The conflict interaction between two complex systems: cyclic migration

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Abstract

We construct and study a discrete time model describing the conflict interaction between two complex systems with non-trivial internal structures. The external conflict interaction is based on the model of alternative interaction between a pair of non-annihilating opponents. The internal conflict dynamics is similar to the one of a predator-prey model. We show that the typical trajectory of the complex system converges to an asymptotic attractive cycle. We propose an interpretation of our model in terms of migration processes.

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1.   Introduction

Since the beginning of 20th century the Lotka-Volterra model of prey-predator interaction is one of the main models for simulation of many processes in population theory and economics. As a rule, continuous models where Lotka-Volterra equations have ratio-depended parameters are studied (see, for example [3, 5, 6, 11, 12, 13, 16, 18, 20]). Logistical and Ricker's models are also studied in some works, for example [5]. In the majority of works the prey-predator interaction is treated only inside of single region, and no migration from one region to another is considered.

In some works [6, 5] models with migration are studied, with a process of migration.

There are also only few works (see [5] and references wherein), in which discrete models are considered, though in reality such processes are more natural, since they take better into account seasonal phenomena (reproduction, migration, etc.).

The main aim of the majority of works is determination of stable points, bifurcation points, asymptotic behavior, and analysis of model's depending on the coefficients of the equations.

In [3] synchronization of population dynamics with natural phenomena (like change of seasons and floods) is studied. In the work [20] dependence of population dynamics on population density and migration is studied. In the work [6] migration is not assumed to be random, but aims at maximization of some function of the population. At last, in [3] the influence of stochastic terms in a Lotka-Volterra model is described, and interesting figures are presented.

In recent works [17, 19] Salam and Takahashi study conflict models, similar to ours. They introduce important and more complex multi-opponent systems. In [17] not only conflict, but cooperation between opponents is studied. The figures, obtained by them, are very similar to Figure 3 of the present work.

In this work we construct a model that joins two most rarely studied variants of Lotka-Volterra model, i.e., a discrete model with migration. Here individuals migrate not randomly, but according to strategies, discussed in section 5.

We construct the model of the conflict interaction between a pair of complex systems A and B. The system is a finite set of positive numbers: $\boldsymbol{P} = (P_1, \ldots, P_N)$ for A and $\boldsymbol{R} = (R_1, \ldots, R_N)$ for B, where N means the quantity of parameters that characterize the system. We study dynamics

in the discrete time. So, the evolution of every system is described by the sequence of vectors with non-negative coordinates $\mathbf{P}^n = (P_1^{(n)}, \ldots, P_N^{(n)})$ for A, and $\boldsymbol{R}^n = (R_1^{(n)}, \ldots, R_N^{(n)})$ for B, $n = 1, 2, \ldots$. The vectors $\boldsymbol{P}$ and $\boldsymbol{R}$ correspond to the moment $n = 0$. Naturally, each system tries to reach the optimal values of its coordinates. In reality, due to the conflict interaction, every coordinate changes in a complicated way. The evolution of all changes is determined by double dependence: by the conflict interaction between systems (which we shortly describe in section 3), and by the mutual "fight" of coordinates (of the prey-predator type interaction) inside every system. We suppose that every system is complex in the sense that its elements may be treated as one of the types: dominant (predators, employers) or dependent (preys, workers). So, every coordinate $P_i^{(n)}$, $R_i^{(n)}$ may be regarded as the quantity (population) of dominant, respectively dependent species at the position i at time n.

The law of evolution inside of each (independent) system is described in section 2. We suppose this law is identical in every system and is based on the well-known discrete Lotka-Volterra equation.

In section 3 we shortly call the main results on conflict interaction between non-annihilating opponents.

In section 4, that includes the main results of the work, we construct a dynamical system describing simultaneous conflict interaction both inside every system and between the systems. The outer interaction is an alternative conflict between non-annihilating systems, whereas the inner one is a prey-predator model of Lotka-Volterra type.

We may join these two types of interactions in a discrete time. Thus, our dynamical system consists of a discrete sequence of states. Two operations happen at any fixed moment of time: redistribution of probabilities to occupy some controversial positions by opponent systems, and quantitative changes (namely population) of all species inside both systems.

The computer modelling of such a complex interaction shows some very interesting phenomena. In this work we limit however ourselves to present only one observation. Namely, under an appropriate choice of parameters and initial data the complex system oscillates. We find a rather wide range for initial data for which the population trajectory in phase-space becomes cyclic. Moreover, we observe the stability of the limit cycle, so it is an attractor.

2.   Traditional models of population dynamics

Malthus proposed in 1798 the population equation

$$\frac{dP}{dt} = (b-d)P \, , \qquad\qquad (2.1)$$

where P is the cumulative number of individuals (species), and b, d are the natural birth and death rates. In reality, one expects exponential solution

$$P(t) = P(0)e^{(b-d)t}$$

describes the ideal population of biological species. The exponential rise, if $b-d > 0$, or decrease, if $b-d < 0$ at most in a local period of time.

Verhulst introduced in 1838 a more realistic equation with saturation terms:

$$\frac{dP}{dt} = (b-d)P - cP^2 \, , \qquad\qquad (2.2)$$

where the coefficient $c > 0$ represents the competition activity of individuals for living resources. The square power corresponds a conception of an alternative law of access to the living resource.
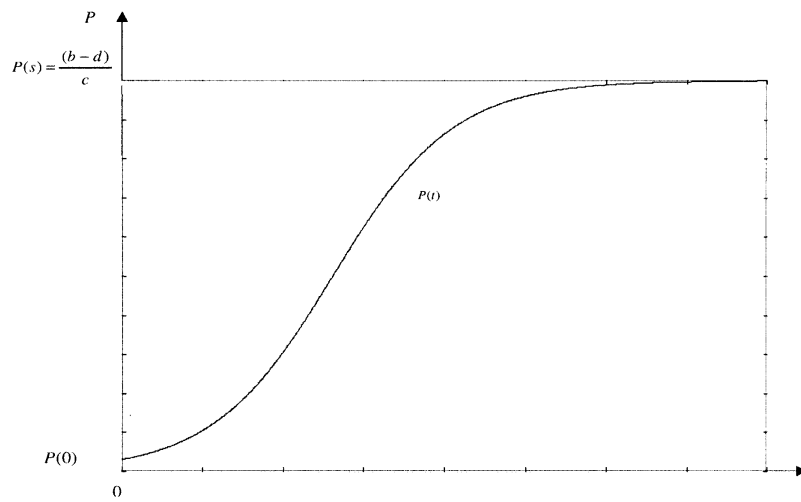
Figure 1

A typical shape of the logistical curve $\frac{dP}{dt} = (b-d)P - cP^2$

The solution of (2.2) describes the 5-shaped logistical curve (see Figure 1) and corresponds better to the actual behavior of many population processes.

The curve starts with a small value $P(0)$, exponentially increases, and then saturate at the capacity $P(s) = \frac{b-d}{c}$, $b - d > 0$.

In the economic context, equation (2.2) can be written as follows

$$\frac{dM}{dt} = (g - l)M - fM^2, \tag{2.3}$$

where M is the capital (money), g and l are the average gain and loss percentages on the capital, and f stands for the coefficient of confrontation between individuals. If $g - l < 0$, the capital decays to 0 exponentially; if $g - l > 0$, at the beginning the capital increases exponentially quick, but then the growth slows down, so that it never reaches asymptotic value of saturation $M(s) = \frac{g-l}{f}$.

