Additive spectral problem (brief survey and some recent results)

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Outlines

Let $A = A^*$, $B = B^*$ and $C = C^*$ be hermitian $n \times n$ matrices. For hermitian matrix X we denote its eigenvalues by

$$\sigma(X) : \sigma_1(X) \geq \sigma_2(X) \geq \ldots \geq \sigma_n(X),$$

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$$\sigma_{i+j-1}(A+B) \leq \sigma_i(A) + \sigma_j(B), \quad i+j \leq n+1,$$
$$\sigma_{i+j-n}(A+B) \geq \sigma_i(A) + \sigma_j(B), \quad i+j \geq n+1,$$
$$\sum_{i \leq p} \sigma_i(A+B) \leq \sum_{j \leq p} \sigma_j(A) + \sum_{k \leq p} \sigma_k(B),$$
$$\sum_{i \in I} \sigma_i(A+B) \leq \sum_{j \in I} \sigma_j(A) + \sum_{k \leq p} \sigma_k(B),$$

where I is any subset of $\{1, 2, \ldots, n\}$ of cardinality p.

In 1962 Alfred Horn found series of necessarily inequalities

$$\sum_{k\in K}\sigma_k(A+B)\leq\sum_{i\in I}\sigma_i(A)+\sum_{j\in J}\sigma_j(B),$$

for some triple of subsets $I, J, K \subset \{1, 2, \ldots, n\}$.

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Conjecture 1.1 (Alfred Horn)

These inequalities form complete list of the restrictions on the spectrums

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A. Knutson, T. Tao, *The honeycomb model of GL(n,C) tensor products. I. Proof of the saturation conjecture*, 1999.

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and by Alexander Klyachko

A. A. Klyachko, *Stable bundles, representation theory and Hermitian operators*, 1999.

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and many others, see
William Fulton, *Eigenvalues, invariant factors, highest weights, and Schubert calculus*, 2000.

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there exists a solution for

$$A_1 + \ldots + A_n = \gamma I,$$
$$A_i = A_i^*, \text{ with given } \quad \sigma(A_i).$$

Let H be separable Hilbert space.
Let $M_i = \{0 = \alpha_0^{(i)} < \alpha_1^{(i)} < \ldots < \alpha_{m_i}^{(i)}\} \subset \mathbb{R}_+$ and $\gamma \in \mathbb{R}_+$.

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$$\mathcal{A}_{M_1,M_2,\dots,M_n;\gamma} = \mathbb{C}\langle a_1,\dots,a_n \big| a_i = a_i^*, (a_i - \alpha_0^{(i)})\dots(a_i - \alpha_{m_i}^{(i)}) = 0, \\ a_1 + a_2 + \dots + a_n = \gamma e\rangle.$$

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$$\mathcal{P}_{M_1,M_2,\dots,M_n;\gamma} = \mathbb{C}\langle p_1^{(1)},\dots,p_{m_n}^{(n)} \mid p_i^{(k)} = p_i^{(k)2} = p_i^{(k)*}, \\ \sum_{i=1}^{n}\sum_{k=1}^{m_i}\alpha_k^{(i)} p_k^{(i)} = \gamma e, p_j^{(i)} p_k^{(i)} = 0\rangle.$$

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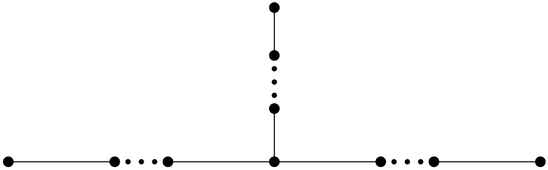
- star-shaped graph Γ



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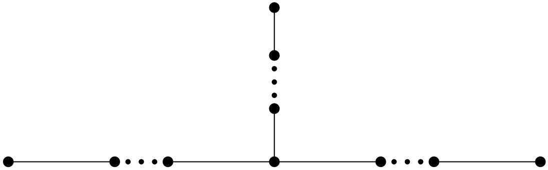

