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ON THE POINT SPECTRUM OF SELF-ADJOINT OPERATORS THAT APPEARS UNDER SINGULAR PERTURBATIONS OF FINITE RANK

M. E. Dudkin[1] **and V. D. Koshmanenko**[2]

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We discuss purely singular finite-rank perturbations of a self-adjoint operator A in a Hilbert space $\mathcal{H}$. The perturbed operators $\tilde{A}$ are defined by the Krein resolvent formula $(\tilde{A}-z)^{-1} = (A-z)^{-1}+B_z$, $\operatorname{Im} z \neq 0$, where B_z are finite-rank operators such that $\operatorname{dom} B_z \cap \operatorname{dom} A = \{0\}$. For an arbitrary system of orthonormal vectors $\{\psi_i\}_{i=1}^{n<\infty}$ satisfying the condition $\operatorname{span}\{\psi_i\} \cap \operatorname{dom} A = \{0\}$ and an arbitrary collection of real numbers $\lambda_i \in \mathbb{R}^1$, we construct an operator $\tilde{A}$ that solves the eigenvalue problem $\tilde{A}\psi_i = \lambda_i \psi_i$, $i = 1, \ldots, n$. We prove the uniqueness of $\tilde{A}$ under the condition that $\operatorname{rank} B_z = n$.

1. Introduction

In a complex separable Hilbert space $\mathcal{H}$ with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$, we consider an unbounded self-adjoint operator $A = A^*$ with the domain of definition $\mathfrak{D}(A) \equiv \operatorname{dom} A$. Another self-adjoint operator $\tilde{A}$ in $\mathcal{H}$ is called [1–8] (cf. [9, 10]) purely singularly perturbed with respect to A; denote $\tilde{A} \in \mathcal{P}_s(A)$ if the domain

$$\mathfrak{D} := \left\{ f \in \mathfrak{D}(A) \cap \mathfrak{D}(\tilde{A}) : Af = \tilde{A}f \right\} \tag{1}$$

is dense in $\mathcal{H}$. It is clear that, for every $\tilde{A} \in \mathcal{P}_s(A)$, there exists a densely defined symmetric operator

$$\dot{A} := A \upharpoonright \mathfrak{D} = \tilde{A} \upharpoonright \mathfrak{D}, \qquad \mathfrak{D}(\dot{A}) = \mathfrak{D}, \tag{2}$$

with nontrivial deficiency indices

$$\mathrm{n}^{\pm}(\dot{A}) = \dim \operatorname{Ker}(\dot{A} \pm i)^* \neq 0.$$

In the present paper, we consider the subclass of operators $\tilde{A} \in \mathcal{P}_s^n(A)$, where

$$n = \mathrm{n}^+(\dot{A}) = \mathrm{n}^-(\dot{A}) < \infty.$$

[1] Kiev Polytechnic Institute, Kiev.

[2] Institute of Mathematics, Ukrainian Academy of Sciences, Kiev.

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We study the problem of the existence and construction of an operator $\tilde{A} \in \mathcal{P}_s^n(A)$ that solves the eigenvalue problem

$$\tilde{A}\psi_i = \lambda_i \psi_i, \quad i = 1, \ldots, n, \tag{3}$$

for arbitrary preassigned real numbers λ_i and a system of orthonormal vectors $\{\psi_i\}_{i=1}^n$ satisfying the condition $\operatorname{span}\{\psi_i\} \cap \operatorname{dom} A = \{0\}$.

The spectrum (in particular, point spectrum) of self-adjoint extensions of symmetric operators with finite deficiency indices in the general form was first studied by M. Krein in [11], where he proved the existence of at least one extension with preassigned eigenvalues in the regularity field of a symmetric operator (see also [12–16]). In this connection, one should also mention the papers [17, 18], where, in particular, the existence of an arbitrary component of the spectrum in lacunas of a symmetric operator was proved in terms of boundary-value spaces and Weyl functions.

We propose to consider the eigenvalue problem for self-adjoint extensions of a symmetric operator from the viewpoint of the theory of singularly perturbed operators. The key point of our result is the fact that points λ_i in (3) are arbitrary, and, in particular, they can belong to the spectrum of the operator A. Note that, in [19], an analogous result was proved in the case where the operator A is positive, $\lambda_i \le 0$, and $\tilde{A}$ is not necessarily a purely singularly perturbed operator.

The statement below is the main result of the present work.

