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Ansatzes and exact solutions for nonlinear Schrödinger and wave equations

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Розглядається ефективний та простий підхід до побудови анзаців для нелійнійного рівняння Шредінгера та нелінійного хвильового рівняння, а також умови їх редукції до звичайних диференціальних рівнянь. Представлено повний опис анзаців деякого типу. Обговорюється зв'язок між розв'язками та лієвською й умовною симетрією цих рівнянь.

We consider construction of ansatzes for nonlinear Schrödinger and wave equations, and conditions of their reduction to ordinary differential equations. Complete description of ansatzes of certain types is presented. The relationship between solutions and both Lie and conditional symmetry of these equations is discussed.

1. Introduction. We are going to use here a straightforward method for construction of exact solutions for partial differential equations (PDE) which sometimes allows to obtain a wider class of exact solutions than the classical Lie method of similarity reduction [1–3]. The idea of this approach focuses on a notion of ansatz – a special substitution which reduces a PDE to another PDE with less number of independent variables or to an ordinary differential equation (ODE) [1, 4, 5]. The Lie method provides ansatzes using subalgebras of an invariance algebra of an equation [1, 2, 3, 6]. We tried to search for ansatzes directly, substituting some general form of ansatz to an equation and then considering conditions of its reduction. This technique is used intensively for two-dimensional equations (see, e.g., [7–13]), and we succeeded to apply it for a four-dimensional equation. The general idea is obvious but the main difficulties here are investigation of compatibility and solution of reduction conditions, which present nontrivial problems.

2. Nonlinear Schrödinger equation. First let us consider the nonlinear Schrödinger equation

$$2iu_t + \triangle u - uF(|u|) = 0. \tag{1}$$

Here u is a complex-valued function, $u = u(t, \vec{x})$, $\vec{x}$ is a n-dimensional vector of space variables, $|u| = \sqrt{uu^*}$, an asterisk designates complex conjugation, $\triangle u = \partial^2 u/\partial x_a^2$, $a = 1, \ldots, n$.

Eq.(1) with an arbitrary function F is invariant under the Galilei algebra with basis operators

$$\begin{aligned} &\partial_t, \quad \partial_a, \quad J_{ab} = x_a\partial_b - x_b\partial_a, \quad M = i(u\partial_u - u^*\partial_{u^*}), \\ &G_a = t\partial_a + ix_a(u\partial_u - u^*\partial_{u^*}), \quad a, b = 1, \ldots, n. \end{aligned} \tag{2}$$

Solutions obtained from the algebra (2) by means of the Lie method are well-known [14–16] and all of them are of the form

$$u = \exp\{if(t, \vec{x})\}\, \varphi(\omega). \tag{3}$$

Such form of a substitution is the most general reducing an arbitrary nonlinear equation (1) to an ODE. The expression (3) where f, ω are some unknown real functions of t and $\vec{x}$ will be an ansatz for Eq.(1) if its substitution reduces (1) to an ODE for a complex function depending on the new variable ω only. Whence we get conditions on the functions f and ω:

$$\begin{aligned} &2f_t + f_a f_a = S(\omega), \quad \triangle f = T(\omega), \\ &\omega_t + f_a\omega_a = X(\omega), \quad \triangle\omega = Y(\omega), \quad \omega_a\omega_a = Z(\omega), \end{aligned} \tag{4}$$

where S, T, X, Y, Z are arbitrary smooth functions.

For $n = 2$, $n = 3$ we had found the general solution of the system (4) up to equivalency of substitutions (3).

For the purpose of reduction of Eq.(1) it is sufficient to consider the system (4) only up to equivalence of the ansatzes (3). We shall call ansatzes equivalent if they lead to the same solutions of the equation.

We deal here with real functions f and ω, so $Z(\omega)$ in (4) must be nonnegative. Whence we can reduce the equation $\omega_a\omega_a = Z(\omega)$ by local transformations to the same form with $Z(\omega) = 0$ or $Z(\omega) = 1$.

1) $Z(\omega) = 0$. In this case $\omega_a = 0$, $\omega = \omega(t)$ and we can put $\omega = t$. The system (4) can be written as

$$2f_t + f_a f_a = S(t), \quad \triangle f = T(t).$$

It is evident that the ansatzes of form (3) are equivalent up to transformations $f \to f + r(\omega)$, so we can put $S(t) = 0$. We come to the system

$$2f_t + f_a f_a = 0, \quad \triangle f = T(t), \tag{5}$$

and the following theorem gives a necessary condition of its compatibility.

Theorem 1. *The system (5) can be compatible only if*

$$T(t) = \theta'(t)/\theta(t), \quad \theta^{(n+1)} = 0.$$

Proof of this theorem can be carried out using differential consequences of (5) and the Hamilton-Cayley theorem. It is rather cumbersome, and its complete version can be found in [17].