Lotka (1907) and Volterra (1901) extended the Verhulst logistical equation to the Lotka-Volterra equations intended for the description of amount changes in populations of two species in interaction. These equations are also known under the name of predator-prey model. We will refer on Lotka-Volterra equations in the following form:

$$\frac{dP}{dt} = aP - bPR - cP^2,$$

$$\frac{dR}{dt} = -dR + ePR - fR^2, \tag{2.4}$$

where all coefficients are nonnegative.

The population of prey is described by the first equation. Without presence of any predators it grows exponentially at the beginning and then comes to the fixed capacity $P(s) = a/c$. The predators, without any prey to feed on, die out. When both species are present, the growth of the prey is limited by the predators, due to the term $-bPR$, and the predators grow if the amount of prey available, i.e. if ePR is large enough.

There are many publications devoted to the analysis of Lotka-Volterra equations (2.4) (see for example [16] and references wherein).

The models with discrete time are also studied. In this case, equations (2.4) have the following view:

$$P_1^{(n)} = P_1^{(n-1)} + P_1^{(n-1)}(a - bP_2^{(n-1)} - cP_1^{(n-1)}),$$

$$P_2^{(n)} = P_2^{(n-1)} + P_2^{(n-1)}(-d + eP_1^{(n-1)} - fP_2^{(n-1)}). \tag{2.5}$$

Typical behaviour of discrete Lotka-Volterra model is shown in Figure 2.

3.   Conflict interaction between non-annihilating opponents

In this section we shortly remind an alternative approach to describe the redistribution of conflicting positions between two opponents, say A and B, concerning an area of common interests.

We consider the simplest case where the existence space of common interests is a finite set of positions $\Omega = \{\omega_1, \ldots, \omega_N\}$, $N \geq 2$. Each of the opponents A and B tries to occupy a position ω_i, $i = 1, \ldots, N$ with a probability $P_A(\omega_i) = p_i \geq 0$ or $P_B(\omega_i) = r_i \geq 0$. The starting distributions of A and B along Ω are arbitrary and normed: $\sum_{i=1}^N p_i = 1 = \sum_{i=1}^N r_i$. A and B can not be present simultaneously in a same position ω_i. The interaction between A and B is considered in discrete time $t \in \mathbb{N}_0$. We introduce the noncommutative conflict composition between stochastic vectors $\boldsymbol{p}^0 = (p_1, \ldots, p_N)$, $\boldsymbol{r}^0 = (r_1, \ldots r_N) \in \mathbb{R}_+^N$:

$$\boldsymbol{p}^1 := \boldsymbol{p}^0 * \boldsymbol{r}^0, \ \ \boldsymbol{r}^1 = \boldsymbol{r}^0 * \boldsymbol{p}^0, \ \ \boldsymbol{p}^0 \equiv \boldsymbol{p}, \ \ \boldsymbol{r}^0 \equiv \boldsymbol{r},$$

where the coordinates of $\boldsymbol{p}^1, \boldsymbol{r}^1$ are defined as follows

$$p_i^{(1)} = \frac{p_i^{(0)}(1 - \alpha r_i^{(0)})}{1 - \alpha \sum_{i=1}^N p_i^{(0)} r_i^{(0)}}, \quad r_i^{(1)} = \frac{r_i^{(0)}(1 - \alpha p_i^{(0)})}{1 - \alpha \sum_{i=1}^N p_i^{(0)} r_i^{(0)}}, \tag{3.1}$$

where the coefficient $-1 \leq \alpha \leq 1$, $\alpha \neq 0$ stands for the activity interaction. At the nth step of the conflict dynamics we get two vectors

$$\boldsymbol{p}^n = \boldsymbol{p}^{n-1} * \boldsymbol{r}^{n-1} \equiv \boldsymbol{p}^0 *^n \boldsymbol{r}^0, \quad \boldsymbol{r}^n = \boldsymbol{r}^{n-1} * \boldsymbol{p}^{n-1} \equiv \boldsymbol{r}^0 *^n \boldsymbol{p}^0$$

with coordinates

$$p_i^{(n)} = \frac{p_i^{(n-1)}(1 - \alpha r_i^{(n-1)})}{z_n}, \quad r_i^{(n)} = \frac{r_i^{(n-1)}(1 - \alpha p_i^{(n-1)})}{z_n}$$

with z_n a normalization coefficient given by

$$z_n = 1 - \alpha(\boldsymbol{p}^{n-1}, \boldsymbol{r}^{n-1}),$$

with $(\cdot, \cdot)$ the inner product in $\mathbb{R}^N$.

The behavior of the state $\{\boldsymbol{p}^n, \boldsymbol{r}^n\}$ at time $t = n$ for $n \to \infty$ has been investigated in [1, 4, 7, 8, 9, 10]. We shortly describe the results.

Theorem 1. *For any pair of non-orthogonal stochastic vectors* $\boldsymbol{p}, \boldsymbol{r} \in \mathbb{R}_+^N$, $(\boldsymbol{p}, \boldsymbol{r}) > 0$, *and fixed interaction intensivity parameter* $\alpha \neq 0$, $-1 \leq \alpha \leq 1$, *with condition* $\alpha \neq \frac{1}{(\boldsymbol{p}, \boldsymbol{r})}$, *sequence of states* $\{\boldsymbol{p}^n, \boldsymbol{r}^n\}$ *tends to the limit state* $\{\boldsymbol{p}^\infty, \boldsymbol{r}^\infty\}$
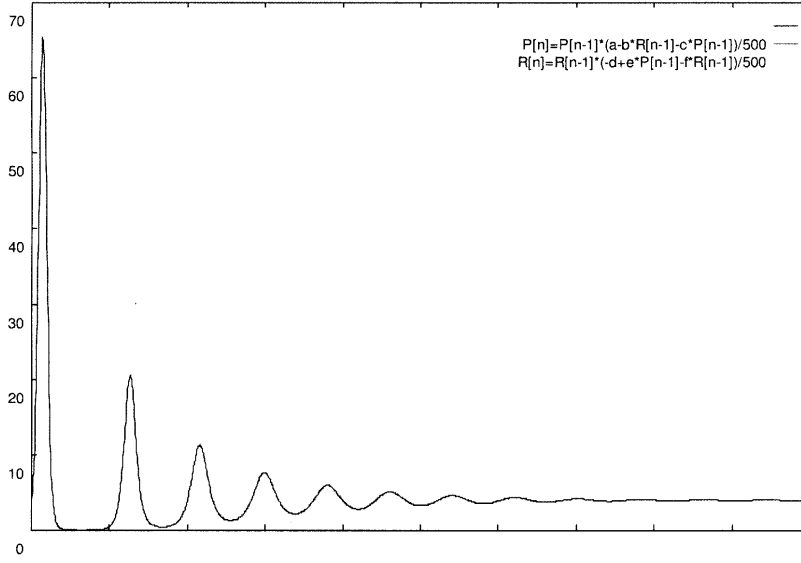
Figure 2

Lotka-Volterra model with discrete time

$$P_1^{(n)} = P_1^{(n-1)}(a - bP_1^{(n-1)} - cP_1^{(n-1)}),$$
$$P_2^{(n)} = P_2^{(n-1)}(-d + eP_2^{(n-1)} - cP_2^{(n-1)}),$$

$a = 0.2, b = 0.006, c = 0.002, d = 0.008, e = 0.002, f = 0,$
$P_1^{(0)} = 3, P_2^{(0)} = 5$

$$p^\infty = \lim_{n\to\infty} p^n, \quad r^\infty = \lim_{n\to\infty} r^n.$$

This limit state is invariant with respect to the conflict interaction:

$$p^\infty = p^\infty \divideontimes r^\infty, \quad r^\infty = r^\infty \divideontimes p^\infty.$$

Moreover,

$$\begin{cases} p^\infty \perp r^\infty, & \textit{if } p \neq r \textit{ and } 0 < \alpha \leq 1 \\ p^\infty = r^\infty, & \textit{in all other cases.} \end{cases}$$

We emphasize that in the case of a purely repulsive interaction, $0 < \alpha \leq 1$, if the starting distributions are different, then the limiting vectors are orthogonal.