- weight $\chi : \Gamma \to \mathbb{R}_+$, $\chi = (\alpha_1^{(1)}, \ldots, \alpha_{m_1}^{(1)}; \ldots; \alpha_1^{(n)}, \ldots, \alpha_{m_n}^{(n)})$

For example if

$$\mathcal{P}_{(\alpha_1^{(1)},\ldots,\alpha_{m_1}^{(1)};\alpha_1^{(2)},\ldots,\alpha_{m_2}^{(2)};\alpha_1^{(3)},\ldots,\alpha_{m_3}^{(3)}),\gamma} =$$

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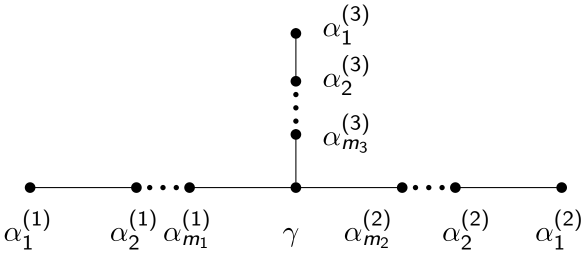

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- for each weight χ to describe the set $\Sigma_{\Gamma,\chi} =$ (all possible γ for which there are representations of $\mathcal{P}_{\Gamma,\chi,\gamma}$);
- for each appropriated pair $(\chi;\gamma)$ to describe all irreducible $*$-representation of $\mathcal{P}_{\Gamma,\chi,\gamma}$.

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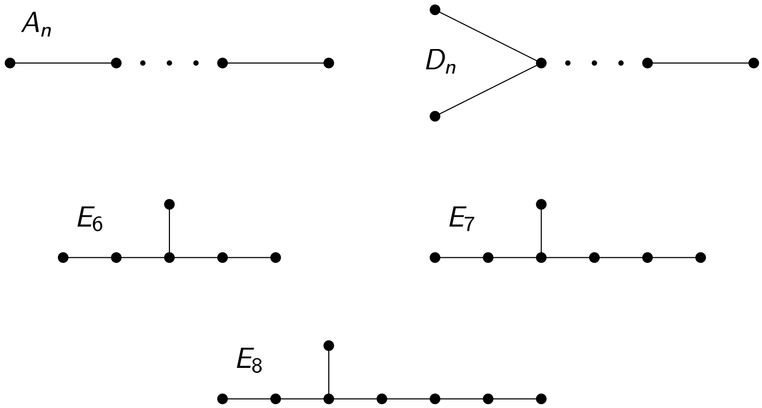

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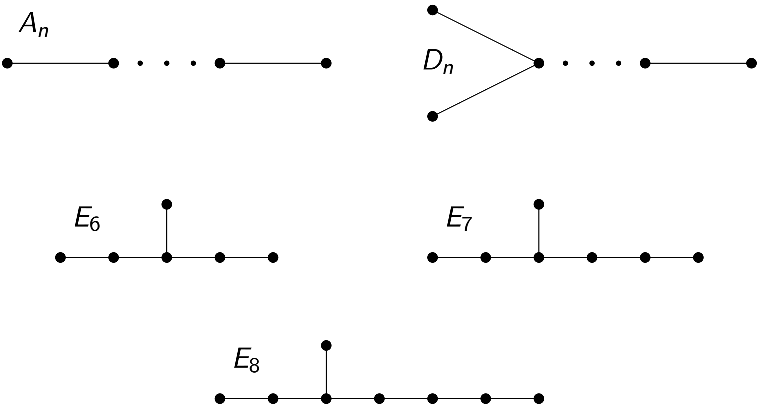

then $\mathcal{P}_{\Gamma,\chi,\gamma}$ is finite dimensional, and complete answers for posed problem are known for all possible weights χ;

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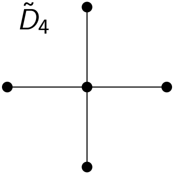
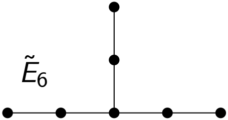
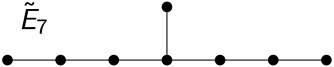
$\tilde{D}_4$



