \mathbf{A}_{k+1}^{k}\psi_{k+2} = 0. \end{aligned}$$

Similarly we obtain the same relation for $j = k+3, \;\; l = k+1, k+2, k+3$. And so on. □

As a consequence we obtain

Proposition 6.

$$A^{(k)}\psi_l = NE_k\delta_{kl}\psi_k, \quad \textit{if} \quad k, l = 1, ..., N. \tag{33}$$

Proof. To check this assertion, it is convenient to present the above considered operators in the form of the operator array,

$$\begin{pmatrix} A_0^1 = A \\ A_2^1 = A \\ \cdot \\ \cdot \\ \cdot \\ A_N^1 = A \end{pmatrix} \begin{pmatrix} A_1^1 & A_2^1 & \cdot & \cdot & \cdot & A_N^1 = A^{(1)} \\ A_1^2 & A_2^2 & \cdot & \cdot & \cdot & A_N^2 = A^{(2)} \\ \cdot & \cdot & A_3^3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ A_1^N & \cdot & \cdot & \cdot & \cdot & A_N^N = A^{(N)} \end{pmatrix}$$

and consequently to look and check the assertion for each operator using the previous propositions. We remark only that $A_1^2 = A_1^3 = \cdots = A_1^N$, $A_2^3 = A_2^4 = \cdots = A_2^N$,... □

We emphasize that each operator $A^{(k)}$ does not depend on the order in which the pairs $\{\psi_j, \delta_{jk}NE_j\}_{j=1}^{N}$ are taking in the construction of this operator. In other words,

$$A^{(k)} = A_{j=N}^{k} = A_{P(j)=N}^{k},$$

where $P : \{1, ..., N\} \to \{i_1, ..., i_N\}$ denotes a permutation in the set of indices $\{1, ..., N\}$, which preserves the index k. Therefore each $A^{(k)}$ admits the representation

$$A^{(k)} = A^{(k,0)} \widetilde{+} t_{kk}^{0} \left\langle \cdot, \omega_2^{k,0}\right\rangle \omega_2^{k,0}, \tag{34}$$

where $A^{(k,0)}$ is constructed by the inverse eigenvalues method from a sequence of pairs $\{\psi_j, 0\}_{j=1, j\neq k}^{N}$, and

$$\omega_2^{k,0} = (\mathbf{A}^{(k,0)} - NE_k)\psi_k,$$
$$t_{kk}^0 = -\frac{1}{\left\langle \psi_k, \omega_2^{k,0} \right\rangle}.$$

Here we again used the additive property of generalized sums. Clearly, each $A^{(k,0)}$ is positive,

$$A^{(k,0)} \geq 0, \tag{35}$$

and moreover it solves the problem

$$A^{(k,0)}\psi_j = 0, \quad j \neq k. \tag{36}$$

Proof of Theorem 4. Directly from (26), (29) and the above propositions it follows that $\widetilde{A}$ is self-adjoint and (22) is fulfilled. So we have to prove only that there absent additional negative eigen-solutions for the operator $\widetilde{A}$ defined by (26).

Assume $\widetilde{A}\psi = E\psi$, $E < 0$, for some vector ψ. Clearly, $\psi \notin \mathcal{N} := \mathrm{span}\{\psi_k\}_{k=1}^N$, and moreover ψ should be orthogonal to this subspace. However each operator $A^{(k)}$ is positive in the subspace $\mathcal{H} \ominus \mathcal{N}$ (see (35), (36)). Therefore $\psi = 0$, and $\widetilde{A}$ solves the problem (22) in the exact sense. □

We remark that the operator T defined by (27) satisfies

$$\mathrm{Rank}\, T \leq 2N.$$

Indeed by our constructions all elements ω_j^k in (27) belong to the subspace

$$\mathcal{N}_{-1} = \mathrm{span}\,\{\mathbf{A}\psi_k,\ (\mathbf{A} - NE_j)\psi_j\}_{k,j=1}^N.$$

In other words, for the range of the operator T, we have

$$\mathcal{R}(T) \subset \mathcal{N}_{-1} \subset \mathcal{H}_{-1}.$$

Evidently, $\dim \mathcal{N}_{-1} \leq 2N$. Therefore $\mathrm{Rank}\, T \leq 2N$ also.