Theorem 1. *For an unbounded self-adjoint operator A in a Hilbert space $\mathcal{H}$, there exists a unique purely singularly perturbed operator $\tilde{A} \in \mathcal{P}_s^n(A)$ that solves the eigenvalue problem (3) for arbitrary preassigned numbers $\lambda_i \in \mathbb{R}^1$, $i = 1, \ldots, n < \infty$, and any set of orthonormal vectors $\{\psi_i\}_{i=1}^n$ satisfying the condition*

$$\operatorname{span}\{\psi_i\}_{i=1}^n \cap \mathfrak{D}(A) = \{0\}. \tag{4}$$

Note that the proof of this theorem is constructive. We successively construct the resolvent of the operator A using a purely singular perturbation of rank one at each step.

2. Singular Perturbations of Rank One

Let $\dot{A} \subset \dot{A}^*$ be a closed symmetric operator with the domain of definition $\mathfrak{D}(\dot{A})$ dense in $\mathcal{H}$. Assume that its deficiency indices are $\mathrm{n}^{\pm}(\dot{A}) = 1$. Then

$$\mathcal{H} = \mathfrak{M}_z \oplus \mathfrak{N}_z, \quad \operatorname{Im} z \neq 0,$$

where

$$\mathfrak{M}_z = (\dot{A} - z)\mathfrak{D}(\dot{A})$$

is the range of values of the operator $\dot{A} - z$ and

$$\mathfrak{N}_z := \mathfrak{M}_z^{\perp} = \operatorname{Ker}(\dot{A}^* - \bar{z})$$

is the deficiency subspace ($\dim \mathfrak{N}_z = 1$).

Let $\mathcal{A}(\dot{A})$ be the set of all self-adjoint extensions of the operator $\dot{A}$. We fix a self-adjoint extension $A \in \mathcal{A}(\dot{A})$. It is clear that every operator $\tilde{A} \neq A$ from the set $\mathcal{A}(\dot{A})$ also belongs to the set $\mathcal{P}_s^1(A)$. In this case, the domain $\mathfrak{D}$ in (1) coincides with $\mathfrak{D}(\dot{A})$. It is known [11, 14] that $\mathfrak{N}_z \cap \mathfrak{D}(A) = \{0\}$.

Theorem 2 [11, 12]**.** *The resolvent of every self-adjoint operator* $\tilde{A} \in \mathcal{A}(\dot{A})$, $\tilde{A} \neq A$, *is determined by the Krein formula*

$$(\tilde{A} - z)^{-1} = (A - z)^{-1} + b_z^{-1}(\cdot, \eta_{\bar{z}})\eta_z, \tag{5}$$

where the vector function η_z *with values in* $\mathfrak{N}_z$ *satisfies the equation*

$$\eta_z = (A - \xi)(A - z)^{-1}\eta_\xi, \qquad \operatorname{Im} z, \operatorname{Im} \xi \neq 0, \tag{6}$$

and the values of the scalar function b_z *satisfy the relations*

$$b_z = b_\xi + (\xi - z)(\eta_\xi, \eta_{\bar{z}}), \qquad \operatorname{Im} z, \operatorname{Im} \xi \neq 0, \tag{7}$$

$$\bar{b}_z = b_{\bar{z}}. \tag{8}$$

Using Theorem 2, we obtain a description of all operators $\tilde{A} \in \mathcal{P}_s^1(A)$ (cf. [2, 5]).

Theorem 3. *An operator* $\tilde{A} \neq A$ *self-adjoint in* $\mathcal{H}$ *belongs to the set* $\mathcal{P}_s^1(A)$ *if and only if, for any* $z_0 \in \mathbb{C}$, $\operatorname{Im} z_0 \neq 0$ *(and, hence, for all* z *of this type), there exist a subspace*

$$\mathfrak{N}_{z_0} \subset \mathcal{H}, \qquad \dim \mathfrak{N}_{z_0} = 1, \quad \mathfrak{N}_{z_0} \cap \mathfrak{D}(A) = \{0\}, \tag{9}$$

and a number

$$b_{z_0} \in \mathbb{C}, \qquad \operatorname{Im} b_{z_0} = -\operatorname{Im} z_0, \tag{10}$$

such that

$$(\tilde{A} - z_0)^{-1} = (A - z_0)^{-1} + b_{z_0}^{-1}(\cdot, \eta_{\bar{z}_0})\eta_{z_0}, \tag{11}$$

where $\eta_{z_0} \in \mathfrak{N}_{z_0}$, $\|\eta_{z_0}\| = 1$, *and*

$$\eta_{\bar{z}_0} = (A - z_0)(A - \bar{z}_0)^{-1}\eta_{z_0}.$$

For an arbitrary point $z \in \mathbb{C}$, $\operatorname{Im} z \neq 0$, *the resolvent of the operator* $\tilde{A}$ *is determined by formula (5), where the functions*

$$\eta_z = (A - z_0)(A - z)^{-1} \eta_{z_0}, \tag{12}$$

$$b_z = b_{z_0} + (z_0 - z)(\eta_{z_0}, \eta_{\bar{z}}) \tag{13}$$

satisfy relations (6)–(8).

