The geometry of Laurent solutions

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Hamiltonian mechanics on Poisson manifolds

Poisson manifold (M, π):

- M differentiable manifold
- π bivector field on M
  - π section of $\wedge^2 TM \to M$
  - π skew-symmetric biderivation of $C^\infty(M)$
  - π locally of the form

$$\pi = \sum_{i<j} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

- $\pi = \{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfies the Jacobi identity

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$$

Example 1: symplectic manifolds

(M, ω) symplectic manifold: $\{F, G\} := \omega(\mathcal{X}_F, \mathcal{X}_G)$, where $\mathcal{X}$ is the Hamiltonian operator,

$$\omega(\mathcal{X}_H, \cdot) = \mathrm{d}H$$

so that

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Example 2: the dual of a Lie algebra

$\mathfrak{g}$ finite-dimensional Lie algebra; $\mathfrak{g}^*$ its dual (vector space)
for $\xi \in \mathfrak{g}^*$ and $F \in C^\infty(\mathfrak{g}^*)$

$$\mathrm{d}_\xi F \in (\mathfrak{g}^*)^* \cong \mathfrak{g}$$

The Poisson structure on $\mathfrak{g}^*$ is given for $F, G \in C^\infty(\mathfrak{g}^*)$ by

$$\forall \xi \in \mathfrak{g} \quad \{F, G\}(\xi) := \langle \xi, [\mathrm{d}_\xi F, \mathrm{d}_\xi G] \rangle$$

Functions in involution

$F, G \in C^\infty(M)$ are in involution if $\{F, G\} = 0$

- F is constant on the integral curves of $\mathcal{X}_G$
- $\mathcal{X}_G$ is tangent to the smooth fibers of $F : M \to \mathbf{R}$
- $[\mathcal{X}_F, \mathcal{X}_G] = 0$

Generalization: $\mathbf{F} = (F_1, \ldots, F_s)$ in involution then

- $\mathcal{X}_{F_i}$ are tangent to the smooth fibers of $\mathbf{F} : M \to \mathbf{R}^s$
- $[\mathcal{X}_{F_i}, \mathcal{X}_{F_j}] = 0$

Example: H Hamiltonian, then F is in involution with H iff F is a constant of motion. Then $\mathcal{X}_H$ is tangent to the smooth fibers of F.

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How many?: if $\mathbf{F} = (F_1, \ldots, F_s)$ in involution and independent (almost everywhere) then $s \leqslant \dim M - \frac{1}{2} Rk \{\cdot, \cdot\}$

The invariant manifolds

$(M, \{\cdot\,,\cdot\}\,, \mathbf{F})$ Liouville integrable

- $\mathbf{F} = (F_1, \ldots, F_s)$
- $\mathbf{F}$ independent and in involution
- $s = \dim M - r$ where $2r := Rk\,\{\cdot\,,\cdot\}$

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Two open subsets of M

- $\mathcal{U}_{\mathbf{F}} = \{m \in M \mid \mathrm{d}_m F_1 \wedge \cdots \wedge \mathrm{d}_m F_s \neq 0\}$ dense
- $M_r = \{m \in M \mid Rk\, \pi_m = 2r\}$
- $\mathcal{U}_{\mathbf{F}} \cap M_r \neq \emptyset$

On $\mathcal{U}_{\mathbf{F}} \cap M_r$

- $\mathcal{X}_{F_1}, \dots, \mathcal{X}_{F_s}$ define integrable distribution of rank r
- leafs are called regular invariant manifolds
- notation: $\mathcal{F}_m$ where $m \in \mathcal{U}_{\mathbf{F}} \cap M_r$

The Liouville Theorem (Poisson version)

Let $(M, \{\cdot\,,\cdot\}\,, \mathbf{F})$ be Liouville integrable system

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- $Rk\,\{\cdot\,,\cdot\} = 2r$

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The Liouville Theorem (Poisson version)

Let $(M, \{\cdot,\cdot\}, \mathbf{F})$ be Liouville integrable system

- $\mathbf{F} = (F_1, \ldots, F_s)$
- $Rk\,\{\cdot,\cdot\} = 2r$

Let $m \in \mathcal{U}_{\mathbf{F}} \cap M_r$. Then

- $\mathcal{F}_m$ is the connected component of the fiber through m of $\mathbf{F}$ restricted to $\mathcal{U}_{\mathbf{F}} \cap M_r$
- The integral curve of each $\mathcal{X}_{F_i}$, starting from m can be determined by quadratures
- If the flow of each $\mathcal{X}_{F_i}$ is complete on $\mathcal{F}_m$, there $\exists$ a diffeo

$$\mathcal{F}_m \longrightarrow \mathbf{T}^{r-q} \times \mathbf{R}^q$$

  which linearizes all $\mathcal{X}_{F_i}$
- In particular, if $\mathcal{F}_m$ is compact, then $\exists$ a diffeomorphism

$$\mathcal{F}_m \longrightarrow \mathbf{T}^r$$

  linearizing all $\mathcal{X}_{F_i}$. $\mathcal{F}_m$ is called a regular Liouville torus.