Therefore each of the vectors p^∞, r^∞ contains by necessity some amount of zero coordinates on different positions ω_i. For example the typical limiting picture for $p^n, r^n \in \mathbb{R}_+^3$ is presented in Figure 3 (comp. with [17, 19]).
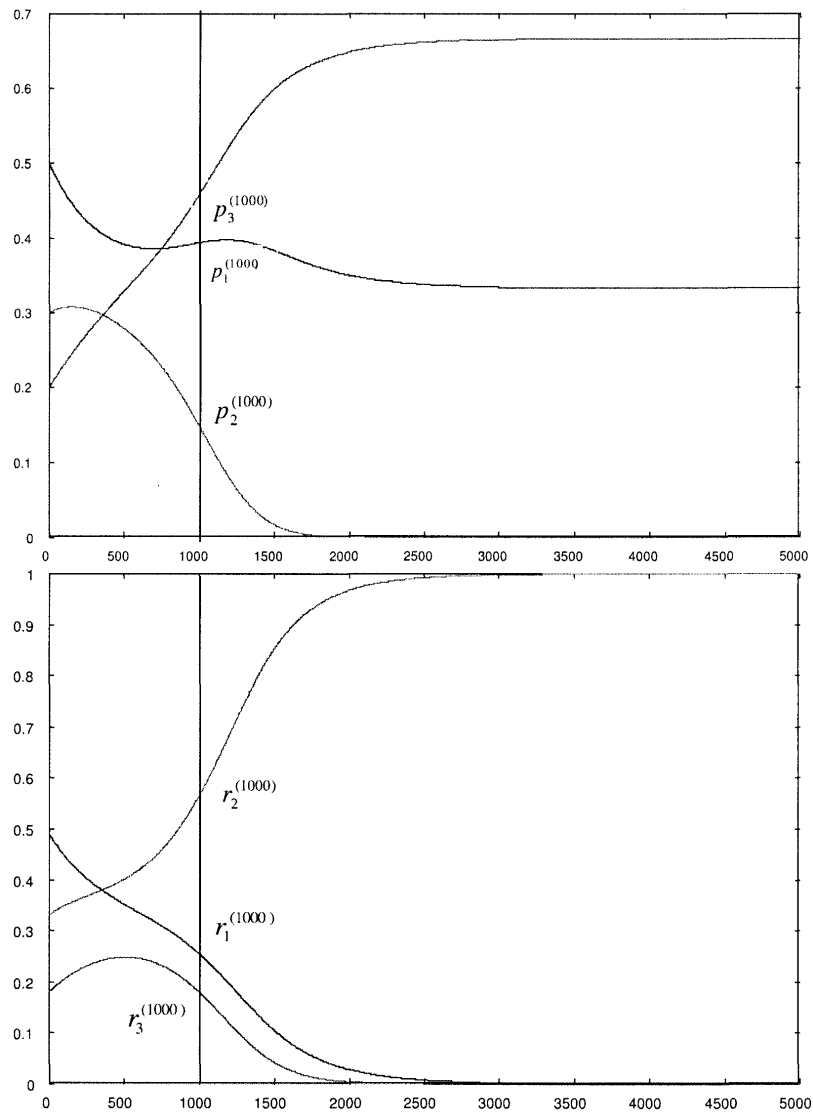
Figure 3

$\alpha = 1$, $p^0 = (0.5; 0.3; 0.2)$, $p^0 = (0.48; 0.34; 0.18)$,
$p^\infty = (0.33; 0; 0.67)$, $r^\infty = (0; 1; 0)$

If we start with a pair of identical vectors, $\boldsymbol{p} = \boldsymbol{r}$, then $\boldsymbol{p}^\infty = \boldsymbol{r}^\infty$ too. That is, all non-zero coordinates of the limiting vectors are equal.

In the general case, $\boldsymbol{p}, \boldsymbol{r} \in \mathbb{R}_+^N$, the coordinates $p_i^{(n)}, r_i^{(n)}$ have at most several oscillations and then reach monotonically their positive or zero limits. The limiting values p_i^∞, r_i^∞ may be described in terms of starting states.

Given a couple of stochastic vectors $\boldsymbol{p}, \boldsymbol{r} \in \mathbb{R}_+^N$, $\boldsymbol{p} \neq \boldsymbol{r}$, $(\boldsymbol{p}, \boldsymbol{r}) > 0$, define

$$D_+ := \sum_{i \in \mathbb{N}_+} d_i, \quad D_- := \sum_{i \in \mathbb{N}_-} d_i,$$

where

$$d_i = p_i - r_i, \quad \mathbb{N}_+ := \{i : d_i > 0\}, \quad \mathbb{N}_- := \{i : d_i < 0\}.$$

Obviously

$$0 < D_+ = -D_- < 1,$$

since $\boldsymbol{p} \neq \boldsymbol{r}$, and $\sum_i p_i - \sum_i r_i = 0 = D_+ + D_-$.

Theorem 2. *Let $\boldsymbol{p} \neq \boldsymbol{r}$, $(\boldsymbol{p}, \boldsymbol{r}) > 0$. In the purely repulsive case, $\alpha = 1$, the coordinates of the limiting vectors $\boldsymbol{p}^\infty, \boldsymbol{r}^\infty$ have the following explicit distributions:*

$$p_i^\infty = \begin{cases} d_i/D, & i \in \mathbb{N}_+ \\ 0, & \text{otherwise,} \end{cases} \qquad r_i^\infty = \begin{cases} -d_i/D, & i \in \mathbb{N}_- \\ 0, & \text{otherwise,} \end{cases} \qquad (3.2)$$

where $D := D_+ = -D_-$.

Remark. *From (3.2) it follows that any transformation $\boldsymbol{p}, \boldsymbol{r} \to \boldsymbol{p}', \boldsymbol{r}'$, which does not change the values d_i and D, preserves the same limiting distribution as for the vectors $\boldsymbol{p}^\infty, \boldsymbol{r}^\infty$. A class of such transformations may be presented by a shift transformation of coordinates, $p_i \to p_i' = p_i + a_i, r_i \to r_i' = r_i + a_i$ with appropriated a_i's.*

In the case $-1 \leq \alpha < 0$ of the pure attractive interaction we have another limiting distribution.

