Integrability & Symmetries of difference equations
The Adler–Bobenko–Suris case

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Continuous ➠ Discrete

Integrable differential equations

✓ Bäcklund transformation, Lax pair

✓ Infinite hierarchies of generalized symmetries

✓ Symmetry reductions to Painlevé equations

Adler-Bobenko-Suris (ABS) Difference Equations

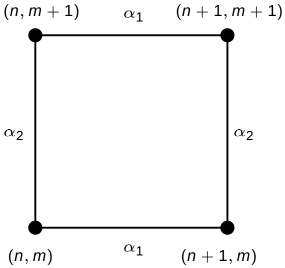
➠ Multidimensional consistent ➔ Bäcklund transformation, Lax pair

➠ Infinite hierarchies of generalized symmetries

➠ Symmetry reductions to discrete & continuous Painlevé equations

General characteristics

$$Q(u_{n,m}\,,\,u_{n+1,m}\,,\,u_{n,m+1}\,,\,u_{n+1,m+1}\,,\,\alpha_1,\alpha_2)\;=\;0$$



An elementary quadrilateral on the lattice

❶ Autonomous difference equations

❷ α_1, α_2 : lattice parameters

❸ Affine linear w.r.t. the values of u

Known equations

⇨ DISCRETE POTENTIAL KORTEWEG – DE VRIES EQUATION (H1)

$$(u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) - \alpha_1 + \alpha_2 = 0$$

Hirota R. (1977)
NONLINEAR PARTIAL DIFFERENCE EQUATIONS. I. A DIFFERENCE ANALOGUE OF THE KORTEWEG-DE VRIES EQUATION

J. Phys. Soc. Japan **43**

⇨ DISCRETE SCHWARZIAN KORTEWEG – DE VRIES EQUATION ($Q1_0$)

$$\alpha_1(u_{n,m} - u_{n,m+1})(u_{n+1,m} - u_{n+1,m+1}) - \alpha_2(u_{n,m} - u_{n+1,m})(u_{n,m+1} - u_{n+1,m+1}) = 0$$

Quispel G., Nijhoff F., Capel H., van der Linden J. (1984)
LINEAR INTEGRAL EQUATIONS AND NONLINEAR DIFFERENCE-DIFFERENCE EQUATIONS

Physica A **125**

New cases

➩ EQUATION H2

$$(u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) + (\alpha_2 - \alpha_1)(u_{n,m} + u_{n+1,m} + u_{n,m+1} + u_{n+1,m+1}) - \alpha_1^2 + \alpha_2^2 = 0$$

➩ EQUATION Q2

$$\begin{aligned}&\alpha_1(u_{n,m} - u_{n,m+1})(u_{n+1,m} - u_{n+1,m+1}) - \alpha_2(u_{n,m} - u_{n+1,m})(u_{n,m+1} - u_{n+1,m+1}) +\\ &\alpha_1\alpha_2(\alpha_1 - \alpha_2)(u_{n,m} + u_{n+1,m} + u_{n,m+1} + u_{n+1,m+1}) - \alpha_1\alpha_2(\alpha_1 - \alpha_2)(\alpha_1^2 - \alpha_1\alpha_2 + \alpha_2^2) = 0\end{aligned}$$

Adler's Equation: the master equation

⇨ Equation Q4

$$a_0 u_{(0,0)} u_{(1,0)} u_{(0,1)} u_{(1,1)}$$
$$+a_1\big(u_{(0,0)} u_{(1,0)} u_{(0,1)} + u_{(1,0)} u_{(0,1)} u_{(1,1)} + u_{(0,1)} u_{(1,1)} u_{(0,0)} + u_{(1,1)} u_{(0,0)} u_{(1,0)}\big)$$
$$+a_2\big(u_{(0,0)} u_{(1,1)} + u_{(1,0)} u_{(0,1)}\big) + \bar{a}_2\big(u_{(0,0)} u_{(1,0)} + u_{(0,1)} u_{(1,1)}\big)$$
$$+\tilde{a}_2\big(u_{(0,0)} u_{(0,1)} + u_{(1,0)} u_{(1,1)}\big) + a_3\big(u_{(0,0)} + u_{(1,0)} + u_{(0,1)} + u_{(1,1)}\big) + a_4 = 0$$

Adler V E (1998)
BÄCKLUND TRANSFORMATION FOR THE KRICHEVER–NOVIKOV EQUATION

Int. Math. Res. Notices **1**

Integrability criterion

Multidimensional consistency

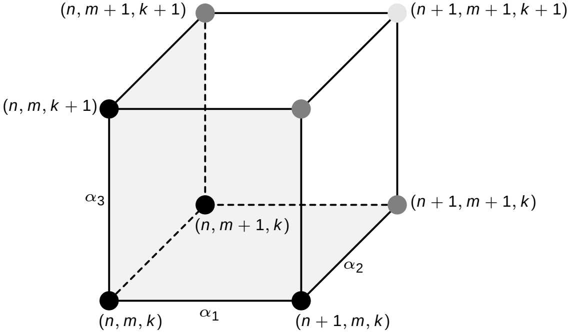
The equation can be extended to a three dimensional lattice in a consistent way.

