Reduction of Multidimensional Non-Linear D'Alembert Equations to Two-Dimensional Equations and Classes of Reduced Equations

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We study conditions of reduction of the multidimensional wave equation

$$\Box u = F(u)$$

- a system of the d'Alembert and Hamilton equations:

$$y_\mu y_\mu = r(y,z), \quad y_\mu z_\mu = q(y,z), \quad z_\mu z_\mu = s(y,z),$$

$$\Box y = R(y,z), \quad \Box z = S(y,z).$$

We prove necessary conditions for compatibility of such system of the reduction conditions.

Possible types of the reduced equations represent interesting classes of two-dimensional parabolic, hyperbolic and elliptic equations. Ansatzes and methods used for reduction of the d'Alembert (n-dimensional wave) equation can be also used for arbitrary Poincaré-invariant equations. This seemingly simple and partial problem involves many important aspects in the studies of the PDE.

We study reduction of the nonlinear d'Alembert equation

$$\Box u = F(u),$$

$$\Box \equiv \partial_{x_0}^2 - \partial_{x_1}^2 - \cdots - \partial_{x_n}^2, \quad u = u(x_0, x_1, \ldots, x_n)$$

by means of the ansatz with two new independent variables

$$u = \varphi(y, z),$$

where y, z are new variables.

n is the number of independent space variables in the initial d'Alembert equation.

We found necessary compatibility conditions for the respective reduction conditions - we proved that the reduced equations may have only a particular form.

The method of symmetry reduction does not provide exhaustive description of all exact solutions for an equation. For this reason it is interesting to look for and to develop other algorithmic methods for search of solutions.

Symmetry Reduction and Direct Method

Direct Method

Gives wider classes of solutions than the method of symmetry reduction by subalgebras of the invariance algebra

But

for majority of equations presents considerable difficulties.

And

Reduction conditions are much more difficult for investigation and solution in the case of equations containing second and/or higher derivatives for all independent variables, and for multidimensional equations.

Previous Work

A similar problem was considered earlier for an ansatz with one independent variable

$$u = \varphi(y), \tag{1}$$

where y is a new independent variable.

Compatibility analysis of the d'Alembert–Hamilton system

$$\Box u = F(u), \quad u_\mu u_\mu = f(u) \tag{2}$$

in the three-dimensional space was done by S.B. Collins.

Later - Cieciura and Grundland; Fushchych, Yehorchenko, Zhdanov; Fushchych, Zhdanov, Revenko; Zhdanov, Panchak

d'Alembert–Hamilton system (2) may be reduced by local transformations to the form

$$\Box u = F(u), \quad u_\mu u_\mu = \lambda, \quad \lambda = 0, \pm 1. \tag{3}$$

Statement 1. *For the system* (3) *(n is arbitrary) to be compatible it is necessary that the function F have the following form:*

$$F = \frac{\lambda \partial_u \Phi}{\Phi}, \quad \partial_u^{n+1}\Phi = 0.$$

Statement 2. *For the system* (3) $(u = u(x_0, x_1, x_2, x_3))$ *to be compatible it is necessary and sufficient that the function F have the following form:*

$$F = \frac{\lambda}{N(u + C)}, \quad N = 0, 1, 2, 3.$$

Necessary compatibility conditions of the system of the d'Alembert–Hamilton equations for two functions or for a complex-valued function.

Substitution of ansatz $u = \varphi(y, z)$ into equation $\Box u = F(u)$ leads to the following equation:

$$\varphi_{yy} y_\mu y_\mu + 2\varphi_{yz} z_\mu y_\mu + \varphi_{zz} z_\mu z_\mu + \varphi_y \Box y + \varphi_z \Box z = F(\varphi)$$

$$\left(y_\mu = \frac{\partial y}{\partial x_\mu}, \quad \varphi_y = \frac{\partial \varphi}{\partial y} \right),$$

whence we get a system of equations:

$$y_\mu y_\mu = r(y, z), \quad y_\mu z_\mu = q(y, z), \quad z_\mu z_\mu = s(y, z),$$

$$\Box y = R(y, z), \quad \Box z = S(y, z).$$

This system is a reduction condition for the multidimensional wave equation to a two-dimensional equation by means of such ansatz.