Define the set $\mathbb{S}_0 := \{k \mid p_k^\infty = r_k^\infty = 0\}$ and set

$$\mathbb{S}^\infty := \{1, \dots, N\} \setminus \mathbb{S}_0.$$

Theorem 3. *In the purely attractive case, $\alpha = -1$, the limiting vectors $\boldsymbol{p}^\infty$, $\boldsymbol{r}^\infty$ are equal and their coordinates have the following distributions:*

$$p_i^\infty = r_i^\infty = \begin{cases} 1/m, & i \in \mathbb{S}^\infty \\ 0 & \text{otherwise,} \end{cases} \tag{3.3}$$

where $m = |\mathbb{S}^\infty|$ denotes the cardinality of the set $\mathbb{S}^\infty$.

In general, it is an open question to give a complete characterization of $\mathbb{S}_0$.

Below we present several sufficient conditions for k to belong to the set $\mathbb{S}_0$. Simultaneously these conditions give some characterization for the points to be in $\mathbb{S}^\infty$.

We will use the following notations:

$$\sigma_i := p_i + r_i, \quad \rho_i := p_i r_i, \quad \sigma_i^1 := p_i^1 + r_i^1 \rho_i^1 := p_i^1 r_i^1. \tag{3.4}$$

Proposition 1. *If*

$$\sigma_i \geq \sigma_k, \quad \rho_i > \rho_k, \quad or \quad \sigma_i > \sigma_k, \rho_i \geq \rho_k, \tag{3.5}$$

then

$$p_k^\infty = r_k^\infty = 0,$$

and therefore $k \in \mathbb{S}_0$.

Proof. By (3.4) we have

$$\sigma_k^1 = p_k^1 + r_k^1 - 1/z(p_k + r_k + 2p_k r_k) = 1/z(\sigma_k + 2\rho_k)$$

where we recall that $z = 1 + (\boldsymbol{p}, \boldsymbol{r})$. Therefore each of the conditions (3.5) implies that $\sigma_i^1 > \sigma_k^1$. Further, since

$$\rho_k^1 = 1/z^2(\rho_k + (\rho_k)^2 + \rho_k \sigma_k), \tag{3.6}$$

again from (3.5) it also follows that $p_i^1 > p_k^1$. Thus, by induction, $\sigma_i^N > \sigma_k^N$ and $\rho_i^N > \rho_k^N$ for all $N \geq 1$. Or, in other words,

$$1 < \frac{p_i}{p_k} < \frac{p_i^1}{p_k^1} < \ldots < \frac{p_i^N}{p_k^N} \ldots,$$

$$1 < \frac{p_i}{p_k} < \frac{r_i^1}{r_k^1} < \ldots < \frac{r_i^N}{r_k^N} \ldots, \quad N = 1, 2, \ldots \tag{3.7}$$

Thus, sequences of the ratios

$$\frac{p_i^N}{p_k^N}, \quad \frac{r_i^N}{r_k^N}$$

are monotone increasing as $N \to \infty$. Assume for a moment that there exists a finite limit,

$$1 < \lim_{N \to \infty} \frac{p_i^N}{p_k^N} = \frac{p_i^\infty}{p_k^\infty} \equiv \frac{p_i^\infty}{p_k^\infty} \cdot \frac{1 + r_i^\infty}{1 + r_k^\infty} = M < \infty.$$

This is only possible if $r_i^\infty = r_k^\infty$, which contradicts (3.7). Thus, $M = \infty$ and therefore $p_k^\infty = 0$, as well as $r_k^\infty = 0$. $\qquad\square$

Let us consider now the critical situation, when for a fixed pair of indices, say i and k, the values $\sigma_k - \sigma_i$, $\rho_k - \rho_i$ have opposite signs, for example, $\sigma_k - \sigma_i > 0$, $\rho_k - \rho_i < 0$. In such a case it is not clear what behavior the coordinates p_i^N, r_i^N and p_k^N, r_k^N will have when $N \to \infty$. We will show that the limits depend on which of the two values, $2\rho_i + \sigma_i$ or $2\rho_k + \sigma_k$, is larger. Moreover we will show that even if p_k is the largest coordinate, it may happen that $p_k^\infty = 0$. Let for example, $p_k = \max_j\{p_j, r_j\}$ and $\sigma_k = p_k + r_k > p_i + r_i = \sigma_i$, however the value of r_k is such that $\rho_k = p_k r_k < p_i r_i = \rho_i$. Then under some additional condition it is possible to have $p_k^\infty = 0$. In fact we have:

Proposition 2. *Let for the coordinates $p_i, r_i, p_k, r_k, i \neq k$, the, following conditions be fulfilled:*

$$\sigma_k > \sigma_i \tag{3.8}$$

but

$$\rho_k < \rho_i. \tag{3.9}$$

Assume

$$2\rho_k + \sigma_k \leq 2\rho_i + \sigma_i. \tag{3.10}$$

Then

$$p_k^\infty = r_k^\infty = 0, \tag{3.11}$$

i.e., $k \in \mathbb{S}_0$

Proof. We will show that (3.8), (3.9), and (3.10) imply

$$p_k^1 + r_k^1 = \sigma_k^1 \leq \sigma_i^1 = p_i^1 + r_i^1 \tag{3.12}$$

and

$$p_k^1 r_k^1 = \rho_k^1 < \rho_i^1 = p_i^1 r_i^1. \tag{3.13}$$

Then (3.11) follows from Proposition 1. In reality (3.12) follows from (3.10) directly, without condition (3.9). So, we have only to prove (3.13).

With this aim we find the representation for ρ_i^1 in terms σ_i and σ_i^1. Since $\sigma_i^1 = 1/z(\sigma_i + 2\rho_i)$ we have

$$p_i = 1/2(z\sigma_i^1 - \sigma_i). \tag{3.14}$$

By (3.6) and (3.14) we get

$$
\begin{aligned}
\rho_i^1 &= 1/z^2(\rho_i + \rho_i^2 + \rho_i\sigma_i) \\
&= \frac{1}{2z^2}(z\sigma_i^1 - \sigma_i)[1 + 1/2(z\sigma_i^1 - \sigma_i) + \sigma_i] \\
&= \frac{1}{4z^2}(z\sigma_i^1 - \sigma_i)(2 + z\sigma_i^1 + \sigma_i) \\
&= \frac{1}{4z^2}[2z\sigma_i^1 + z^2(\sigma_i^1)^2 + z\sigma_i^1\sigma_i - 2\sigma_i - z\sigma_i^1\sigma_i - \sigma_i^1] \\
&= \frac{1}{4z^2}[2z\sigma_i^1 + z^2(\sigma_i^1)^2 - \sigma_i^2 - 2\sigma_i].
\end{aligned}
$$

Therefore

$$\rho_k^1 - \rho_i^1 = 1/z^2[\rho_k(1 + \rho_k + \sigma_k) - \rho_i(1 + \rho_i + \sigma_i)].$$

Thus, we have

$$
\begin{aligned}
\rho_k^1 - \rho_i^1 &= 1/4z^2[2z(\sigma_k^1 - \sigma_i^1) + z^2((\sigma_k^1)^2 - (\sigma_i^1)^2) \\
&\quad + ((\sigma_i)^2 - (\sigma_k)^2) + 2(\sigma_i - \sigma_k)] < 0
\end{aligned}
$$

due to starting condition (3.9), and (3.12). Thus $\rho_k^1 < \rho_i^1$, i.e., (3.13) is true. $\square$

We stress that (3.11) is true in spite of $\sigma_k > \sigma_i$. Of course, if $\sigma_k < \sigma_i$ and $\rho_k < \rho_i$, then (3.11) holds without any additional condition of the form (3.10).