2) $Z(\omega) = 1$. It had been established in [18] that when $n = 3$, $\triangle\omega = N/\omega$, $N = 0,\ 1,\ 2$ ($N = 0, 1$ for $n = 2$). Up to equivalency of ansatzes we can put $X(\omega) = 0$.

Theorem 2. *The system of equations*

$$\begin{aligned} &2f_t + f_a f_a = S(\omega), \quad \triangle f = T(\omega), \\ &f_a\omega_a + \omega_t = 0, \quad \omega_a\omega_a = 1, \quad \triangle\omega = N/\omega, \end{aligned} \tag{6}$$

where $N = 0, 1$ with $n = 2$, $N = 0,\ 1,\ 2$ with $n = 3$ is compatible iff $T(\omega) = 0$; $S(\omega) = c_1\omega + c_2$, $N = 0$; $S(\omega) = c_1/\omega^2 + c_2$, $N = 1$; $S(\omega) = c_1$, $N = 2$; c_1, c_2 are arbitrary constants.

Theorem 3. *The system (4) is invariant with respect to the operators*

$$\partial_a, \quad J_{ab} = x_a\partial_b - x_b\partial_a, \quad \widehat{G}_a = t\partial_a + x_a\partial_f. \tag{7}$$

Thus, we can search for its general solution up to transformations generated by operators (7):

$$x_a \to \alpha_{ab}x_b + \beta_a, \quad x_a \to g_a t + x_a, \tag{8}$$

α_{ab}, β_a, g_a are constants, $\alpha_{ac}\alpha_{cb} = \delta_{ab}$ (the Kronecker symbol).

Further we adduce all solutions of the system (4), which are nonequivalent up to transformations (8).

I. $Z(\omega) = 0,\ \omega = t$:

$$1) \quad n = 3, \quad f = \frac{1}{2}\left\{\frac{x_1^2}{t + A_1} + \frac{x_2^2}{t + A_2} + \frac{x_3^2}{t + A_3}\right\}; \tag{9}$$

$$2)\quad n=2,\,3,\quad f=\frac{1}{2}\left\{\frac{x_1^2}{t+B_1}+\frac{x_2^2}{t+B_2}\right\};\tag{10}$$

$$3)\quad n=2,\,3,\quad f=\frac{x_1^2}{2t+c_1};$$

$$4)\quad n=2,\,3,\quad f=c_2x_1+c_3-\frac{1}{2}c_2^2t.$$

II. $Z(\omega)=1$:

$$1)\quad n=2,\,3,\quad \omega=x_1+at^2,\quad f=-2atx_1-\frac{4}{3}a^2t^3+bt;$$

$$2)\quad n=2,\,3,\quad \omega=(x_1^2+x_2^2)^{1/2},\quad f=c\,\tan^{-1}\frac{x_1}{x_2}+dt;$$

$$3)\quad n=3,\quad \omega=(x_1^2+x_2^2+x_3^2)^{1/2},\quad f=et.$$

Here A_i, B_i, C_i, a, b, c, d, e are arbitrary constants.

The ansatz (3) reduces Eq.(1) to the following ODE:

$$-2S(\omega)\varphi+iT(\omega)\varphi+2iX(\omega)\dot\varphi+Y(\omega)\dot\varphi+Z(\omega)\ddot\varphi=\varphi F(|\varphi|).\tag{11}$$

It follows from compatibility conditions of the system (4) that two types of Eq.(11) are possible:

1) If $\omega_a\omega_a=Z(\omega)=0$, we take $\omega=t$ and Eq.(11) will be of the form

$$i(2\dot\varphi+T(t)\varphi)=\varphi F(|\varphi|),\tag{12}$$

where $T=\displaystyle\sum_{i=1}^{m}\frac{1}{t+B_i}$, m may take values from 1 to n; or $T=0$.

Eq.(12) can be easily solved in quadratures: if $T\neq 0$ then

$$\varphi=r\exp\frac{i}{2}\left\{\sum_{i=1}^{m}\frac{x_l^2}{t+B_l}-\int F(r)dt\right\},$$

$$r=C[(t+B_1)\ldots(t+B_m)]^{1/2}$$

or if $T=0$, $f=c_1x_1+c_2-\dfrac{1}{2}c_1^2t$ then

$$\varphi=c\exp i\left\{c_1x_1-\frac{1}{2}F(c)t+c_2-c_1^2\frac{t}{2}\right\}.$$

2) If $\omega_a\omega_a = Z(\omega) = 1$ then Eq.(11) will be of the form

$$-2S(\omega)\varphi + \frac{N}{\omega}\dot{\varphi} + \ddot{\varphi} = \varphi F(|\varphi|). \tag{13}$$

Eq.(13) in general obviously cannot be solved in quadratures. Some of its particular solutions were given in [14–16].