Example

On $\mathbf{C}^2$ consider the (complex) bivector field

$$\{\cdot\,,\cdot\} := \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}.$$

The real part and imaginary parts of $\{\cdot\,,\cdot\}$ are (real) Poisson structures on $\mathbf{R}^4$.
Let $z_k = x_k + iy_k$, then

$$\begin{aligned}
\{\cdot\,,\cdot\}_{\Re} &:= 4\,\Re\left(\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}\right) \\
&= \Re\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial y_1}\right) \wedge \left(\frac{\partial}{\partial x_2} - i\frac{\partial}{\partial y_2}\right) \\
&= \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} - \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2}.
\end{aligned}$$

Similarly

$$\{\cdot\,,\cdot\}_{\Im} := \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_2} - \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_1}.$$

So, on $\mathbf{R}^4$ two Poisson structures,

$$\{\cdot\,,\cdot\}_{\Re} = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} - \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2}.$$

$$\{\cdot\,,\cdot\}_{\Im} := \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_2} - \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_1}.$$

Let $F = G + iH$ be a holomorphic function on $\mathbf{C}^2$,

$$\{\cdot, G\}_{\Re} = \frac{\partial G}{\partial x_2}\frac{\partial}{\partial x_1} - \frac{\partial G}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial G}{\partial y_2}\frac{\partial}{\partial y_1} + \frac{\partial G}{\partial y_1}\frac{\partial}{\partial y_2}$$

$$\{\cdot, H\}_{\Im} = \frac{\partial H}{\partial y_2}\frac{\partial}{\partial x_1} - \frac{\partial H}{\partial x_1}\frac{\partial}{\partial y_2} - \frac{\partial H}{\partial y_1}\frac{\partial}{\partial x_2} + \frac{\partial H}{\partial x_2}\frac{\partial}{\partial y_1}$$

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Cauchy-Riemann equations $\Rightarrow \{\cdot, G\}_{\Re} = \{\cdot, H\}_{\Im}$, hence

$$\{H, G\}_{\Re} = \{H, H\}_{\Im} = 0 \qquad \{G, H\}_{\Im} = \{G, G\}_{\Re} = 0$$

The invariant manifolds are Riemann surfaces (with punctures)

$$\{(z_1, z_2) \in \mathbf{C}^2 \mid F(z_1, z_2) = c\}$$

The Action-Angle Theorem (Laurent, Miranda, V.)

Let $(M, \{\cdot, \cdot\}, \mathbf{F})$ be Liouville integrable system, of dimension n and rank $2r$. Suppose that $\mathcal{F}_m$ is a regular Liouville torus, where $m \in \mathcal{U}_{\mathbf{F}} \cap M_r$. There exist on a neighborhood U of $\mathcal{F}_m$

- $\mathbf{R}$-valued smooth functions $(p_1, \dots p_{n-r})$
- $\mathbf{R}/\mathbf{Z}$-valued smooth functions $(\theta_1, \dots, \theta_r)$

such that

- $(\theta_1, \dots, \theta_r, p_1, \dots, p_{n-r}) : U \simeq \mathbf{T}^r \times B^{n-r}$ is a diffeomorphism
- In terms of these coordinates

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial p_i},$$

- the functions $p_1, \dots, p_{n-r}$ depend on the functions $F_1, \dots, F_{n-r}$ only.

Complex integrable systems

$(M, \Pi, \mathbf{F})$ where

- M is a complex algebraic manifold, here $\mathbf{C}^n$
- Π is an algebraic Poisson structure, here polynomial
- $\mathbf{F} = (F_1, \ldots, F_s)$ algebraic functions, here polynomial

As before, integrable means that

- $\mathbf{F}$ in involution
- $\mathbf{F}$ independent
- $s = \dim M - \frac{1}{2} Rk \{\cdot\, , \cdot\}$

Good or bad definition?

Complex integrability, what remains

Roughly speaking, all *local* properties remain true:

- The integrable vector fields commute, $\left[\mathcal{X}_{F_i}, \mathcal{X}_{F_j}\right] = 0$;
- These vector fields are tangent to the smooth fibers of the map
$$\begin{array}{cccc} \mathbf{F} & : & M & \rightarrow & \mathbf{C}^s \\ & & m & \mapsto & (F_1(m), \ldots, F_s(m)) \end{array}$$
- These vector fields define an integrable distribution on $\mathcal{U}_{\mathbf{F}} \cap M_r$, which is here a Zariski open subset (hence dense)
- For $m \in \mathcal{U}_{\mathbf{F}} \cap M_r$, the integral curves of every $\mathcal{X}_{F_i}$ can be determined by quadratures.

Complex integrability, what goes bad

In general, things go bad *semi-locally*, *globally* and *at infinity*

- The fibers of $\mathbf{F} : \mathbf{C}^n \to \mathbf{C}^s$ are never compact, hence never tori $\mathbf{C}^n/\Lambda$
- In fact $\mathbf{C}^n$ does not have compact holomorphic submanifolds
- Except in trivial cases, flow is never complete
- In general the integral curves to the vector fields $\mathcal{X}_{F_i}$ will be multi-valued

In my opinion, the last point is the worst one.