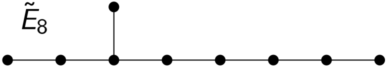
$\tilde{E}_6$



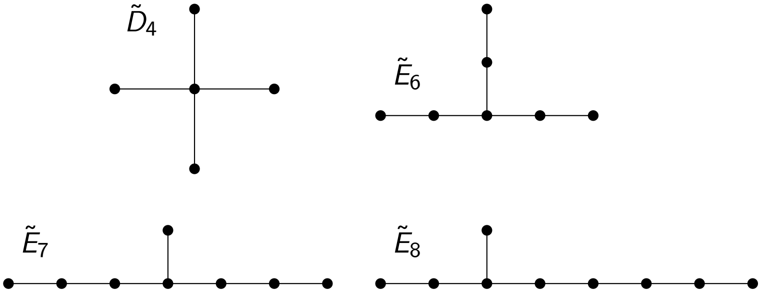
$\tilde{E}_7$



$\tilde{E}_8$



if Γ is extended Dynkin graph



then the algebra $\mathcal{P}_{\Gamma,\chi,\gamma}$ is infinite dimensional and of polynomial growth

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M. A. Vlasenko, A. S. Mellit, and Yu. S. Samoilenko, *On algebras generated with linearly dependent generators that have given spectra*, 2005
S. Albeverio, V. Ostrovskiy, Yu. Samoilenko, *Journal of Algebra*, 2006.

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$$\xi = \xi_r \ldots \xi_1$$

of arrows s.t. $t(\xi_p) = s(\xi_{p+1}), \quad 1 \leq p < r$.
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Path algebra $\mathbb{C}Q$ of a quiver Q is algebra spanned by all paths in Q with multiplication given by composition

$$xy = \begin{cases} \text{obvious composition (if } t(y) = s(x)) \\ 0 \qquad\qquad\qquad\qquad\qquad \text{(otherwise)} \end{cases}$$

A *representation* X of a quiver Q is given by a vector space X_i for each vertex $i \in Q_0$ and linear operator $X_\rho : X_{s(\rho)} \to X_{t(\rho)}$ for each arrow $\rho \in Q_1$.

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A *morphism* $\theta : X \to X'$ is given by linear maps $\theta_i : X_i \to X_i$ for each $i \in Q_0$, satisfying $X'_\rho \theta_{s(\rho)} = \theta_{t(\rho)} X_\rho$ for each $\rho \in Q_1$.

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Proposition 1 (see for example Crawley-Boevey)

Representations of quiver Q $\Leftrightarrow$ left $\mathbb{C}Q$-modules.

Theorem 1 (Gabriel $\sim$ 1973)

The classification of all indecomposable representations of Q is

- **finite** *problem, if Q is Dynkin quiver;*
- **tame** *problem, if Q is an extenden Dynkin quiver;*
- **wild** *problem, in all other cases*

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The deformed preprojective algebra of weight $\lambda \in \mathbb{C}^{Q_0}$ is

$$\Pi^\lambda(Q) = \mathbb{C}\overline{Q} / \left(\sum_{a \in Q} [a, a^*] - \sum_{i \in Q_0} \lambda_i e_i \right).$$

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Theorem 2

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There also exist interconnection between $\mathcal{P}_{\Gamma,\chi,\gamma}$ and orthoscalar representation of quivers.

Kruglyak S. A., Roiter A. V. *Locally scalar representations of graphs in the category of Hilbert spaces*, 2004.

A powerfull tool to investigate representations of quiver are Coxeter functors which allow to build series of representations starting from simplest representation

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Similar functors were built for algebras $\mathcal{P}_{\Gamma,\chi,\gamma}$ by Kruglyak and Roiter. Namely there are exist two functors linear S (which generate representation in the same space) and hyperbolical T (which, strictly speaking, build representation in new space).