Another explicit solution of the problem (3) with an operator T of rank $2N$ can be obtained as follows [29]. Assume for a moment that all the vectors $\psi_j \in \mathcal{D}(A)$. Then we can introduce in $\mathcal{H}$ the $N-$dimensional subspace $\mathcal{N} = \mathrm{span}\ \{\psi_j\}_{j=1}^N$ and the operator

$$T' = \sum_{i=1}^{N} E_i(\cdot, \psi_i)\psi_i.$$

Let $P = \sum_{i=1}^N (\cdot, \psi_i)\psi_i$ be the orthogonal projection onto $\mathcal{N}$ in $\mathcal{H}$. Then the operator

$$\widetilde{A} = (I-P)A(I-P) + T' = A + T,$$
$$T = PAP - PA - AP + T'$$

obviously solves the problem (3). In a general situation, where $\psi_j \in \mathcal{H}_1(A)$, the explicit solution of (3) can be presented in the form $\widetilde{A} = A \tilde{+} T$, where T has the form

$$T = T' + \sum_{i=1}^{N} \left[\left(\sum_{k=1}^{N} (\cdot, \psi_k) \langle \mathbf{A}\psi_k, \psi_i \rangle \psi_i \right) - \langle \cdot, \mathbf{A}\psi_i \rangle \psi_i - \langle \cdot, \psi_i \rangle \mathbf{A}\psi_i \right].$$

Finally we remark that among all other operators $\widetilde{A}$ that give a solution of the negative eigenvalue problem (3) our construction (see (26)–(29)) possesses the rather specific property. Namely, each operator $A^{(k)}$, $k = 1, ..., N$, solves the following negative eigenvalue problem: $A^{(k)}\psi_j = E'_j\ \psi_j$, where ψ_j are the same vectors as in (3), while $E'_k = NE_k$ and $E'_j = 0$, if $j \neq k$. We note that each $A^{(k)}$ is a rank N perturbation of A (see (23)–(25)) and by our construction ($A^{(k)} = \lim_{j \to k} A^k_j$) at any step the operator A^k_j (which is a rank j perturbation of A) also solves the negative eigenvalues problem. Moreover each operator of the form $A^{(k_1,...,k_i)} = A^{(k_1)} + ... + A^{(k_i)}$, $k_i \leq N$, also solves the negative eigenvalues problem, $A^{(k_1,...,k_i)}\psi_j = E'_j\ \psi_j$, where $E'_j = NE_j$, if at last one of the indices $k_1, ..., k_i$ coincides with j and $E'_j = 0$ otherwise.

5. A description of T

Let $A \geq 0$ be as above. Let $\widetilde{A} \in \mathcal{P}_{\mathrm{ad}}(A)$. This means that $\widetilde{A}$ can be represented by the generalized sum, $\widetilde{A} = A \tilde{+} T$, of A and a bounded self-adjoint operator T : $\mathcal{H}_1(A) \to \mathcal{H}_{-1}(A)$ (see Appendix). Assume that $\widetilde{A}$ solves the negative eigenvalues problem (3) with $E_j < 0$, $j = 1, ..., N$, in the exact sense, i.e., the set $\{\psi_j,\ E_j\}_{j=1}^N$ exhausts all eigen-solutions of equation (3) with negative eigenvalues. In this section we find a general description of the operators T.

Assume for a moment that the range

$$\mathcal{R}(T) \subset H_{-1}(A)$$

(see (9)) and that T, as an operator from $H_1(A)$ to $H_{-1}(A)$, is bounded,

$$|Q_T[f]| = |\langle Tf, f \rangle| \leq M \, \|f\|^2_{H_1}, \tag{37}$$

where $Q_T[f]$ stands for the quadratic form of T. Then by the Birman-Schwinger principle [14] the operator $\widetilde{A}_\alpha = A \tilde{+} \alpha T$ does not have any negative eigenvalues if $0 \leq |\alpha| \leq \frac{1}{M}$. Indeed, (37) implies

$$\alpha Q_T[f] \leq \langle \mathbf{A}f, f \rangle, \quad \text{if} \quad 0 \leq |\alpha| \leq \frac{1}{M},$$

since $\langle \mathbf{A}f, f \rangle = \|f\|^2_{H_1}$. Therefore in this case,

$$\langle \mathbf{A}f, f \rangle + \alpha \langle Tf, f \rangle \geq 0.$$

This means that the quadratic form of $\widetilde{A}_\alpha$ is positive and hence $\widetilde{A}_\alpha$ does not possess any negative eigenvalue.

Conversely, if $\widetilde{A} = A \tilde{+} T$ is positive, $\widetilde{A} \geq 0$, then

$$-Q_T[f] \leq \|f\|^2_{H_1} \equiv \langle \mathbf{A}f, f \rangle. \tag{38}$$

Therefore, in this case T may be considered as an operator (not necessarily bounded) acting from $H_1(A)$ to $H_{-1}(A)$ and which obeys the inequality

$$-Q_T[f] \leq \langle \mathbf{A}f, f \rangle, \quad f \in H_1(A) \cap \mathcal{D}(T). \tag{39}$$