4.   Model of conflict interaction between complex systems

In this section we construct a dynamical model of conflict interaction between a pair of complex systems. Each of the systems is subjected to the inner conflict between their elements. For simplicity, we assume both systems to be similar and described by discrete prey-predator models of type (2.5). We introduce the conflict interaction between these systems

using an approach developed in [1,2,4,7,8,9,10]. With such a rather complex situation we may obtain a wide spectrum of evolutions. In this work we study qualitative characteristics of the behavior of corresponding dynamical systems for some choice of parameters a, b, c, d, e, f, α (see (2.5), (3.1)) and values of initial populations of species P_i, R_i.

The coefficient α, that shows intensity of the interaction between systems, has an important effect. The increasing α from zero to unit causes the appearance of a series of bifurcations. For $\alpha = 0$ we have two copies of independent Lotka-Volterra models. For small values of α both systems behave like pure Lotka-Volterra systems, coming them to a stable state.

Under fixed parameters and the starting coordinates $a = 0.2$, $b = 0.006$, $c = 0.002$, $d = 0.008$, $e = 0.002$, $f = 0$, $P_1^{(0)} = 3$, $P_2^{(0)} = 10$, $R_1^{(0)} = 5$, $R_2^{(0)} = 20$ we have first bifurcation point at $\alpha \approx 0.0056781739$. The coordinates $P_i^{(n)}$ $(R_i^{(n)})$ oscillate and a cycle of a small period appears.

The following increase of a shows the appearance of new bifurcation points that are characterized by an increasing value of the cyclic period. For the value a $\alpha = 0.4815545975$ a cycle of infinite period appears. This means that all coordinates rapidly reach the stable state. In this case some species may disappear, even if they had some stable positive values in a pure ($\alpha = 0$) Lotka-Volterra model.

The role of the coefficients a, b, c, d, e, f and initial quantity of the species P_i, R_i in a pure Lotka-Volterra model is well-known and described (see, e.g., [16, 12]). Partially, coefficients a, d govern the increase of the pray population when predators are absent and the predator population decreasing when prays are absent. In turn, the coefficients b, e are responsible respectively for the pray quantity decreasing with an increasing number of predators, and increase of the predator population with an increase of the number of prays. The last coefficients in each of the equations give the limitation of increasing of both populations. In other words, each population "makes pressure" on itself, it does not permit an infinite reproducibility.

Questions about stable points, orbits, asymptotic behavior of orbits are well described for the classical Lotka-Volterra model. We shall recall that usually there are at least three equilibrium points. They are refereed in literature as follows (see, e.g. [16]):

(1) trivial $(0, 0)$;

(2) axial $(a/b, 0)$;

(3) inner positive

$$\left(\frac{a}{b} - \frac{b}{c} \frac{ae - cd}{be + cf}, \frac{ae - cd}{be + cf} \right).$$

(4.1)

An equilibrium point is called stable point if after a sudden change of population it comes back to an equilibrium point some time later. This may happen monotonically, or with some oscillations.

We should note that under the existence of stable points the behavior of the system is well defined by coefficients a, b, c, d, e, f. But under the absence of stable points, the behavior of the system is defined by the initial data P_i, R_i. Depending on how close the initial data are situated with respect to the equilibrium point, the system may evaluate in a different way.

The role of all these coefficients is preserved in the case of our model. But now their influence is much more complex. We present here only first steps in this direction. We shall discuss not only stability zones, as it was pointed above, but also the values of the coefficients for which the system oscillates along some closed cycles.

The state of our dynamical system is fixed by a pair of vectors $\boldsymbol{P}^n = (P_1^{(n)}, \ldots, P_N^{(n)})$, $\boldsymbol{R}^n = (R_1^{(n)}, \ldots, R_N^{(n)})$ with non-negative coefficients, where $n = 0, 1, \ldots$ denotes the discrete time, $N \geq 2$ stands for the number of conflict positions. Here we study the most simple situation, when every system consists of only two agents: pray and predator, i.e. $N = 2$. The complex conflict transformation is denoted by the mapping

$$\begin{pmatrix} \boldsymbol{P}^n \\ \boldsymbol{R}^n \end{pmatrix} \xrightarrow{F} \begin{pmatrix} \boldsymbol{P}^{n+1} \\ \boldsymbol{R}^{n+1} \end{pmatrix},$$

where F is the composition of four operations, the specific mathematical transformations: $F = [\mathcal{N}^{-1} * \mathcal{N}] U$.

Let us describe them in an explicit form for the first step.

The first operation U describes the interaction between elements inside every system separately according to the pray-predator model. Corresponding mathematical transformation of vectors (the interaction composition) $\{\boldsymbol{P}^0, \boldsymbol{R}^0\} \xrightarrow{U} \{\tilde{\boldsymbol{P}}^0, \tilde{\boldsymbol{R}}^0\}$ is described by the system of equations of the form (2.5):

$$\tilde{P}_1^{(0)} = P_1^{(0)} + P_1^{(0)}(a - bP_2^{(0)} - cP_1^{(0)}),$$
$$\tilde{P}_2^{(0)} = P_2^{(0)} + P_2^{(0)}(-d + eP_1^{(0)} - cP_2^{(0)}),$$

and

$$\tilde{R}_1^{(0)} = R_1^{(0)} + R_1^{(0)}(a - bR_2^{(0)} - cR_1^{(0)}),$$
$$\tilde{R}_2^{(0)} = R_2^{(0)} + R_2^{(0)}(-d + eR_1^{(0)} - fR_2^{(0)}),$$

where the passage to new values of coordinates is pointed by tilde, but not by changing of upper index, likely to (2.5).

The following operation involves the interaction $*$ (see (3.1)) between previous systems according to the theory of the alternative conflict for non-annihilating opponents (see, e.g. [1,2,4,7,8,9, 10]). To describe this operation we at first have to normalize the vectors $\tilde{\boldsymbol{P}}^0 = (\tilde{P}_1^{(0)}, \tilde{P}_2^{(0)})$, $\tilde{\boldsymbol{R}}^0 = (\tilde{R}_1^{(0)}, \tilde{R}_2^{(0)})$, i.e., to work with stochastic vectors.

We use the following notation for normalization: $\mathcal{N}\{\tilde{\boldsymbol{P}}^0, \tilde{\boldsymbol{R}}^0\} = \{\boldsymbol{p}^0, \boldsymbol{r}^0\}$, where the coordinates of the stochastic vectors $\boldsymbol{p}^0$, $\boldsymbol{r}^0$ are determined by formulae

$$p_1^{(0)} = \frac{\tilde{P}_1^{(0)}}{\tilde{z}_P^{(0)}}, \quad p_2^{(0)} = \frac{\tilde{P}_2^{(0)}}{\tilde{z}_P^{(0)}}, \quad r_1^{(0)} = \frac{\tilde{R}_1^{(0)}}{\tilde{z}_R^{(0)}}, \quad r_2^{(0)} = \frac{\tilde{R}_2^{(0)}}{\tilde{z}_R^{(0)}},$$

where $\tilde{z}_P^{(0)} = \tilde{P}_1^{(0)} + \tilde{P}_2^{(0)}$, $\tilde{z}_R^{(0)} = \tilde{R}_1^{(0)} + \tilde{R}_2^{(0)}$.