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$$S : (\chi; \gamma) \longmapsto (\chi'; \gamma'),$$
$$\chi' = (\alpha^{(1)}_{m_1} - \alpha^{(1)}_{m_1-1}, \dots, \alpha^{(1)}_{m_1}; \dots; \alpha^{(n)}_{m_n} - \alpha^{(n)}_{m_n-1}, \dots, \alpha^{(n)}_{m_n}),$$
$$\gamma' = \alpha^{(1)}_{m_1} + \dots + \alpha^{(n)}_{m_n} - \gamma;$$
$$T : (\chi; \gamma) \longmapsto (\chi''; \gamma),$$
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Now we are going to study the dynamic of Coxeter functors for the case where Γ is an extenden Dynkin graph.

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$$\omega_{\tilde{D}_4}(\chi) = \frac{1}{2}(\alpha_1^{(1)} + \alpha_1^{(2)} + \alpha_1^{(3)} + \alpha_1^{(4)}),$$
$$\omega_{\tilde{E}_6}(\chi) = \frac{1}{3}(\alpha_1^{(1)} + \alpha_2^{(1)} + \alpha_1^{(2)} + \alpha_2^{(2)} + \alpha_1^{(3)} + \alpha_2^{(3)}),$$
$$\omega_{\tilde{E}_7}(\chi) = \frac{1}{4}(\alpha_1^{(1)} + \alpha_2^{(1)} + \alpha_3^{(1)} + \alpha_1^{(2)} + \alpha_2^{(2)} + \alpha_3^{(2)} + 2\alpha_1^{(3)}),$$
$$\omega_{\tilde{E}_8}(\chi) = \frac{1}{6}(\alpha_1^{(1)} + \alpha_2^{(1)} + \alpha_3^{(1)} + \alpha_4^{(1)} + \alpha_5^{(1)} + 2\alpha_1^{(2)} + 2\alpha_2^{(2)} + 3\alpha_1^{(3)}).$$

these hyperplanes are invariant in the sense

$$S : (\chi; \omega(\chi)) \longmapsto (\chi'; \omega(\chi')),$$
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V. L. Ostrovskyi, *Special characters on star graphs and representations of $*$-algebras*

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Proposition 2

The action of $(ST)^k$ functor on the pair $(\chi;\gamma)$ could be written in the following way:

$$(ST)^k(\chi;\gamma) = (f_{1,k}(\chi) - (\omega_\Gamma(\chi)-\gamma)f_{2,k}(\chi_\Gamma);\psi_{1,k} - (\omega_\Gamma(\chi)-\gamma)\psi_{2,k}),$$

where the characters $f_{1,k}(\chi)$ and $f_{2,k}(\chi_\Gamma)$, and the numbers $\psi_{1,k}$ and $\psi_{2,k}$ satisfy the following properties:

(i) *if $k_1 \equiv k_2 \pmod{p_\Gamma(p_\Gamma-1)}$ then $f_{1,k_1}(\chi) = f_{1,k_2}(\chi)$ and $\psi_{1,k_1} = \psi_{1,k_2}$;*

(ii) *the components of $f_{2,k}(\chi_\Gamma)$ and the numbers $\psi_{2,k}$ are defined in the following way:*

$$f_{2,k}(\chi_\Gamma)_i^{(j)} = \left[\frac{(\chi_\Gamma)_i^{(j)}}{p_\Gamma-1}k\right], \quad \psi_{2,k} = \left[\frac{p_\Gamma}{p_\Gamma-1}k\right];$$

(iii) $f_{1,p_\Gamma(p_\Gamma-1)k} = \chi$, $f_{2,p_\Gamma(p_\Gamma-1)k} = kp_\Gamma\chi_\Gamma$, $\psi_{1,k}=\gamma$, *and* $\psi_{2,k} = kp_\Gamma^2$.