- Classification of integrable cases

Adler V.E., Bobenko A.I., Suris Y.B. (2003)
CLASSIFICATION OF INTEGRABLE EQUATIONS ON QUAD GRAPHS. THE CONSISTENCY APPROACH

Commun. Math. Phys. **233**

Multidimensional consistency



Auto-Bäcklund transformation and Lax pair

① CONSISTENCY → AUTO–BÄCKLUND TRANSFORMATION

Bobenko A., Suris Yu. (2002)
INTEGRABLE SYSTEMS ON QUAD GRAPHS

Int. Math. Res. Notices **11**

Atkinson (2008)
BÄCKLUND TRANSFORMATIONS FOR INTEGRABLE LATTICE EQUATIONS

J. Phys. A: Math. Theor. **41**

② CONSISTENCY → LAX PAIR

Nijhoff F. (2002)
LAX PAIR FOR THE ADLER (LATTICE KRICHEVER-NOVIKOV) SYSTEM

Phys. Lett. A **297**

Bobenko A., Suris Yu. (2002)
INTEGRABLE SYSTEMS ON QUAD GRAPHS

Int. Math. Res. Notices **11**

Auto-Bäcklund transformation and Lax pair revised

③ AUTO–BÄCKLUND TRANSFORMATION → LAX PAIR

- **Equation** $$Q(u_{n,m}\,,\, u_{n+1,m}\,,\, u_{n,m+1}\,,\, u_{n+1,m+1}\,,\, \alpha_1, \alpha_2) = 0$$

- **Auto-Bäcklund** $$\begin{cases} \mathbb{B}^1 := Q(u_{n,m}\,,\, u_{n+1,m}\,,\, \tilde{u}_{n,m}\,,\, \tilde{u}_{n+1,m}\,,\, \alpha_1, \lambda) = 0 \\ \mathbb{B}^2 := Q(u_{n,m}\,,\, \tilde{u}_{n,m}\,,\, u_{n,m+1}\,,\, \tilde{u}_{n,m+1}\,,\, \lambda, \alpha_2) = 0 \end{cases}$$

- **Lax pair**

$$\Psi_1 = \frac{1}{h_1}\begin{pmatrix} \mathbb{B}^1_{,4} & -\mathbb{B}^1_{,34} \\ \mathbb{B}^1 & -\mathbb{B}^1_{,3} \end{pmatrix}\Psi\,, \quad \Psi_2 = \frac{1}{h_2}\begin{pmatrix} \mathbb{B}^2_{,4} & -\mathbb{B}^2_{,24} \\ \mathbb{B}^2 & -\mathbb{B}^2_{,2} \end{pmatrix}\Psi$$

Symmetries of difference equations

Local symmetries

- Symmetry generator $\mathbf{v} = \eta(n, m, u_{n,m}\,,\, u_{n+1,m}\,,\, u_{n-1,m}\,, \ldots)\,\partial_{u_{n,m}}$
- Prolongation of the symmetry generator

$$\mathrm{pr}\,\mathbf{v} = \mathbf{v} + \eta(n+1, m, u_{n+1,m}\,,\, u_{n+2,m}\,,\, u_{n,m}\,, \ldots)\,\partial_{u_{n+1,m}} + \ldots$$

- Reductions
- Initial value problem

Extended symmetries

Symmetries acting on the lattice parameters as well

- Symmetry generator

$$\mathbf{v} = \eta(n, m, u_{n,m}\,,\, u_{n+1,m}\,,\, u_{n-1,m}\,, \ldots)\,\partial_{u_{n,m}} + \xi(\alpha_1, \alpha_2)\,\partial_{\alpha_1} + \zeta(\alpha_1, \alpha_2)\,\partial_{\alpha_2}$$

- Master symmetries
- Reductions

Symmetry analysis

Generalized symmetries

$$\mathbf{v}_0 = \mathrm{H}(u_{n,m}\,,\, u_{n+1,m}\,,\, u_{n-1,m}\,,\alpha_1)\,\partial_{u_{n,m}}$$

$$\mathbf{w}_0 = \mathrm{H}(u_{n,m}\,,\, u_{n,m+1}\,,\, u_{n,m-1}\,,\alpha_2)\,\partial_{u_{n,m}}$$