3. Nonlinear wave equation. We can apply the results for the Schrödinger equation (1) to describe all ansatzes of the form

$$u = f(x)\varphi(\omega) \tag{14}$$

with $\omega = \alpha_\mu x_\mu,\ \alpha_\mu\alpha_\mu = 0$ for a nonlinear wave equation

$$\Box u = \lambda u^k, \tag{15}$$

where $u = u(x_0, x_1, x_2, x_3)$ is a real function; $k \neq 1$, λ are parameters; the summation over repeated Greek indices is as follows: $x_\mu x_\mu \equiv x_0^2 - x_1^2 - x_2^2 - x_3^2$.

Further for simplicity of presentation we shall take $\omega = x_0 + x_3$. In this case the ansatz (14) will reduce Eq.(15) to an ODE if $f(x)$ satisfies the following conditions:

$$\Box f = f^k T(\omega), \quad 2(f_0 - f_3) = f^k Y(\omega). \tag{16}$$

Here $Y(\omega)$ must not vanish. By means of a substitution of the form $f \to \gamma(\omega) f$ (ansatzes (14) are equivalent up to such substitutions) we can get the system (16) with $Y = 2/(1-k)$. Then from the second equation of (16) we get

$$f = \left[\Phi(\omega, x_1, x_2) + \frac{1}{2}(x_0 - x_3)\right]^{1/(1-k)}. \tag{17}$$

Substitution of (17) into the first equation of (16) gives the following system for the function Φ:

$$\Phi_{11} + \Phi_{22} = T(\omega)(1-k), \quad 2\Phi_\omega - \Phi_1^2 - \Phi_2^2 = 0.$$

Using the results for the system (4) we get solutions for different $T(\omega)$ with which the system (16) can be compatible:

$$\Phi = -\frac{1}{2}\sum_{i=1}^{m}\frac{x_i^2}{\omega + B_i}, \quad T = \frac{1}{k-1}\sum_{i=1}^{m}\frac{1}{\omega + B_i}; \quad (m = 1 \text{ or } 2)$$

$$\Phi = B_1 x_1 + B_2 + \frac{B_1^2}{2}\omega, \quad T = 0; \ B_i \text{ are constants.}$$

Now Eq.(15) can be reduced to the ODE $\varphi' \dfrac{2}{1-k} + T(\omega)\varphi = \lambda\varphi^k$, which is solvable in quadratures: e.g., let $T = \dfrac{1}{k-1}\sum\limits_{i=1}^{2} \dfrac{1}{\omega + B_i}$. Then

$$\varphi = \sqrt{\rho}\left[\frac{\lambda(1-k)^2}{2}\int \rho^{\frac{k-1}{2}} d\omega\right]^{\frac{1}{1-k}}, \quad \rho = (\omega + B_1)(\omega + B_2).$$

These results can be easily generalized for the cases when ω is a solution of the system $\Box\omega = 0$, $\omega_\mu\omega_\mu = 0$ (see e.g. [19]) or when $u = u(x_0, x_1, \ldots, x_n)$, $n > 3$.

Reduction and solutions for Eq.(15) when u is a complex function are considered in [20].

4. Relation between symmetry and reduction of partial differential equations. In general an ansatz which reduces a PDE to another PDE with less number of independent variables or to an ODE corresponds to some Q-conditional symmetry of that equation. The notion of conditional symmetry was introduced in [21], and many examples of such symmetries for considered equations are given in [7–13].

Definition. *Let us consider a PDE*

$$F(x_1, u, \underset{1}{u}, \ldots, \underset{r}{u}) = 0, \tag{18}$$

where x is a vector of independent variables, U is some function, $\underset{k}{u}$ is a set of k-th order partial derivatives. We shall say that Eq.(18) is Q-conditionally invariant with respect to a set of operators $\{Q_a = \xi^{ab}(x,u)\partial_b + \eta^a(x,u)\partial_u\}$ if the system containing Eq.(18) and the additional conditions

$$L_a = \xi^{ab}(x,u)u_b - \eta^a(x,u) = 0 \tag{19}$$

is compatible and invariant with respect to these operators.

Operators of conditional invariance can be defined up to an arbitrary multiplier,and such invariance is essential when Q_a are not proportional to some operators of Lie invariance.

It can be proved that in the case of Q-conditional invariance a solution of the system (19) gives an ansatz which will reduce Eq.(18). Very often investigation of reduction conditions or Q-conditional invariance gives more ansatzes than the classical Lie method. However, all ansatzes described above correspond to Lie symmetry operators of Eqs. (1) and (15). So we proved that ansatzes (3) and (14) yield no essential Q-conditional invariance for these equations. This fact does not disprove the idea that the direct method of reduction is more general than the classical Lie method, though it is usually more difficult to apply.

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