Proof. *Necessity.* If $\tilde{A} \in \mathcal{P}_s^1(A)$, then $\tilde{A} \in \mathcal{A}(\dot{A})$, where $\dot{A} := A \upharpoonright \mathfrak{D}$ and $\mathfrak{D}$ is defined according to (1). Conditions (9)–(11) and relations (12) and (13) are satisfied by virtue of (5)–(8). In particular, relation (10) follows from (7) and (8). Indeed, according to (7), we obtain

$$b_{\bar{z}_0} = b_{z_0} + (z_0 - \bar{z}_0)(\eta_{z_0}, \eta_{z_0}).$$

Hence, by virtue of (8), we get

$$\overline{b}_{z_0} - b_{z_0} = -2i \operatorname{Im} b_{z_0} = 2i \operatorname{Im} z_0,$$

i.e., $\operatorname{Im} b_{z_0} = -\operatorname{Im} z_0$; here, the vector η_{z_0} is normalized to 1 without loss of generality.

Sufficiency. Let us prove that the right-hand side of (11) defines a self-adjoint operator $\tilde{A} \in \mathcal{P}_s^1(A)$. For this purpose, we consider the operator function

$$\tilde{R}(z) = (A - z)^{-1} + b_z^{-1}(\cdot, \eta_{\bar{z}})\eta_z, \tag{14}$$

where η_z and b_z are defined by (12) and (13), and prove that this function is the resolvent of a self-adjoint operator $\tilde{A} \in \mathcal{P}_s^1(A)$, i.e., $(\tilde{A} - z)^{-1} = \tilde{R}(z)$.

First, we verify that $\tilde{R}(z)$ is a pseudoresolvent [20, p. 533], i.e., that it satisfies the Hilbert identity

$$\tilde{R}(z) - \tilde{R}(\xi) = (z - \xi)\tilde{R}(z)\tilde{R}(\xi), \qquad \operatorname{Im} z, \operatorname{Im} \xi \neq 0. \tag{15}$$

Taking (14) into account, we rewrite (15) in the form

$$R(z) + b_z^{-1}(\cdot, \eta_{\bar{z}})\eta_z - R(\xi) - b_\xi^{-1}(\cdot, \eta_{\bar{\xi}})\eta_\xi = (z - \xi)\Big[R(z) + b_z^{-1}(\cdot, \eta_{\bar{z}})\eta_z\Big] \cdot \Big[R(\xi) + b_\xi^{-1}(\cdot, \eta_{\bar{\xi}})\eta_\xi\Big], \tag{16}$$

where $R(z) = (A - z)^{-1}$. Using the Hilbert identity for the self-adjoint operator A, we get

$$\begin{aligned} & b_z^{-1}(\cdot, \eta_{\bar{z}})\eta_z - b_\xi^{-1}(\cdot, \eta_{\bar{\xi}})\eta_\xi \\ & \qquad = (z - \xi) b_\xi^{-1}(\cdot, \eta_{\bar{\xi}}) R(z)\eta_\xi + (z - \xi) b_z^{-1}\big(\cdot, R(\xi)\eta_{\bar{z}}\big)\eta_z + (z - \xi) b_z^{-1} b_\xi^{-1}(\cdot, \eta_{\bar{\xi}})(\eta_\xi, \eta_{\bar{z}})\eta_z. \end{aligned} \tag{17}$$

By virtue of (12), we obtain

$$\eta_z - \eta_\xi = (z-\xi)(A-z)^{-1}\eta_\xi, \qquad \operatorname{Im} z, \operatorname{Im}\xi \neq 0.$$

Therefore, relation (17) reduces to the form

$$0 = b_\xi^{-1}(\cdot, \eta_{\bar\xi})\eta_z - b_z^{-1}(\cdot, \eta_{\bar\xi})\eta_z + (z-\xi)b_z^{-1}b_\xi^{-1}(\cdot, \eta_{\bar\xi})(\eta_\xi, \eta_{\bar z})\eta_z. \tag{18}$$

On the other hand, the right-hand side of (18) is equal to zero by virtue of (13). Thus, identity (15) is true.