Theorem 5. *Let an operator* $\widetilde{A'} = A\widetilde{+}T'$ *belong to the class* $\mathcal{P}_{\rm ad}(A)$ *(see Appendix). Then it solves the negative eigenvalues problem (3) with* $E_j < 0$, $j = 1, ..., N$, *in the exact sense if and only if* T' *admits the representation*

$$T' = T_N + T_{\rm ad}, \tag{40}$$

where T_N *has the form (18) and the quadratic form* $Q_{T_{\rm ad}}[f] = \langle T_{\rm ad}f, f\rangle$ *obeys the inequality*

$$-Q_{T_{\rm ad}}[f] \leq \langle \mathbf{A}f, f\rangle \tag{41}$$

and, besides,

$$\operatorname{Ker} T_{\rm ad} \supset \mathcal{N} := \operatorname{span}\{\psi_j\}_{j=1}^N . \tag{42}$$

Proof. Let $\widetilde{A'} = A\widetilde{+}T'$ solve the negative eigenvalues problem (3) with $E_j < 0$, $j = 1, ..., N$, in the exact sense. Consider the operator $A_N = A\widetilde{+}T_N$ constructed in Section 3 that also solves the same problem. Define

$$T_{\rm ad} = \widetilde{\mathbf{A}'} - \mathbf{A}_N : \mathcal{H}_1(A) \to \mathcal{H}_{-1}(A),$$

where $\widetilde{\mathbf{A}'} = \mathbf{A}+T'$ and $\mathbf{A}_N = \mathbf{A} + T_N$. So $T_{\rm ad} = T' - T_N$, and we can write

$$\widetilde{A'} = A\widetilde{+}T' = A\widetilde{+}(T_{\rm ad} + T_N).$$

We will show that $T_{\rm ad}$ satisfies (41) and (42). Indeed (42) is fulfilled since both $\widetilde{A'}$ and A_N solve the same negative eigenvalues problem and therefore $T_{\rm ad}$ equals zero on the subspace

$$\mathcal{N} = \operatorname{span}\{\psi_j\}_{j=1}^N .$$

Thus we have only to prove that the quadratic form of $Q_{T_{\rm ad}}[f]$ obeys the inequality (41) on the subspace $H_1^\perp(A)$ which is defined as the closure of $\mathcal{H}_1^\perp(A) = \mathcal{H}_1(A)\ominus \mathcal{N}$ in $H_1(A)$. By the additive property of generalized operator sums (see Appendix) we have

$$\widetilde{A'} = A\widetilde{+}(T_{\rm ad} + T_N) = \left(A\widetilde{+}T_N\right)\widetilde{+}T_{\rm ad} = A_N\widetilde{+}T_{\rm ad}. \tag{43}$$

Therefore

$$\widetilde{A'} = \widetilde{A'}^{\restriction} \oplus \sum_{j=1}^N E_j P_{\psi_j} = \left(A_N^{\restriction} \oplus \sum_{j=1}^N E_j P_{\psi_j}\right)\widetilde{+}T_{\rm ad} = \left(A_N^{\restriction}\widetilde{+}T_{\rm ad}^{\restriction}\right) \oplus \sum_{j=1}^N E_j P_{\psi_j},$$

where $\restriction$ stands for the restriction onto subspace $\mathcal{H}_1^\perp(A)$ and where we used that $T_{\rm ad} = T_{\rm ad}^{\restriction} \oplus 0$. We note now that both $\widetilde{A'}^{\restriction}$ and $A_N^{\restriction}$ are positive, since $\widetilde{A'}$ and A_N solve the problem (3) in the exact sense. Thus the operator $\widetilde{A'}^{\restriction}$ can be viewed as a perturbation of the positive operator $A_N^{\restriction}$ by $T_{\rm ad}^{\restriction}$. By the arguments before this theorem,

$$-\left\langle T_1^{\restriction} f, f\right\rangle \leq \left\langle \mathbf{A}_N^{\restriction} f, f\right\rangle .$$

Hence we can conclude that

$$-Q_{T_{\rm ad}\upharpoonright}[f] \leq \left\langle \mathbf{A}^{\upharpoonright} f, f \right\rangle, \quad f \in H_1^{\perp}(A) \cap \mathcal{D}(T_1^{\upharpoonright}),$$

since, by the constructions of Section 3, it is obvious that $A_N \leq A$. Therefore the quadratic form of the operator $T_{\rm ad}$ obeys the inequality (41).

Conversely, let T' admit the representation (40), $T' = T_N + T_{\rm ad}$, where T_N is constructed in Section 3 and $T_{\rm ad}$ obeys inequality (41) and satisfies (42). Then $\widetilde{A}_{\rm ad} := A\widetilde{+}T_{\rm ad}$ is positive and coincides with A on $\mathcal{N} \subset {\rm Ker}\, T_{\rm ad}$. Now we note that if we will construct the operator T_N (see Section 3) starting from $\widetilde{A}_{\rm ad}$, instead of A, we get the same operator since this construction involves only the values of A on the subspace $\mathcal{N}$ where both A and $\widetilde{A}_{\rm ad}$ coincide. Thus we can write

$$T_N(\widetilde{A}_{\rm ad}) = T_N(A) = T_N.$$

Therefore by the additive property of the generalized sum,

$$\widetilde{A'} = A\widetilde{+}T' = (A\widetilde{+}T_{\rm ad})\widetilde{+}T_N.$$