The next step exactly corresponds to the conflict interaction between systems. We introduce new stochastic vectors $\{\boldsymbol{p}^1, \boldsymbol{r}^1\}$ with coordinates:

$$p_j^{(1)} = \frac{p_j^{(0)}(1 - \alpha r_j^{(0)})}{1 - \alpha \sum_{i=1}^2 p_i^{(0)} r_1^{(0)}}, \quad r_j^{(1)} = \frac{r_j^{(0)}(1 - \alpha p_j^{(0)})}{1 - \alpha \sum_{i=1}^2 p_i^{(0)} r_1^{(0)}}, \quad j = 1, 2.$$

Finally, we have to come back to the non-normalized vectors, which characterize quantitatively populations in both regions after inner and outer conflicts operations. So, at time $n = 1$ we have the following vectors $\mathcal{N}^{-1}\{\boldsymbol{p}^1, \boldsymbol{r}^1\} = \{\boldsymbol{P}^1, \boldsymbol{R}^1\}$, where

$$\boldsymbol{P}^1 = (P_1^{(1)}, P_2^{(1)}), \quad \boldsymbol{R}^1 = (R_1^{(1)}, R_2^{(1)}),$$

and where

$$P_j^{(1)} = p_j^{(1)} \tilde{z}_p^{(0)}, \quad R_j^{(1)} = r_j^{(1)} \tilde{z}_R^{(0)}, \quad j = 1, 2.$$

We can repeat this procedure starting from $\{\boldsymbol{P}^1, \boldsymbol{R}^1\}$. So we get $\{\boldsymbol{P}^2, \boldsymbol{R}^2\}$. And so on for any nth step.

To find the equilibrium points in the case of the complex conflict interaction described above, we have to solve the following system of

equations for P_1, P_2, R_1, R_2:

$$\begin{cases} (a+1-bR_1-cP_1)(Z_2-\alpha R_2(-d+1+eP_2-fR_2))Z_1 = Z, \\ (-d+1+eP_1-fR_1)(Z_2-\alpha P_2(a+1-bR_2-cP_2))Z_1 = Z, \\ (a+1-bR_2-cP_2)(Z_1-\alpha R_1(-d+1+eP_1-fR_1))Z_2 = Z, \\ (-d+1+eP_2-fR_2)(Z_1-\alpha P_1(a+1-bR_1-cP_1))Z_2 = Z, \end{cases}$$

where

$$\begin{aligned} Z_1 &= P_1(a+1-bR_1-cP_1) + R_1(-d+1+eP_1-fR_1), \\ Z_2 &= P_2(a+1-bR_2-cP_2) + R_2(-d+1+eP_2-fR_2), \\ Z &= Z_1 Z_2 - \alpha[P_1 P_2(a+1-bR_2-cP_2)(a+1-bR_1-cP_1) \\ &\quad + R_1 R_2(-d+1+eP_1-fR_1)(-d+1+eP_2-fR_2)]. \end{aligned}$$

We note that in the case $\alpha = 0$ we have two copies of pure Lotka-Volterra models and the corresponding system of equations has at least three equilibrium points (trivial, axial, inner positive).

For the case $\alpha \neq 0$ it is difficult to obtain exact solutions. Let us obtain some insights by numerical approximation.

Partially, we found that there exist equilibrium points and the limit cycles for a wide set of parameter values and initial data (see Figure 4-7).

Moreover, we established the shift effect for the equilibrium point. Namely, we observe that the inner positive equilibrium point (it exist in any system and may be found by formula (4.1)) is shifted after the application of the conflict interaction between systems. We see by (4.1) that stabilization of discrete Lotka-Volterra model with parameters $a = 0.2$, $b = 0.006$, $c = 0.002$, $d = 0.008$, $e = 0.002$, $f = 0$, $\alpha = 0.007$, $P_1^{(0)} = 3$, $P_2^{(0)} = 10$, $R_1^{(0)} = 5$, $R_2^{(0)} = 20$ occurs when $P_1 = 4$, $P_2 = 32$. This may be easily verified by putting these initial data into corresponding equations. In this case we have trivial dynamics.

Let us consider the case of discrete Lotka-Volterra model with the conflict interaction between systems. We take the values of the coefficients $a = 0.2$, $b = 0.006$, $c = 0.002$, $d = 0.008$, $e = 0.002$, $f = 0$, $\alpha = 0.005$. Now the equilibrium point has the coordinates $P_1 = 4.043507$, $P_2 = 32.100629$. The dynamics is constant with these initial data.

In case of larger α, when oscillations appear, the equilibrium point may also be easily found if we put the initial data in both systems to be equal. In this case the behavior is like in the case of a pure Lotka-Volterra model, and stabilization occurs. However, the stable point is shifted, for
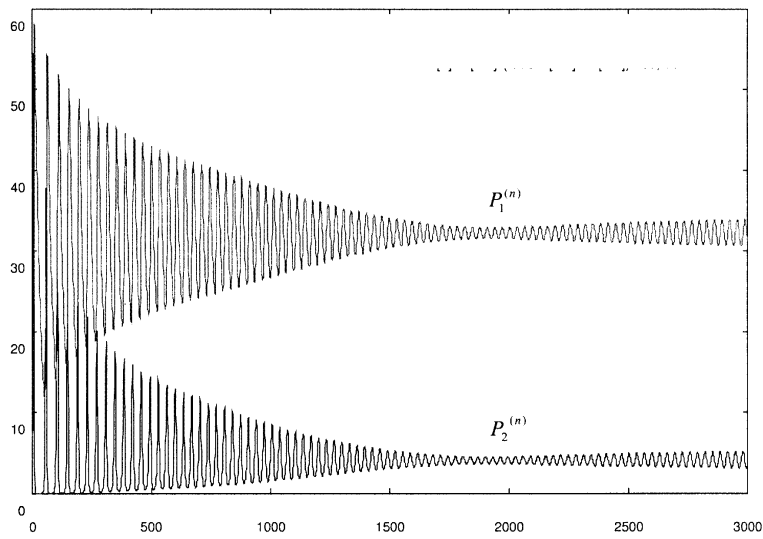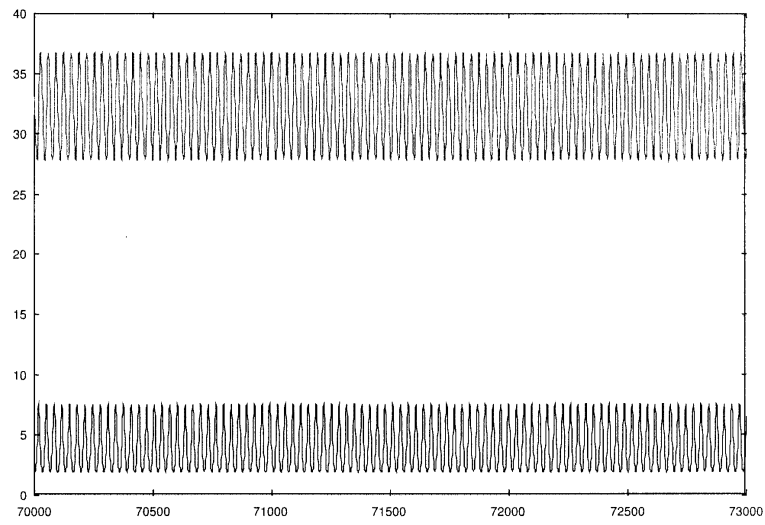
Figure 4