The pseudoresolvent $\tilde R(z)$ is the resolvent of a certain densely defined closed operator (see [20, p. 533] and Theorem 7.7.1 in [21]) if and only if $\operatorname{Ker}\tilde R(z_0) = \{0\}$ for at least one point $z_0 \in \mathbb{C}$, $\operatorname{Im} z_0 \neq 0$. For all $0 \neq f \perp \eta_{\bar z_0}$, by virtue of (11) we get

$$\tilde R(z_0)f = (A-z_0)^{-1}f \neq 0.$$

For the vector $\eta_{\bar z_0}$, we have

$$\tilde R(z_0)\eta_{\bar z_0} = (A-z_0)^{-1}\eta_{\bar z_0} + b_{z_0}^{-1}\|\eta_{\bar z_0}\|^2\eta_{z_0} \neq 0$$

because $(A-z_0)^{-1}\eta_{\bar z_0} \in \mathfrak{D}(A)$, and $\eta_{z_0} \notin \mathfrak{D}(A)$ by virtue of (9). Hence, $\tilde R(z) = (\tilde A - z)^{-1}$ is the resolvent of the closed operator $\tilde A$ in $\mathcal{H}$. In fact, $\tilde A$ is a self-adjoint operator. To verify this, it is necessary to prove (see Theorem 7.7.3 in [21] and [20, p. 533]) that

$$\left(\tilde R(z)\right)^* = \tilde R(\bar z). \tag{19}$$

Equality (19) is valid because relation (10) yields (8), which, in turn, yields

$$\left(\tilde R(z)\right)^* = (A-z)^{-1} + b_{\bar z}^{-1}(\cdot, \eta_z)\eta_{\bar z} = \tilde R(\bar z).$$

Thus, $\tilde R(z)$ is the resolvent of the self-adjoint operator $\tilde A$ satisfying relation (11). It remains to prove that the domain $\mathfrak{D}$ defined by (1) is dense in $\mathcal{H}$. Denote

$$\mathfrak{M}_{z_0} := (\tilde A - z_0)\mathfrak{D} = (A-z_0)\mathfrak{D}. \tag{20}$$

It follows from (11) that $\mathfrak{M}_{z_0}^{\perp} = \mathfrak{N}_{z_0}$. Let $\varphi \perp \mathfrak{D}$. Then, by virtue of (20), for all $f \in \mathfrak{D}$, $f = (A-z_0)^{-1}h$, $h \in \mathfrak{M}_{z_0}$, we get

$$0 = (\varphi, f) = \left(\varphi, (A-z_0)^{-1}h\right) = \left((A-\bar z_0)^{-1}\varphi, h\right).$$

This means that $(A-\bar{z}_0)^{-1}\varphi \in \mathfrak{N}_{z_0}$. However, according to (9), this is possible only for $\varphi = 0$. Thus, we have proved that $\tilde{A} \in \mathcal{P}_s^1(A)$.

Theorem 3 is proved.

In the case $n = 1$, Theorem 1 can be reformulated as follows (cf. Theorem 2 in [19]):

Theorem 4. *For an arbitrary self-adjoint unbounded operator A in the Hilbert space $\mathcal{H}$, there exists a uniquely defined purely singularly perturbed operator $\tilde{A} \in \mathcal{P}_s^1(A)$ that solves the problem*

$$\tilde{A}\psi = \lambda\psi \tag{21}$$

for any preassigned vector $\psi \in \mathcal{H} \setminus \mathfrak{D}(A)$ and arbitrary number $\lambda \in \mathbb{R}^1$.