The system

$$y_\mu y_\mu = r(y,z), \quad y_\mu z_\mu = q(y,z), \quad z_\mu z_\mu = s(y,z),$$

$$\Box y = R(y,z), \quad \Box z = S(y,z).$$

depending on the sign of the expression $rs - q^2$, may be reduced by local transformations to one of the following types:

1) elliptic case: $rs - q^2 > 0$, $v = v(y,z)$ is a complex–valued function,

$$\Box v = V(v, v^*), \quad \Box v^* = V^*(v, v^*),$$

$$v^*_\mu v_\mu = h(v, v^*), \quad v_\mu v_\mu = 0, \quad v^*_\mu v^*_\mu = 0$$

(the reduced equation is of the elliptic type);

2) hyperbolic case: $rs - q^2 < 0$, $v = v(y,z)$, $w = w(y,z)$ are real functions,

$$\Box v = V(v,w), \quad \Box w = W(v,w),$$

$$w_\mu w_\mu = h(v,w), \quad v_\mu v_\mu = 0, \quad w_\mu w_\mu = 0$$

(the reduced equation is of the hyperbolic type);

3) parabolic case: $rs - q^2 = 0$, $r^2 + s^2 + q^2 \neq 0$, $v(y,z)$, $w(y,z)$ are real functions,

$$\Box v = V(v,w), \quad \Box w = W(v,w),$$

$$v_\mu w_\mu = 0, \quad v_\mu v_\mu = \lambda \ (\lambda = \pm 1), \quad w_\mu w_\mu = 0$$

(if $W \neq 0$, then the reduced equation is of the parabolic type);

4) first-order equations: $(r = s = q = 0)$, $y \to v$, $z \to w$

$$v_\mu v_\mu = w_\mu w_\mu = v_\mu w_\mu = 0,$$

$$\Box v = V(v,w), \quad \Box w = W(v,w)$$

.

Theorem 1. *System*

$$\Box v = V(v, v^*), \quad \Box v^* = V^*(v, v^*),$$

$$v^*_\mu v_\mu = h(v, v^*), \quad v_\mu v_\mu = 0, \quad v^*_\mu v^*_\mu = 0$$

is compatible if and only if

$$V = \frac{h(v, v^*)\partial_{v^*}\Phi}{\Phi}, \quad \partial_{v^*} \equiv \frac{\partial}{\partial v^*},$$

where Φ *is an arbitrary function for which the following condition is satisfied*

$$(h\partial_{v^*})^{n+1}\Phi = 0.$$

The function h may be represented in the form $h = \frac{1}{R_{vv^*}}$ (it is nonzero), where R is an arbitrary sufficiently smooth function, R_v, R_{v^*} are partial derivatives by the respective variables.

Then the function Φ may be represented in the form $\Phi = \sum\limits_{k=0}^{n} f_k(v) R_v^k$, where $f_k(v)$ are arbitrary functions, and

$$V = \frac{\sum\limits_{k=1}^{n} k f_k(v) R_v^{k-1}}{\sum\limits_{k=0}^{n} f_k(v) R_v^k}.$$

The respective reduced equation will have the form

$$h(v, v^*)\left(\phi_{vv^*} + \phi_v \frac{\partial_{v^*}\Phi}{\Phi} + \phi_{v^*}\frac{\partial_v \Phi^*}{\Phi^*}\right) = F(\phi).$$

This equation may also be rewritten as an equation with two real independent variables ($v = \omega + \theta$, $v^* = \omega - \theta$):

$$2\widetilde{h}(\omega, \theta)(\phi_{\omega\omega} + \phi_{\theta\theta}) + \Omega(\omega, \theta)\phi_\omega + \Theta(\omega, \theta)\phi_\theta = F(\phi).$$

Theorem 2. *System*

$$\Box v = V(v,w), \quad \Box w = W(v,w),$$

$$w_\mu w_\mu = h(v,w), \quad v_\mu v_\mu = 0, \quad w_\mu w_\mu = 0$$

is compatible if and only if

$$V = \frac{h(v,w)\partial_w \Phi}{\Phi}, \quad W = \frac{h(v,w)\partial_v \Psi}{\Psi},$$

where the functions Φ*,* Ψ *for which the following conditions are satisfied*

$$(h\partial_v)^{n+1}\Psi = 0, \quad (h\partial_w)^{n+1}\Phi = 0.$$

The function h may be presented in the form $h = \frac{1}{R_{vw}}$ (it is anyway nonzero), where R is an arbitrary sufficiently smooth function, R_v, R_w are partial derivatives by the respective variables. Then the functions Φ, Ψ may be represented in the form

$$\Phi = \sum_{k=0}^{n} f_k(v) R_v^k, \quad \Psi = \sum_{k=0}^{n} g_k(w) R_w^k,$$

where $f_k(v)$, $g_k(w)$ are arbitrary functions,

$$V = \frac{\sum\limits_{k=1}^{n} k f_k(v) R_v^{k-1}}{\sum\limits_{k=0}^{n} f_k(v) R_v^k}, \quad W = \frac{\sum\limits_{k=1}^{n} k g_k(w) R_w^{k-1}}{\sum\limits_{k=0}^{n} g_k(w) R_w^k}.$$