The existence of the strong bifurcation produces oscillations of the large amplitude, $a = 0.2$, $b = 0.006$, $c = 0.002$, $d = 0.008$, $e = 0.002$, $f = 0$, the conflict interaction coefficient $a = 0.01$, $P_1^{(0)} = 3$, $P_2^{(0)} = 5$, $R_1^{(0)} = 7$, $R_2^{(0)} = 10$
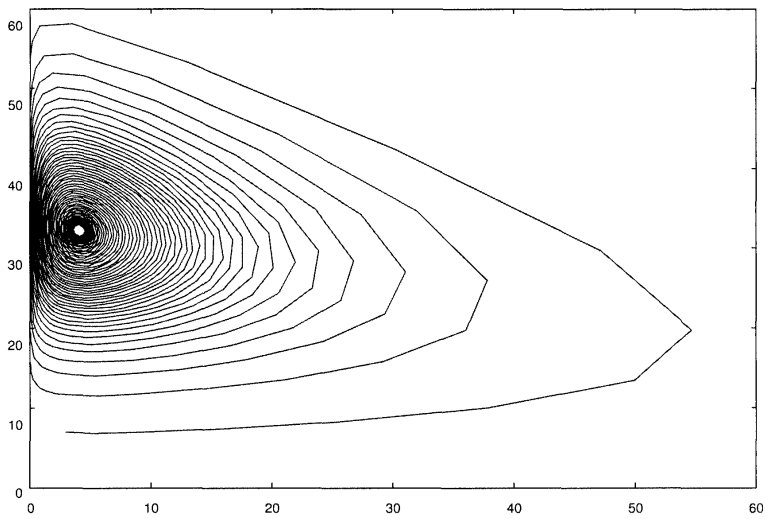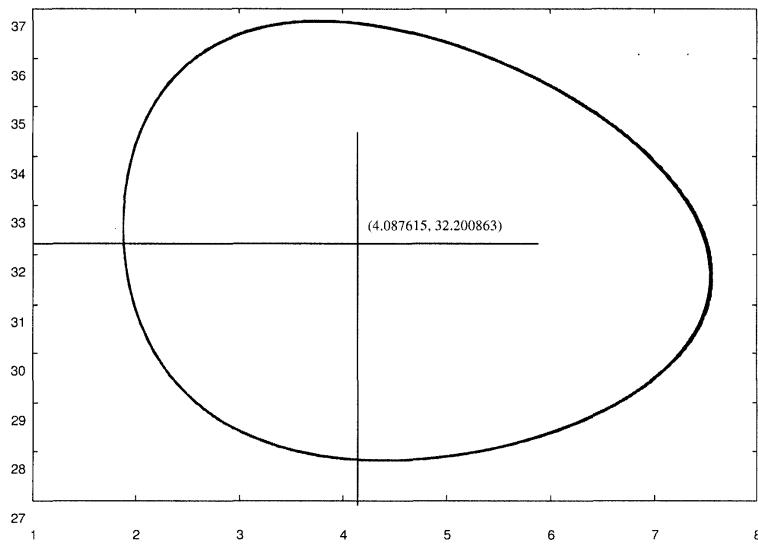

Figure 5

The corresponding phase-space $(P_1^{(0)}, P_2^{(0)})$

Figure 6

The existence of the stable oscillations of the conflict interaction between Lotka-Volterra systems after 70000 steps of iteration


(4.087615, 32.200863)

Figure 7

The limiting cycle in the corresponding phase-space $(P_1^{(0)}, P_2^{(0)})$ after 70000 steps of iteration. The unstable equilibrium point is shown

example, when $\alpha = 0.01$ (see Figure 4-7) the equilibrium point is $P_1 = R_1 = 4.087615$, $P_2 = R_2 = 32.200863$.

Thus, if we have some pray-predator system and want to change the population inside this system, we may create an analogous "artificial" system, introduce the conflict interaction and obtain the desired shift of the equilibrium point. Apparently a stronger shift of the stable equilibrium point in the case of an "ensemble" of larger amount of Lotka-Volterra systems. So, we observed the interesting phenomenon: the equilibrium point of an isolated system is shifted if we come to the case when identical systems are united as an "ensemble".

However, this equilibrium point is unstable, any perturbation of initial data causes the receding of the system from the equilibrium point.

One of more interesting observations concerns the limit cycles. It is known that no such kind of orbits in discrete Lotka-Volterra model is possible. But under the effect of the outer conflict, as we see at the pictures, the dynamical system reaches the limit cycle starting both from an inside or outside point with respect to the orbit. Partially, in Figure 10,11 we present the model, that starts at $P_1^{(0)} = 4$, $P_2^{(0)} = 32$. As it was pointed above, in case of a pure Lotka-Volterra model, with these initial data there is no dynamics. However, in the case of the model with the outer conflict the process tends to a limit cycle.
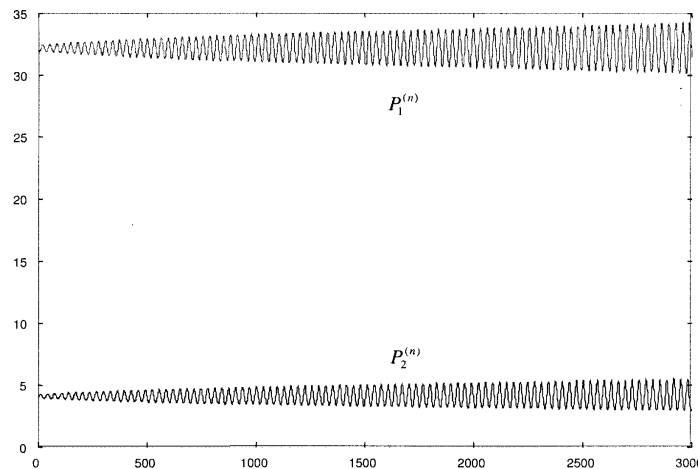
Figure 8

Conflict interaction between Lotka-Volterra systems. The starting parameters are the same as in Figure 4. Initial data are inside the limit cycle (Figure 7)
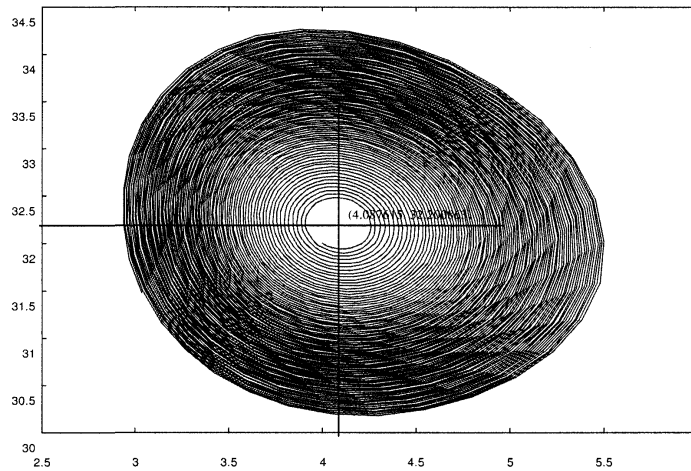
Figure 9

The corresponding phase-space $(P_1^{(0)}, P_2^{(0)})$. Trajectory tends to the limit cycle, which is an attractor. Initial data are inside the limit cycle (Figure 7)

5.   Interpretation

In many works on mathematical biology and economics [3, 5, 6, 11, 12, 13, 16, 18, 20] the modelling of population dynamics or economical processes is based on Lotka-Volterra equations. As a rule, continuous, not discrete, models are studied. In some works the migration process is considered. It takes place between different regions, inside which an interaction of the Lotka-Volterra type is present. For example, in [5] the migration rate between regions has some fixed probability.