Proof. We fix $z_0 \in \mathbb{C}$, $\operatorname{Im} z_0 \neq 0$, and set

$$\eta_{z_0} := (A-\lambda)(A-z_0)^{-1}\psi, \tag{22}$$

$$b_{z_0} := (\lambda - z_0)(\psi, \eta_{\bar{z}_0}). \tag{23}$$

For arbitrary $z \in \mathbb{C}$, $\operatorname{Im} z \neq 0$, we define functions η_z and b_z according to formulas (12) and (13). Using a functional calculus for the operator A, we get

$$\eta_z = (A-\lambda)(A-z)^{-1}\psi = \psi + (z-\lambda)(A-z)^{-1}\psi, \tag{24}$$

$$b_z := (\lambda - z)(\psi, \eta_{\bar{z}}). \tag{25}$$

Consider an operator function $\tilde{R}(z)$ of the form (14). Using Theorem 3, we verify that this function is the resolvent of a self-adjoint operator. For this purpose, it suffices to prove that the functions η_z and b_z satisfy relations (6)–(8). Equation (6) can easily be verified by using (22) and (24). Equality (7) can also be immediately established using the Hilbert identity for the resolvent of the operator A. If follows from (24) and (25) that

$$\bar{b}_z = (\lambda - \bar{z})(\eta_{\bar{z}}, \psi) = (\lambda - \bar{z})\Big((A-\lambda)(A-\bar{z})^{-1}\psi, \psi\Big) = b_{\bar{z}},$$

i.e., relation (8) is also true. Hence, $\tilde{R}(z) = (\tilde{A} - z)^{-1}$, where $\tilde{A}$ is a self-adjoint operator in $\mathcal{H}$. The fact that $\tilde{A}$ belongs to $\mathcal{P}_s^1$ can be established as follows: We set $\mathfrak{N}_{z_0} := \{c\eta_{z_0}\}_{c\in\mathbb{C}}$. The condition $\mathfrak{N}_{z_0} \cap \mathfrak{D}(A) = \{0\}$ follows from representation (24), and

$$\eta_z = \psi + (z-\lambda)(A-z)^{-1}\psi \notin \mathcal{D}(A)$$

because $\psi \notin \mathcal{D}(A)$. Equality (10) is a consequence of relation (13) and the self-adjointness of $\tilde{A}$ if the vector ψ is normalized so that $\|\eta_{z_0}\| = 1$. By virtue of Theorem 3, we have $\tilde{A} \in \mathcal{P}_s^1(A)$. The resolvent of this operator has the form

$$(\tilde{A}-z)^{-1} = (A-z)^{-1} + \frac{1}{(\lambda-z)(\psi,\eta_{\bar{z}})}(\cdot,\eta_{\bar{z}})\eta_z, \tag{26}$$

where η_z is defined by the vector ψ according to (24).

By virtue of (24), relation (26) yields

$$(\tilde{A}-z)^{-1}\psi = (A-z)^{-1}\psi + \frac{1}{\lambda-z}\eta_z = (A-z)^{-1}\psi + \frac{1}{\lambda-z}\Big(\psi + (z-\lambda)(A-z)^{-1}\psi\Big) = \frac{1}{\lambda-z}\psi.$$

Hence, the operator $\tilde{A}$ solves problem (21).

The operator $\tilde{A}\in\mathcal{P}_s^1$ that solves problem (21) is unique because representation (11), together with the condition

$$(\tilde{A}-z_0)^{-1}\psi = \frac{1}{\lambda-z_0}\psi,$$

uniquely fixes the number b_{z_0} and the vector η_{z_0} (to within the phase factor $e^{-\theta}$, $0\le\theta<2\pi$).

3. Proof of Theorem 1

We prove Theorem 1 by induction with the use of Theorem 4. By analogy with (14), we introduce an operator function $R_1(z)$, changing the notation

$$R_1(z) = (A-z)^{-1} + b_1^{-1}(z)\big(\cdot,\eta_1(\bar{z})\big)\eta_1(z), \qquad \operatorname{Im} z \ne 0, \tag{27}$$

where $R_1(z)\equiv\tilde{R}(z)$ [see (26)] is the resolvent of the operator $A_1 = \tilde{A}$ and [see (24) and (25)]

$$\eta_1(z) \equiv \eta_z = (A-\lambda_1)(A-z)^{-1}\psi_1, \tag{28}$$

$$b_1(z) \equiv b_z = (\lambda_1-z)\big(\psi_1,\eta_1(\bar{z})\big) \tag{29}$$

with $\psi=\psi_1$ and $\lambda=\lambda_1$. According to the proof of Theorem 4, the operator function

$$\tilde{R}(z) := R_2(z) = R_1(z) + b_2^{-1}(z)\big(\cdot,\eta_2(\bar{z})\big)\eta_2(z),$$