Now we remark that by the construction of Section 3 the operator $(A\widetilde{+}T_{\rm ad})\widetilde{+}T_N$ solves the starting negative eigenvalues problem. □

6. Appendix. Generalized operator sum. Additive representation

Here we introduce two families of operators $\widetilde{A}$, which we denote by $\mathcal{P}_{\rm ws}(A)$ and $\mathcal{P}_{\rm ad}(A)$, and discuss the question about the additive representation of $\widetilde{A}$ in the form of a generalized sum, $\widetilde{A} = A\widetilde{+}T$, with $T : \mathcal{H}_1 \to \mathcal{H}_{-1}$.

Given $A \geq 0$ in $\mathcal{H}$ (with obvious changes, one can assume that A is bounded from below) introduce an A-scale of the Hilbert spaces,

$$\mathcal{H}_{-s} \supset \mathcal{H}_0 \equiv \mathcal{H} \supset \mathcal{H}_s, \quad s > 0, \tag{44}$$

where $\mathcal{H}_s \equiv \mathcal{H}_s(A) = \mathcal{D}(A^{s/2})$ in the norm $\|\varphi\|_s := \|(A+I)^{s/2}\varphi\|$, and I stands for the identity, and $\mathcal{H}_{-s} \equiv \mathcal{H}_{-s}(A)$ is the dual space with respect to $\mathcal{H}_s$ ($\mathcal{H}_{-s}$ is the completion of $\mathcal{H}$ in the norm $\|f\|_{-s} := \|(A+I)^{-s/2}f\|$). Obviously, $D = A+I$ is unitary as a map from $\mathcal{H}_2$ to $\mathcal{H}$, and moreover both maps $D : \mathcal{H}_1 \to \mathcal{H}_{-1}$ and $D : \mathcal{H} \to \mathcal{H}_{-2}$ are isometric. So the operator

$$\mathbf{D} := (A+I)^{\rm cl} \equiv \mathbf{A} + I : \mathcal{H} \to \mathcal{H}_{-2}$$

is unitary, where cl stands for the closure. The following relations hold (for details see [10], [13]):

$$\begin{gathered}\langle f, \varphi\rangle = (f, \varphi), \quad f \in \mathcal{H}, \quad \varphi \in \mathcal{H}_s, \quad s \geq 0, \\ (\omega, \mathbf{D}\varphi)_{-1} = \langle \omega, \varphi\rangle = (\mathbf{D}^{-1}\omega, \varphi)_1, \quad \omega \in \mathcal{H}_{-1}, \quad \varphi \in \mathcal{H}_1, \\ (\omega, \mathbf{D}f)_{-2} = (\mathbf{D}^{-1}\omega, f), \quad \omega \in \mathcal{H}_{-2}, \quad f \in \mathcal{H},\end{gathered} \tag{45}$$

where $(\cdot,\cdot)_s$ denotes the inner product in $\mathcal{H}_s$ and $\langle \cdot,\cdot\rangle$ stands for the dual inner product (the pairing) in the $A-$scale.

By definition (see [2,19,20,21,25]), a self-adjoint operator $\widetilde{A}\neq A$ is said to be a *(pure) singular perturbation* of A if $\widetilde{A}$ coincides with A on some linear subset $\mathcal{D}$ dense in $\mathcal{H}$. The class of all such operators is denoted by $\mathcal{P}_{\rm s}(A)$. Thus $\widetilde{A}\neq A$ belongs to the class $\mathcal{P}_{\rm s}(A)$ iff the set

$$\mathcal{D} := \{f \in \mathcal{D}\ (\widetilde{A}) \cap \mathcal{D}\ (A) : Af = \widetilde{A}f\} \text{ is dense in } \mathcal{H}. \tag{46}$$

We say that $\widetilde{A}\neq A$ belongs to the class of *weak (pure) singular perturbations* of A, and write

$$\widetilde{A} \in \mathcal{P}_{\rm ws}(A), \tag{47}$$

if besides (46) the following condition holds:

$$\mathcal{D}(\widetilde{A}) \subset \mathcal{H}_1. \tag{48}$$

Otherwise we write

$$\widetilde{A} \in \mathcal{P}_{\rm ss}(A) \tag{49}$$

for a *strong (pure) singular perturbations* of A. We say that $\widetilde{A}$ is *a rank N weak* (not necessarily pure) *singular perturbation* of A if $\mathcal{D}(\widetilde{A}) \subset \mathcal{H}_1$ and the resolvent difference $R_z(\widetilde{A}) - R_z(A)$ for one $z \in \rho(\widetilde{A}) \cap \rho(A)$, and therefore for all such z, is a rank N operator.