The respective reduced equation will have the form

$$h(v,w)\left(\phi_{vw} + \phi_v \frac{\partial_w \Phi}{\Phi} + \phi_w \frac{\partial_v \Psi}{\Psi}\right) = F(\phi).$$

Theorem 3. *System*

$$\Box v = V(v,w), \quad \Box w = W(v,w),$$

$$v_\mu w_\mu = 0, \quad v_\mu v_\mu = \lambda \ (\lambda = \pm 1), \quad w_\mu w_\mu = 0$$

is compatible if and only if

$$V = \frac{\lambda \partial_v \Phi}{\Phi}, \quad \partial_v^{n+1} \Phi = 0, \quad W \equiv 0.$$

We cannot reduce our wave equation by means of ansatz $u = \varphi(y,z)$ to a parabolic equation – in this case one of the variables will enter the reduced ordinary differential equation of the first order as a parameter.

Compatibility and solutions of such system for $n = 3$ were considered by Fushchych, Zhdanov, Revenko; for this case necessary and sufficient compatibility conditions, as well as a general solution, were found.

The first order system is compatible only in the case if $V = W \equiv 0$, that is the reduced equation may be only an algebraic equation $F(u)$=0. Thus we cannot reduce our wave equation by means of ansatz $u = \varphi(y,z)$ to a first-order equation.

Brief description of proof of Theorem 2 for the hyperbolic case

We will operate with matrices of dimension $(n+1)\times(n+1)$ of the second variable of the functions v and w:

$$\hat{V} = \{v_{\mu\nu}\}, \quad \hat{W} = \{w_{\mu\nu}\}.$$

With respect to operations with these matrices we utilise summation arrangements customary for the Minkowsky space: $v_0 = i\partial_{x_0}$, $v_a = -i\partial_{x_a}(a = 1, \ldots, n)$, $v_\mu v_\mu = v_0^2 - v_1^2 - \cdots - v_n^2$.

Lemma 1. *If the functions v and w are solutions of the system*

$$\Box v = V(v,w), \quad \Box w = W(v,w),$$

$$w_\mu w_\mu = h(v,w), \quad v_\mu v_\mu = 0, \quad w_\mu w_\mu = 0,$$

then the following relations are satisfied for them for any k:

$$\mathrm{tr}\hat{V} = \frac{(-1)^k}{(k-1)!}(h(v,w)\partial_w)^{k+1}V(v,w),$$
$$\mathrm{tr}\hat{W} = \frac{(-1)^k}{(k-1)!}(h(v,w)\partial_v)^{k+1}W(v,w).$$

Lemma 2. *If the functions v and w are solutions of the system*

$$\Box v = V(v,w), \quad \Box w = W(v,w),$$

$$w_\mu w_\mu = h(v,w), \quad v_\mu v_\mu = 0, \quad w_\mu w_\mu = 0,$$

then $\det\hat{V} = 0$, $\det\hat{W} = 0$.

Lemma 3. *Let $M_k(\hat{V})$ be the sum of principal minors of the order k for the matrix $\hat{V}$. If the functions v and w are solutions of the system then the following relations are satisfied for them for any k:*

$$M_k(\hat{V}) = \frac{(h(v,w)\partial_w)^k\Phi}{k!\Phi}, \quad M_k(\hat{W}) = \frac{(h(v,w)\partial_v)^k\Psi}{k!\Psi},$$

where the functions Φ, Ψ satisfy the following conditions

$$(h\partial_v)^{n+1}\Psi = 0, \quad (h\partial_w)^{n+1}\Phi = 0.$$

These lemmas may be proved with the method of mathematical induction with utilisation of the Hamilton–Cayley theorem (E is a unit matrix of the dimension $(n+1)\times(n+1)$).

$$\sum_{k=0}^{n-1}(-1)^k M_k \hat{V}^{n-k} + (-1)^n E\det\hat{V} = 0.$$

It is evident that the statement of Theorem 2 is a direct consequence of Lemma 3 for $k = 1$.