where $\eta_2(z)$ and $b_2(z)$ are defined by formulas (28) and (29) with λ_2, ψ_2, and A_1 instead of λ_1, ψ_1, and A, respectively, is the resolvent of the unique operator $A_2\in\mathcal{P}_s^1(A_1)$ that solves the problem $A_2\psi_2 = \lambda_2\psi_2$ only if $\psi_2\notin\mathfrak{D}(A_1)$. The last fact follows from condition (4). Indeed, using relation (27), we obtain a description of the domain of definition of the operator A_1, namely

$$\mathfrak{D}(A_1) = \big\{h\in\mathcal{H}:\ h = f + c(z)\eta_1(z),\ f\in\mathfrak{D}(A)\big\},$$

where

$$c(z) = b_1^{-1}(z)\big((A-\lambda_1)f, \psi_1\big).$$

If we now assume that

$$\psi_2 = f + c(z)\eta_1(z),$$

then, by virtue of the equality

$$\eta_1(z) = \psi_1 + (z-\lambda_1)(A-z)^{-1}\psi_1,$$

this means that

$$\psi_2 - c(z)\psi_1 = f + (z-\lambda_1)c(z)(A-z)^{-1}\psi_1 \in \mathfrak{D}(A),$$

which contradicts condition (4). Hence, $\psi_2 \notin \mathfrak{D}(A_1)$ and $A_2 \in \mathcal{P}_s^1(A_1)$. Let us verify that A_2 solves the problem $A_2\psi_1 = \lambda_1\psi_1$. Indeed, by virtue of the equality $A_1\psi_1 = \lambda_1\psi_1$, it follows from (27) that

$$(A_2-z)^{-1}\psi_1 = \frac{1}{z-\lambda_1}\psi_1$$

because, by virtue of the fact that $\psi_1 \perp \psi_2$, we have

$$\big(\psi_1, \eta_2(\bar{z})\big) = \Big((A_1-\lambda_2)\psi_1, (A_1-z)^{-1}\psi_2\Big) = 0.$$

It is easy to verify that analogous reasoning is valid at an arbitrary kth step, $1 < k \le n$. By induction, the operator function

$$\tilde{R}(z) \equiv R_n(z) = (A_{n-1}-z)^{-1} + b_n^{-1}(z)\big(\cdot, \eta_n(\bar{z})\big)\eta_n(z),$$

where $\eta_n(z)$ and $b_n(z)$ are defined according to formulas (28) and (29) with ψ_n, λ_n, and the operator A_{n-1}, is the resolvent of a self-adjoint operator $A_n \in \mathcal{P}_s^1(A_{n-1})$ that solves problem (3).

It remains to prove that A_n belongs to $\mathcal{P}_s^n(A)$ and is unique.

By construction, we have

$$(A_n - z)^{-1} = (A-z)^{-1} + B_n(z), \tag{30}$$

where $\operatorname{rank} B_n(z) = n$. Indeed,

$$B_n(z) = \sum_{k=1}^{n} b_k^{-1}(z)\big(\cdot, \eta_k(\bar{z})\big)\eta_k(z), \tag{31}$$

where $b_k(z)$ and $\eta_k(z)$ are defined by formulas (28) and (29) [or (24) and (25)] with ψ_k, λ_k, and the operator A_{k-1}. One can easily verify that, by virtue of (4), all vectors $\eta_k(z)$ are linearly independent and do not belong to $\mathfrak{D}(A)$. Therefore, by virtue of Theorem A1 in [1], the domain

$$\mathfrak{D} = (A-z)^{-1}\,\mathrm{Ker}\,B_n(z) = (A_n - z)^{-1}\,\mathrm{Ker}\,B_n(z)$$

is dense in $\mathcal{H}$, and the symmetric operator

$$\dot{A} = A \upharpoonright \mathfrak{D} = A_n \upharpoonright \mathfrak{D}$$

has the deficiency indices

$$\mathrm{n}^+(\dot{A}) = \mathrm{n}^-(\dot{A}) = n.$$

Thus, $A_n \in \mathcal{P}_s^n(A)$. The uniqueness of A_n is a consequence of (30) and (31) because, on the set of n linearly independent vectors ψ_i (note that $\mathrm{span}\{\psi_i\} \cap \mathrm{Ker}\,B_n(z) = \{0\}$), the operator $B_n(z)$ has fixed values, namely

$$B_n(z)\psi_i = \frac{1}{\lambda_i - z}\psi_i - (A-z)^{-1}\psi_i, \qquad i = 1, \ldots, n,$$

and the resolvent $R_n(z)$ coincides with $R(z)$ on the subspace $\mathrm{Ker}\,B_n(z)$.

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