An important fact is that each operator $\widetilde{A} \in \mathcal{P}_{\rm ws}(A)$, ($\widetilde{A} \neq A_\infty$, where A_∞ denotes the Friedrichs extension of the symmetric operator $A\restriction\mathcal{D}$) admits an additive representation in the form of a generalized sum, $\widetilde{A} = A\tilde{+}T$, with T acting in the A-scale, $T : \mathcal{H}_1\to\mathcal{H}_{-1}$ (see Theorem 3 in [26], Theorem 7 in [27]). Before formulating this result, we need in some preparations.

Let $0 \neq T$ be a closed symmetric operator acting in the $A-$scale, from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$. Note that the adjoint operator T^* is defined with respect to the dual inner product $\langle\cdot,\cdot\rangle$ between $\mathcal{H}_1$ and $\mathcal{H}_{-1}$, so T^* acts also from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$. Thus

$$\langle T\varphi, \psi\rangle = \langle\varphi, T\psi\rangle, \quad \varphi,\psi \in \mathcal{D}(T) \subset \mathcal{D}(T^*) \subset \mathcal{H}_{-1}.$$

We say that an operator $T : \mathcal{H}_1 \to \mathcal{H}_{-1}$ is $\mathcal{H}-$*singular* if the range $\mathcal{R}(T)$ contains elements which do not belong to $\mathcal{H}$. An operator T is said to be *pure singular* with respect to $\mathcal{H}$ or shortly, *pure $\mathcal{H}-$singular* (see [25,21,27,6]), if its range essentially belongs to the space $\mathcal{H}_{-1}$, i.e., if

$$\mathcal{R}(T)^{{\rm cl},-1} \cap \mathcal{H} = \{0\},$$

where cl,-1 stands for the closure in $\mathcal{H}_{-1}$. It is known (see [26] Theorem A) that T is pure $\mathcal{H}-$singular if the set

$$\operatorname{Ker} T \text{ is dense in } \mathcal{H}. \tag{50}$$

Since T is a closed operator, the set $\mathcal{M}_1 = \operatorname{Ker} T$ is a closed subspace in $\mathcal{H}_1$. We write

$$T \in \mathcal{H}_{-1} - \text{class}, \tag{51}$$

if the set

$$\operatorname{Ker} T \cap \mathcal{D}(A) \text{ is dense in } \mathcal{M}_1. \tag{52}$$

Obviously the set Ker $T \cap \mathcal{D}(A)$ is a proper closed subspace in $\mathcal{H}_2$. We denote it by $\mathcal{M}_2$. Now (50) and (52) can be rewritten as

$$\mathcal{M}_2^{\mathrm{cl},1} = \mathcal{M}_1, \quad \mathcal{M}_1^{\mathrm{cl},0} = \mathcal{H}_2, \tag{53}$$

where cl, 0 (cl,1) stands for the closure in $\mathcal{H}$, (resp. in $\mathcal{H}_1$). Of course from (53) it also follows that $\mathcal{M}_2^{\mathrm{cl},0} = \mathcal{H}_2$.

We will interpret the operators T as objects carrying singular perturbations of A (see [2,21,6,25]). For the construction of the perturbed operator $\widetilde{A}$ we use the generalized operator sum approach which extends the well-known form-sum method. Let us recall this construction (for more details see [10],[12], [11,25,23,18]).

Given a symmetric operator $T : \mathcal{H}_1 \to \mathcal{H}_{-1}$ let us define the *generalized operator sum,* $\widetilde{A} = A\tilde{+}T$, as the "restriction" of the operator sum $\mathbf{A} + T : \mathcal{H}_1 \to \mathcal{H}_{-1}$ onto $\mathcal{H}$, where recall, $\mathbf{A}$ stands for the closure of A as an operator from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$.

Precisely,

$$\mathcal{D}(\widetilde{A}) = \{\varphi \in \mathcal{H}_1 \cap \mathcal{D}(T) : \mathbf{A}\varphi + T\varphi \in \mathcal{H}\}, \quad \widetilde{A}\varphi = \mathbf{A}\varphi + T\varphi. \tag{54}$$

We note that separately each component $\mathbf{A}\varphi$ and $T\varphi$ in general belong to $\mathcal{H}_{-1}$. It is easily seen that $\widetilde{A}$ always is a Hermitian (symmetric) operator, i.e., $(\widetilde{A}\varphi, \psi) = (\varphi, \widetilde{A}\psi)$, $\varphi, \psi \in \mathcal{D}(\widetilde{A})$, but in general, it is not necessarily densely defined in $\mathcal{H}$. Surely if $\mathcal{D}(T) \subseteq \mathcal{D}(A)$ and $\mathcal{R}(T) \subseteq \mathcal{H}$ then $A\tilde{+}T$ coincides with the usual operator sum $A+T$. It has been shown [10,11,12] (see also [27,18]) that the sum in the sense of quadratic forms is a particular case of the generalized operator sum.