Note 1. Equation

$$\Box v = V(v, v^*), \quad \Box v^* = V^*(v, v^*),$$

$$v^*_\mu v_\mu = h(v, v^*), \quad v_\mu v_\mu = 0, \quad v^*_\mu v^*_\mu = 0$$

may be rewritten for a pair of real functions $\omega = \mathrm{Re}\, v$, $\theta = \mathrm{Im}\, v$. Though in this case necessary the respective compatibility conditions would have extremely cumbersome form.

Note 2. Transition from

$$y_\mu y_\mu = r(y, z), \quad y_\mu z_\mu = q(y, z), \quad z_\mu z_\mu = s(y, z),$$

$$\Box y = R(y, z), \quad \Box z = S(y, z).$$

to

$$\Box v = V(v, v^*), \quad \Box v^* = V^*(v, v^*),$$

$$v^*_\mu v_\mu = h(v, v^*), \quad v_\mu v_\mu = 0, \quad v^*_\mu v^*_\mu = 0;$$

$$\Box v = V(v, w), \quad \Box w = W(v, w),$$

$$w_\mu w_\mu = h(v, w), \quad v_\mu v_\mu = 0, \quad w_\mu w_\mu = 0,$$

etc. is convenient only from the point of view of investigation of compatibility. The sign of the expression $rs - q^2$ may change for various y, z, and the transition is being considered only within the region where this sign is constant.

Examples of solutions of the system of d'Alembert–Hamilton equations. Let us adduce explicit solutions of systems of the type

$$y_\mu y_\mu = r(y,z), \quad y_\mu z_\mu = q(y,z), \quad z_\mu z_\mu = s(y,z),$$
$$\Box y = R(y,z), \quad \Box z = S(y,z).$$

and the respective reduced equations. Parameters a_μ, b_μ, c_μ, d_μ ($\mu = \overline{0,3}$) satisfy the conditions:

$$-a^2 = b^2 = c^2 = d^2 = -1 \quad (a^2 \equiv a_0^2 - a_1^2 - \cdots - a_3^2),$$
$$ab = ac = ad = bc = bd = cd = 0;$$

y, z are functions of x_0, x_1, x_2, x_3.

1) $y = ax, \quad z = dx, \quad \varphi_{yy} - \varphi_{zz} = F(\varphi);$

2) $y = ax, \quad z = \left((bx)^2 + (cx)^2 + (dx)^2\right)^{1/2},$

$$\varphi_{yy} - \varphi_{zz} - \frac{2}{z}\varphi_z = F(\varphi);$$

In this case the reduced equation is a so-called radial wave equation,

3) $y = bx + \Phi(ax + dx), \quad z = cx, \quad -\varphi_{zz} - \varphi_{yy} = F(\varphi);$

4) $y = \left((bx)^2 + (cx^2)\right)^{1/2}, \quad z = ax + dx, \quad -\varphi_{yy} - \dfrac{1}{y}\varphi_y = F(\varphi).$

Classes of the Reduced Equations

$$h(v, v^*)\left(\phi_{vv^*} + \phi_v \frac{\partial_{v^*}\Phi}{\Phi} + \phi_{v^*}\frac{\partial_v \Phi^*}{\Phi^*}\right) = F(\phi),$$

where

$$(h\partial_{v^*})^{n+1}\Phi = 0.$$

$$h(v, w)\left(\phi_{vw} + \phi_v \frac{\partial_w \Phi}{\Phi} + \phi_w \frac{\partial_v \Psi}{\Psi}\right) = F(\phi),$$

where

$$(h\partial_v)^{n+1}\Psi = 0, \quad (h\partial_w)^{n+1}\Phi = 0.$$

Further Research

1. Study of Lie and conditional symmetry of the system of the reduction conditions.
2. Investigation of Lie and conditional symmetry of the reduced equations. Finding exact solutions of the reduced equations.
3. Relation of the equivalence group of the class of the reduced equations with symmetry of the initial equation.
4. Group classification of the reduced equations.
5. Finding of sufficient compatibility conditions and of a general solution of the compatibility conditions for lower dimensions ($n = 2, 3$).
6. Finding and investigation of compatibility conditions and classes of the reduced equations for other types of equations, in particular, for Poincaré–invariant scalar equations.

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