We study discrete Lotka-Volterra models with an additional interaction between them. That may be interpreted as a some kind of correlation between the habitants of different regions. We suppose that discrete models are more natural, partially it is clear that birth and death of individuals happen at some fixed moments of time.

It is well known that in the classical discrete pray-predator model a stable point exist. The amount of prays and predators tends to this point in the phase-space. In this case we observe the following dynamics, after several period of oscillations the populations stabilize (see Figure 2). Thus, we have an attracting point in phase-space. Such a dynamics exists inside every region when "migration" is absent.

When we introduce an additional interaction between the habitants of different regions a redistribution process appears which we interpret as a migration. In some of our complex models there is no stable point, the amount of prays and predators in both regions oscillates along fixed orbits.Appearently these orbits in a phase-space are attractors.

We note that explicit formulas of conflict interaction between non-annihilating opponents which describe the redistribution of populations are given by (3.1). The individuals of a certain kind migrate to the region, where their amount more numerous.

Is the "migration strategy" which is described in our model a natural one? We suppose that in many cases individuals may be right behaving in such a way. If we consider a pray-predator model, it is clear that every separated individual is unable to estimate all factors that have an influence on the population dynamics like vital resources inside region, real amount of own and alternative population, current population dynamics. In other words, the individual "does not know" the parameters of the Lotka-Volterra equations and their current influence on the population dynamics.

However the individual has the group reflex and will migrate to the region, where, as he supposes, the vital conditions are best (his population should be concentrated there). He suggests, right there are the resources, possibilities for reproduction, better conditions to organize large groups. Formula (3.1) just describes this tendency.

Similar motivations may be proposed in case of the work migration. Here the unemployed may be regarded as playing the role of "prays", employees as playing the role of "predators". People, who seek for work and migrate to another country, do not know, as a rule, the real situation in the opposite region. They prefer to migrate to the country where the majority of their friends migrated (group reflex).

Similar, but opposite picture happens with employees who inverts their capital to the region with a higher profit.

So, at the cost of migration accelerates the increasing of one of species' population in one of the regions. But at the same time there is an effect of the inner pray-predator "fight" inside every system. Partially, the population influences itself at the cost of the last term in Lotka-Volterra equation. As a result, some time later the backward migration starts.

In the Figure 6 we may see the effect of delay, when the amount of prays inside the region decreases, but the predators continue migration to

this region, until their amount starts decreasing by following the Lotka-Volterra model.

We emphasize, that in our model, in comparison with discrete Lotka-Volterra model, a cyclic oscillations of populations are observed. Moreover, a cyclic attractor exists in the phase-space, and the pray-predator trajectory tends to this orbit both from inside or outside point with respect to this cycle (Figure 8, 9).

We remark that in our model the normalization was fulfilled by the amount of habitants of the region, so the component of the corresponding vector may be large both at the cost of large population of fixed individuals and at the cost of small whole population of the region. So, a migration to the region with a lot of "free space" is also possible.

We also studied model with the attracting interaction ($\alpha < 0$). In this case we obtained formally a similar dynamics, but now individuals migrate to the region where they are less numerous. Such a migration strategy might be also natural for some species, e.g. for individuals who hunt separately, control large territory and have confrontation with relatives.

References

[1] S. Albeverio, M. Bodnarchyk and V. Koshmanenko, Dynamics of discrete conflict interactions between non-annihilating opponents, *MFAT*, Vol. 11 (4) (2005), pp. 309–319.

[2] S. Albeverio, V. Koshmanenko, M. Pratsiovytyi and G. Torbin, Spectral properties of image measures under infinite conflict interaction, *Positivity*, N 10, (2006), pp. 39–49.

[3] M. Bandyopadhyay and J. Chattopadhayay, Ratio-dependent predator-prey model: effects of environmental fluctuation and stability, *Nonlinearity*, Vol. 18 (2005), pp. 913–936.

[4] M. V. Bodnarchyk, V. D. Koshmanenko and N. V. Kharchenko, Properties of limit states of dynamical conflict system, *Nonlinear Oscillations*, Vol. 7 (4) (2004), pp. 446–461 (in Ukrainian).

[5] A. Colato and S. S. Mizrahi, Effects of random migration in population dynamics, *Physical Review E*, Vol. 64 (2001), pp. 1–14.

[6] R. Cressman, V. Krivan and J. Garay, Ideal free distributions, evolutionary games, and population dynamics in multiple-species environment, *The American Naturalist*, Vol. 164 (4) (2004), pp. 473–489.

[7] V. Koshmanenko, On the conflict theorem for a pair of stochastic vectors, *Ukrainian Math. J.*, Vol. 55 (4) (2003), pp. 555–560.

[8] V. Koshmanenko, The theorem of conflict for probability measures, *Math. Methods of Operations Research*, Vol. 59 (2) (2004), pp. 303–313.

[9] V. D. Koshmanenko and N. V. Kharchenko, Invariant points of dynamical conflict system in the space of piecewise uniformly distributed measures, *Ukrainian Math. J.*, Vol. 56 (7) (2004), pp. 927–938.

[10] V. Koshmanenko and N. Kharchenko, Spectral properties of image measures after conflict interactions, *Theory of Stochastic Processes*, Vol. 10 (26) (3-4) (2004), pp. 73–81.

[11] Y. Kuang, Basic properties of mathematical population models, *J. Biomath*, Vol. 17 (2002), pp. 129–142.

[12] Y. Kuang and E. Beretta, Global qualitative analysis of a ratio-dependent predator-prey system, *J. Math. Biol.*, Vol. 36 (1998), pp. 389–406.

[13] Y. Lonzonn, S. Solomon, J. Goldenberg and D. Mazarsky, World-size global markets lead to economic instability, *Acrificial life*, (2003), pp. 357–370.

[14] A. J. Lotka, Relation between birth rates and death rates, *Science*, Vol. 26 (1907), pp. 21–22.

[15] T. R. Malthus, *An Essay on the Principle of Population*, reprinted by Macmillan, 1894.

[16] J. D. Murray, *Mathematical Biology I: An Introduction*, Springer, 2002.

[17] K. Md. M. Salam and K. I. Takahashi, Mathematical model of conflict and cooperation with non-annihilating multi-opponent, *J. Interdisciplinary Math.*, in press.

[18] L. Stone and R. Olinky, Phenomena in ecological systems, *Experimental Chaos: 6th Experimental Chaos Conference*, pp. 476-487, 2003.

[19] K. I. Takahashi and K. Md. M. Salam, Mathematical model of conflict with non-annihilating multi-opponent, *J. Interdisciplinary Math.*, Vol. 9 (3) (2006), pp. 459–473.

[20] J. Tufto, Effects of releasing maladapted individuals: a demographic-evolutionary model, *The American Naturalist*, Vol. 158 (4) (2001), pp. 331–340.

[21] P. P. Verhulst, Notice sur la loi que la population suit dans son accroissement, in *Correspondence mathematique et physique publiee par A. Quetelet*, Vol. 10 (1838), pp. 113–121.

[22] V. Volterra, Sui tentativi di applicazione della matematiche alle scienze biologiche e sociali, *Giornale degli Economisti*, Vol. 23 (1901), pp. 436–458.