We remark that if for an operator T from the $\mathcal{H}_{-1}-$class, the generalized sum $\widetilde{A} = A\tilde{+}T$ is a self-adjoint operator in $\mathcal{H}$, then necessarily $\widetilde{A} \in \mathcal{P}_{\mathrm{ws}}(A)$, since obviously the set $\mathcal{D}$ defined by (46) is dense in $\mathcal{H}$ (see (50)–(53)), and $\mathcal{D}(\widetilde{A}) \subset \mathcal{H}_1$ due to (54). The inverse implication is true under natural conditions.

We have the following result.

Theorem 6. [26,27]. *Each operator* $\widetilde{A} \in \mathcal{P}_{\mathrm{ws}}(A)$, $\widetilde{A} \neq A_\infty$, *under the condition that both pairs,* A, A_∞ *and* $\widetilde{A}$, A_∞ , *are relatively prime (see [1]) with respect to* $\overset{\circ}{A} := A \restriction \mathcal{D} = \widetilde{A} \restriction \mathcal{D}$, *admits a representation in the form of a generalized sum,* $\widetilde{A} = A\tilde{+}T$ *with a uniquely defined (bounded) self-adjoint operator* $T : \mathcal{H}_1 \to \mathcal{H}_{-1}$ *which belongs to the* $\mathcal{H}_{-1}-$*class.*

We note that the above definition of a generalized operator sum could be extended to a case where A is replaced by another self-adjoint operator C in $\mathcal{H}$ such that $\mathcal{H}_1(C)$ differs from $\mathcal{H}_1(A) \equiv \mathcal{H}_1$. Indeed, assume that the domain $\mathcal{D}(C) \subseteq \mathcal{H}_1$ and that C is closable as a map from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$. Let $\mathbf{C} := C^{\mathrm{cl}} : \mathcal{H}_1 \to \mathcal{H}_{-1}$ denote this closure. Then similarly to the above case we define $\widetilde{C} = C\tilde{+}T$ as the "restriction" of the operator sum $\mathbf{C} + T : \mathcal{H}_1 \to \mathcal{H}_{-1}$ onto $\mathcal{H}$. Thus $\mathcal{D}(\widetilde{C}) = \{\varphi \in \mathcal{D}(\mathbf{C}) \cap \mathcal{D}(T) : \mathbf{C}\varphi + T\varphi \in \mathcal{H}\}$, $\widetilde{C}\varphi = \mathbf{C}\varphi + T\varphi$.

In our constructions we use the *additive property* of generalized sums,

$$A\tilde{+}(T_1 + T_2) = (A\tilde{+}T_1)\tilde{+}T_2 = (A\tilde{+}T_2)\tilde{+}T_1, \tag{55}$$

which is valued for bounded operators T_1, T_2, acting from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$ and such that both domains $\mathcal{D}(A\tilde{+}T_1)$, $\mathcal{D}(A\tilde{+}T_2)$ are dense in $\mathcal{H}_1$. Indeed, (55) is true since in this a case the closure of $A\tilde{+}T_1$ ($A\tilde{+}T_2$), as an operator from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$, coincides with $\mathbf{A}+T_1$ (resp., with $\mathbf{A}+T_2$), and therefore using the extended definition (54) one can construct the generalized sum $(A\tilde{+}T_1)\tilde{+}T_2$ (resp., $(A\tilde{+}T_2)\tilde{+}T_1$) in the case where A is replaced by C equals to $A\tilde{+}T_1$ (resp., $A\tilde{+}T_2$).

The next problem which can be posed is to find conditions on a symmetric operator T such that $A\tilde{+}T$ is essentially self-adjoint. In [4,6,18,21,26], sufficient conditions of self-adjointness for $A\tilde{+}T$ have been obtained. In this paper we use the following result.

Theorem 7. [4,21]. *Let $T \in \mathcal{H}_{-1}$ − class and one of the conditions is fulfilled: (a) the operator $\mathcal{T} = (\mathbf{A}+I)^{-1}T$ in $\mathcal{H}_1$ has a pure point spectrum, (b) the domain $\mathcal{D}(A) \subset \mathcal{R}(\mathcal{T}+I)$. Then the generalized sum $\widetilde{A} = A\tilde{+}T$ is essentially self-adjoint.*

In particular, $\widetilde{A} = A\tilde{+}T$ is self-adjoint under the condition that $\mathcal{T} = (\mathbf{A}+I)^{-1}T$ is compact in $\mathcal{H}_1$.

Finally we will discuss the question about the additive representation for operators $\widetilde{A}$ which is bounded from below but does not necessarily belong to the family $\mathcal{P}_{\rm ws}(A)$.