Extended generalized symmetries

$$\mathbf{V} = n\,\mathrm{H}(u_{n,m}\,,\, u_{n+1,m}\,,\, u_{n-1,m}\,,\alpha_1)\,\partial_{u_{n,m}} - r(\alpha_1)\,\partial_{\alpha_1}$$

$$\mathbf{W} = m\,\mathrm{H}(u_{n,m}\,,\, u_{n,m+1}\,,\, u_{n,m-1}\,,\alpha_2)\,\partial_{u_{n,m}} - r(\alpha_2)\,\partial_{\alpha_2}$$

Tongas A., Tsoubelis D., XP (2007)
AFFINE LINEAR AND D_4 SYMMETRIC LATTICE EQUATIONS: SYMMETRY ANALYSIS AND REDUCTIONS

J. Phys. A: Math. Theor. **40**

Infinite hierarchies of symmetries

EXTENDED GENERALIZED SYMMETRIES → MASTER SYMMETRIES

- Infinite hierarchies of generalized symmetries can be constructed successively

$$\{\mathbf{v}_k\}\,, \quad k = 0, 1, 2, \ldots \qquad\qquad \{\mathbf{w}_k\}\,, \quad k = 0, 1, 2, \ldots$$

$$\mathbf{v}_{k+1} = [\mathbf{V}, \mathbf{v}_k] \qquad\qquad \mathbf{w}_{k+1} = [\mathbf{W}, \mathbf{w}_k]$$

Rasin O., Hydon P. (2007)
SYMMETRIES OF INTEGRABLE DIFFERENCE EQUATIONS

Stud. Appl. Math. **49**

Reductions to continuous Painlevé equations

EXTENDED GENERALIZED SYMMETRIES → REDUCTIONS

CONTINUOUSLY INVARIANT SOLUTIONS

Solutions remaining invariant under the action of both of the extended generalized symmetries

① Such solutions are determined by an integrable system of PDEs

② Reductions to **continuous** Painlevé equations

Tsoubelis D., XP (2008)
CONTINUOUS SYMMETRY REDUCTIONS OF THE ADLER-BOBENKO-SURIS EQUATIONS

in preparation

Example

⇨ The discrete potential KdV equation

$$(u_{n,m}-u_{n+1,m+1})(u_{n+1,m}-u_{n,m+1})-\alpha_1+\alpha_2=0$$

⇨ System of PDEs

$$\begin{aligned}
\frac{\partial u_{n+1,m}}{\partial \alpha_2} &= \frac{u_{n+1,m}-u_{n,m+1}}{\alpha_1-\alpha_2}\left(m-(u_{n+1,m}-u_{n,m+1})\frac{\partial u_{n,m}}{\partial \alpha_2}\right)\\
\frac{\partial u_{n,m+1}}{\partial \alpha_1} &= \frac{u_{n+1,m}-u_{n,m+1}}{\alpha_1-\alpha_2}\left(n+(u_{n+1,m}-u_{n,m+1})\frac{\partial u_{n,m}}{\partial \alpha_1}\right)\\
\frac{\partial^2 u_{n,m}}{\partial \alpha_1\partial \alpha_2} &= \frac{1}{\alpha_1-\alpha_2}\left(2(u_{n+1,m}-u_{n,m+1})\frac{\partial u_{n,m}}{\partial \alpha_1}\frac{\partial u_{n,m}}{\partial \alpha_2}+n\frac{\partial u_{n,m}}{\partial \alpha_2}-m\frac{\partial u_{n,m}}{\partial \alpha_1}\right)
\end{aligned}$$

Nijhoff F., Hone A., Joshi N.(2000)
ON A SCHWARZIAN PDE ASSOCIATED WITH THE KDV HIERARCHY

Phys. Lett. A **267**

Tongas A., Tsoubelis D., XP (2001)
A FAMILY OF INTEGRABLE NONLINEAR EQUATIONS OF HYPERBOLIC TYPE

J. Math. Phys. **42**

Example

➩ The discrete potential KdV equation

$$\left(u_{n,m}-u_{n+1,m+1}\right)\left(u_{n+1,m}-u_{n,m+1}\right)-\alpha_1+\alpha_2=0$$

➩ Continuously invariant solutions related to Painlevé VI

$$u_{n,m}=S_{n,m}\left(\frac{\alpha_1}{\alpha_2}\right)\times\left(\alpha_1\alpha_2\right)^{\left(1+2(-1)^{n+m}\mu\right)/4}$$

➩ Function $S_{n,m}$ is determined by solutions of Painlevé VI with parameters

$$\mathrm{A}=\frac{n^2}{2},\ \mathrm{B}=-\frac{m^2}{2},\ \Gamma=\lambda,\ \Delta=\frac{1}{2}-\frac{1}{4}\left(1+2(-1)^{n+m}\mu\right)^2$$