Theorem 3 (K.Y. 2006)

The set $\Sigma_{\widetilde{D}_4,\chi}$ is a union of the following sets

$$\Sigma_1 = \left\{ \frac{\alpha}{2} - \frac{\alpha}{2(4n-1)} \;\middle|\; n < \frac{\alpha_4}{4\alpha_4 - \alpha},\ n < \frac{\alpha - \alpha_1}{\alpha - 4\alpha_1},\ n \in \mathbb{N} \right\},$$

$$\Sigma_2^i = \left\{ \frac{\alpha}{2} - \frac{\alpha_i}{2n} \;\middle|\; n < \frac{\alpha_i}{2\alpha_i + 2\alpha_4 - \alpha},\ n < \frac{\alpha_i}{\alpha_i - \alpha_1},\ n < \frac{\alpha_i}{4\alpha_i - \alpha},\ n \in \mathbb{N} \right\},$$

$$\Sigma_3 = \left\{ \frac{\alpha}{2} - \frac{\alpha - 2\alpha_1}{2(2n+1)} \;\middle|\; n < \frac{\alpha - \alpha_1}{\alpha - 4\alpha_1},\ n < \frac{\alpha_2 + \alpha_3}{2(\alpha_4 - \alpha_1)},\ n(4\alpha_i - \alpha) < \alpha_i \right\},$$

$$\Sigma_4 = \left\{ \frac{\alpha}{2} - \frac{\alpha}{2(4n+1)} \;\middle|\; n < \frac{\alpha - \alpha_4}{4\alpha_4 - \alpha},\ n < \frac{\alpha_1}{\alpha - 4\alpha_1},\ n \in \mathbb{N} \right\},$$

$$\Sigma_5^i = \left\{ \frac{\alpha}{2} - \frac{\alpha - 2\alpha_i}{2(2n+1)} \;\middle|\; n < \frac{\alpha_1}{\alpha - 2\alpha_i - 2\alpha_1},\ n < \frac{\alpha_i}{\alpha - 4\alpha_i},\ n < \frac{\alpha - \alpha_4 - \alpha_i}{2(\alpha_4 - \alpha_i)} \right\},$$

$$\Sigma_\infty = \left\{ \frac{\alpha}{2} - \frac{\alpha - 2\alpha_4}{2(2n-1)} \;\middle|\; n \in \mathbb{N} \right\},$$

$$\Sigma_0 = \left\{ \frac{\alpha}{2} - \frac{\alpha_1}{2n} \;\middle|\; n < \frac{\alpha_1}{\alpha_1 + \alpha_4 - \alpha_2 - \alpha_3},\ n \in \mathbb{N} \right\}.$$

There are exact formulas for representation of algebras $\mathcal{P}_{\tilde{D}_4,\chi,\gamma}$. In other words the description of all irreducible quadruples of projections s.t.

$$\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = \gamma I.$$

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A. Yusenko, *On quadruples of projections that satisfy linear relation*, 2009 (to appear).

Let Γ be extended Dynkin graph and χ be the weight on Γ. Next few statements describes structure properties of $\Sigma_{\Gamma,\chi}$.

K.Y. *On existence of $*$-representations of certain algebras related to extended Dynkin graphs*

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The set $\Sigma_{\Gamma,\chi}$ is infinite if and only if all components of weight satisfies two conditions: $\chi_i < \omega_\Gamma(\chi)$ and $\chi_i' < \omega_\Gamma(\chi')$.

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Corollary 1

Let χ be the weight on Γ such that the conditions of previous theorem are satisfied. Then there is a representation of algebra $\mathcal{P}_{\Gamma,\chi,\omega_\Gamma(\chi)}$ on hyperplane $\gamma = \omega_\Gamma(\chi)$

Theorem 5

If the set $\Sigma_{\Gamma,\chi}$ is infinite then it contains the only limit point.

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Corollary 2

Let Γ be extended Dynkin graph. The algebras $\mathcal{P}_{\Gamma,\chi,\gamma}$ are of **tame** *representation type when $\chi_i < \omega_\Gamma(\chi)$ and $\chi'_i < \omega_\Gamma(\chi')$ otherwise they are of* **finite** *representation type.*

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Thank you very much for your attention.