Let $A \geq 0$ be as above and $\widetilde{A} = \widetilde{A}^*$ be a self-adjoint operator bounded from below. We say that $\widetilde{A}$ belongs to the class of additive perturbations of A, and write

$$\widetilde{A} \in \mathcal{P}_{\rm ad}(A), \tag{56}$$

if $\widetilde{A}$ admits a representation as a generalized operator sum, $\widetilde{A} = A\tilde{+}T$, with some bounded symmetric operator $T : \mathcal{H}_1 \to \mathcal{H}_{-1}$, which does not necessarily belong to the $\mathcal{H}_{-1}$-class. We want to answer the following question. Under what conditions $\widetilde{A}$ is an additive perturbation of A, i.e., when $\widetilde{A} \in \mathcal{P}_{\rm ad}(A)$?

Let $\left\{\widetilde{\mathcal{H}}_k\right\}_{k\in\mathbf{R}^1}$ be an $\widetilde{A}$−scale of Hilbert spaces, where $\widetilde{\mathcal{H}}_k \equiv \mathcal{H}_k(\widetilde{A})$ is the completion of $\mathcal{D}(\widetilde{A})$ in the norm $\|f\|_{\widetilde{\mathcal{H}}_k} := \|(\widetilde{A}+\widetilde{m})^{k/2}f\|^{1/2}$, with $\widetilde{m} \geq 1$ such that

$$\|f\|^2_{\widetilde{\mathcal{H}}_1} \equiv (\widetilde{A}f, f) + \widetilde{m}\,\|f\|^2 \geq \|f\|^2 . \tag{57}$$

Thus we have the rigged Hilbert space for each fixed $s > 0$,

$$\widetilde{\mathcal{H}}_{-s} \supset \mathcal{H}_0 \equiv \mathcal{H} \supset \widetilde{\mathcal{H}}_s,$$

where the space $\widetilde{\mathcal{H}}_{-s} \equiv \mathcal{H}_{-s}(\widetilde{A})$ coincides with the dual space to $\widetilde{\mathcal{H}}_s$. Let $\widetilde{\mathbf{D}} : \widetilde{\mathcal{H}}_1 \to \widetilde{\mathcal{H}}_{-1}$ denote the canonical unitary isomorphism [10], [13]. By the construction, $\widetilde{\mathbf{D}} \restriction \widetilde{\mathcal{H}}_2$ coincides with $\widetilde{A} + \widetilde{m}I$ (for details see [10]),

$$\widetilde{A}f = (\widetilde{\mathbf{D}} - \widetilde{m}I)f, \quad \mathcal{D}(\widetilde{A}) = \left\{f \in \widetilde{\mathcal{H}}_1 : \widetilde{\mathbf{D}}f \in \mathcal{H}\right\}. \tag{58}$$

So the operator $\widetilde{A}$ is uniquely associated with the rigged Hilbert space $\widetilde{\mathcal{H}}_{-1} \supset \mathcal{H} \supset \widetilde{\mathcal{H}}_1$.

We have a criterion for $\widetilde{A} \in \mathcal{P}_{\rm ad}(A)$.

Theorem 8. *Let $\widetilde{A}$ be a self-adjoint operator in $\mathcal{H}$ bounded from below, $\widetilde{A} \geq \widetilde{m} \geq 1$. Assume*

$$\mathcal{H}_1(\widetilde{A}) \supseteq \mathcal{H}_1(A) \tag{59}$$

in the sense of the dense continuous embedding. Then there exists a bounded self-adjoint operator $T : \mathcal{H}_1 \to \mathcal{H}_{-1}$ which satisfies the inequality

$$-\langle Tf, f\rangle \leq \langle \mathbf{A}f, f\rangle + \lambda \|f\|^2, \quad \lambda = \widetilde{m} - 1 \geq 0, \quad f \in \mathcal{H}_1, \tag{60}$$

and such that $\widetilde{A}$ admits an additive representation in the form of a generalized sum, $\widetilde{A} = A\widetilde{+}T$, i.e., $\widetilde{A} \in \mathcal{P}_{\mathrm{ad}}(A)$. Conversely, if $\widetilde{A} \in \mathcal{P}_{\mathrm{ad}}(A)$, $\widetilde{A} = A\widetilde{+}T$, where $T : \mathcal{H}_1 \to \mathcal{H}_{-1}$, is a bounded self-adjoint operator which satisfies the inequality (60), then (59) is fulfilled.

Proof. Let $q_{\widetilde{A}}[f]$ denote the closure of the quadratic form $\left\langle \widetilde{\mathbf{A}}f, f\right\rangle$, $f \in \mathcal{D}(\widetilde{A})$ in $\mathcal{H}$. Clearly the domain $\mathcal{D}(q_{\widetilde{A}})$ coincides with $\mathcal{H}_1(\widetilde{A}) \equiv \widetilde{\mathcal{H}}_1$. Therefore due to (59) the form $q_{\widetilde{A}}$ is defined on $\mathcal{H}_1$. Moreover $q_{\widetilde{A}}$ is continuous on $\mathcal{H}_1$. Indeed if $f_n \to 0$ in $\mathcal{H}_1$, then by (59) $f_n \to 0$ in $\mathcal{H}_1(\widetilde{A})$, and in $\mathcal{H}$ too. Thus due to (57), $q_{\widetilde{A}}[f_n] \to 0$. By this reason the difference

$$q[f] = q_{\widetilde{A}}[f] - q_A[f]$$

is continuous on $\mathcal{H}_1(A)$, where we denote $q_A[f] = \langle \mathbf{A}f, f\rangle$.

Let us prove now the representation $= A\widetilde{+}T$. Consider the operator $\widetilde{\mathbf{D}} : \widetilde{\mathcal{H}}_1 \to \widetilde{\mathcal{H}}_{-1}$.Obviously it coincides with $\widetilde{\mathbf{A}} - \widetilde{m}I$ where $\widetilde{\mathbf{A}}$ denotes the closure of $\widetilde{A}$ as a map from $\widetilde{\mathcal{H}}_1$ to $\widetilde{\mathcal{H}}_{-1}$. Further due to

$$q_{\widetilde{A}}[f] = q_A[f] + q[f] = \left\langle \widetilde{\mathbf{A}}f, f\right\rangle = \langle \mathbf{A}f, f\rangle + \langle Tf, f\rangle, \quad f \in \mathcal{H}_1(A), \tag{61}$$

the restriction of the operator $\widetilde{\mathbf{D}} - \widetilde{m}I$ onto $\mathcal{H}_1(A)$ coincides with $\mathbf{A}+T$. Therefore the operator, self-adjoint in $\mathcal{H}$, associated with the rigged Hilbert space $\widetilde{\mathcal{H}}_1 \subset \mathcal{H} \subset \widetilde{\mathcal{H}}_{-1}$ and constructed by the standard procedure,

$$\widetilde{A} = (\widetilde{\mathbf{D}} - \widetilde{m}I) \restriction \mathcal{D}(\widetilde{A}), \quad \mathcal{D}(\widetilde{A}) = \{f \in \widetilde{\mathcal{H}}_1 : (\widetilde{\mathbf{D}} - \widetilde{m}I)f \in \mathcal{H}\},$$

coincides with $A\widetilde{+}T$. Thus $\widetilde{A} \in \mathcal{P}_{\mathrm{ad}}(A)$.

Moreover, by (57),

$$\|f\|^2_{\widetilde{\mathcal{H}}_1} - \|f\|^2 = q_{\widetilde{A}}[f] + (\widetilde{m} - 1)\|f\|^2 \geq 0,$$

and hence due to (61),

$$q_{\widetilde{A}}[f] - q[f] + \lambda \|f\|^2 = q_A[f] + \lambda \|f\|^2 \geq -q[f]$$

with $\lambda = \widetilde{m} - 1$. So we get the inequality (60).

Conversely, let $\widetilde{A} = A\widetilde{+}T$ be a self-adjoint operator in $\mathcal{H}$, where $T : \mathcal{H}_1 \to \mathcal{H}_{-1}$, satisfies (60) with some $\lambda \geq 0$. Then $\widetilde{A}$ is bounded from below, $\widetilde{A} \geq \widetilde{m} \geq 1$, $\widetilde{m} = \lambda + 1$. It follows from the fact that due to (61), $q_{\widetilde{A}}[f] + \widetilde{m}\ \|f\|^2 \geq \|f\|^2$, where $q_{\widetilde{A}}[f] := \langle \mathbf{A}f, f\rangle + \langle Tf, f\rangle$. Therefore $\widetilde{\mathcal{H}}_1$ is the completion of $\mathcal{H}_1$ in the norm $\|f\|^2_{\widetilde{\mathcal{H}}_1} = \widetilde{q}[f] + \widetilde{m}\,\|f\|^2$ and thus the space $\mathcal{H}_1$ is densely and continuously embedded into $\widetilde{\mathcal{H}}_1$. □

Corollary 1. *Let $A \geq 0$ and $\widetilde{A} \geq \widetilde{m}$ denote self-adjoint operators associated with the triplets $\mathcal{H}_{-1} \supset \mathcal{H} \supset \mathcal{H}_1$ and $\widetilde{\mathcal{H}}_{-1} \supset \mathcal{H} \supset \widetilde{\mathcal{H}}_1$ resp. Then $\widetilde{A} \in \mathcal{P}_{\rm ad}(A)$, i.e., $\widetilde{A} = A \widetilde{+} T$, with a bounded self-adjoint operator $T : \mathcal{H}_1 \to \mathcal{H}_{-1}$, which satisfies the inequality (60) if and only if the condition*

$$\mathcal{H}_{-1} \supseteq \widetilde{\mathcal{H}}_{-1} \supset \mathcal{H} \supset \widetilde{\mathcal{H}}_1 \supseteq \mathcal{H}_1,$$

is fulfilled in the sense of the dense continuous embedding.

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