Example $(\mathbf{C}^2, \{x, y\} = 1, H = y^2 - x^5)$

Hamiltonian vector field

$$\mathcal{X}_H : \begin{cases} \dot{x} &= 2y \\ \dot{y} &= 5x^4 \end{cases}$$

For $c \in \mathbf{C}^*$, the vector field $\mathcal{X}_H$ is tangent to the smooth fiber

$$\mathbf{F}_c = \{(x, y) \in \mathbf{C}^2 \mid y^2 = x^5 + c\}$$

- $\mathbf{F}_c$ is a smooth algebraic curve (Riemann surface)
- $\mathbf{F}_c$ is a topological surface of genus two
- $\mathbf{F}_c$ admits a smooth compactification by adding one point ∞
- $\mathcal{X}_H$ is a nowhere vanishing holomorphic vector field on $\mathbf{F}_c$

? What happens at infinity ?

Example $(\mathbf{C}^2, \{x, y\} = 1, H = y^2 - x^5)$

Local parameter z at infinity, defined by: $x = 1/z^2$ (recall $y^2 = x^5 + c$)

$$x = \frac{1}{z^2}, \qquad y = \frac{1}{z^5}\left(1 + \frac{cz^{10}}{2} + \cdots\right).$$

So $\mathcal{X}_H$ is given, in a neighborhood of infinity, by (recall $\dot{x} = 2y$)

$$\dot{z} = -\frac{1}{z^2}(1 + \mathcal{O}(z)).$$

This is bad

- $\mathcal{X}_H(z) \to \infty$ when $z \to 0$, so $\mathcal{X}_H$ goes to hell at ∞
- solution is $z(t) = \sqrt[3]{-3t(1 + \mathcal{O}(t))}$ multi-valued function

Detecting the problem: Laurent solutions

Following Kowalevski: look for (strict) Laurent solutions to the differential equation. $\dot{x} = 2y$; $\dot{y} = 5x^4$ written as

$$\ddot{x} = 10x^4$$

i.e., look for $a \neq 0$ and $n \in \mathbf{Z}$ with $n > 0$ ("strict") such that

$$x(t) = \frac{a}{t^n}(1 + \mathcal{O}(t))$$

be the start of a formal solution. A simple substitution gives

$$\frac{n(n+1)a}{t^{n+2}}(1 + \mathcal{O}(t)) = \frac{10a^4}{t^{4n}}(1 + \mathcal{O}(t)).$$

This forces $n = 2/3 \notin \mathbf{Z}$ (since $a \neq 0$), so no strict Laurent solutions!

An example which works: $H = y^2 - x^3$

The Hamiltonian vector field

$$\mathcal{X}_H : \begin{cases} \dot{x} &= 2y \\ \dot{y} &= 3x^2 \end{cases}$$

which we write as $\ddot{x} = 6x^2$ has Laurent solutions

$$x(t) = \frac{1}{t^2}(1 + a_k t^k + \mathcal{O}(t^{k+1}))$$

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$$x(t) = \frac{1}{t^2}(1 + a_k t^k + \mathcal{O}(t^{k+1}))$$

with $a_k = 0$ for $1 \leqslant k \leqslant 5$, and most interestingly, for $k = 6$, the parameter a_6 can be chosen arbitrarily!

$$\begin{aligned} x(t) &= \frac{1}{t^2}(1 + at^6 + \mathcal{O}(t^7)) \\ \ddot{x}(t) &= \frac{6}{t^4} + 12at^2 + \mathcal{O}(t^3) \\ 6x^2(t) &= 6\left(\frac{1}{t^4} + 2at^2 + \mathcal{O}(t^3)\right) \end{aligned}$$

The meaning of the free parameter

$$\begin{aligned} x(t) &= \frac{1}{t^2} + at^4 + \mathcal{O}(t^5), \\ y(t) &= \frac{\dot{x}(t)}{2} = -\frac{1}{t^3} + 2at^3 + \mathcal{O}(t^4). \end{aligned}$$

Substituted in the Hamiltonian $H = y^2 - x^3$, one finds

$$\begin{aligned} y^2(t) - x^3(t) &= \frac{1}{t^6} - 4a + \mathcal{O}(t) - \left(\frac{1}{t^6} + 3a + \mathcal{O}(t)\right) \\ &= -7a + \mathcal{O}(t). \end{aligned}$$

In fact, since H is a constant of motion, $H(x(t), y(t))$ is independent of t, so

$$H(x(t), y(t)) = c$$

where $c \in \mathbf{C}$ is a constant which depends only on the initial condition. So $-7a = c$.

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Conclusion: The free parameter allows us to have a Laurent solution for every invariant manifold $\mathcal{F}_c$!

A.c.i. systems (Adler-van Moerbeke)

$(M, \{\cdot\,,\cdot\}\,, \mathbf{F})$ complex integrable system

Algebraically completely integrable (a.c.i.) means

- For generic c, the iso-level set $\mathbf{F}^{-1}(c)$ is an affine part of an Abelian variety $\mathbf{C}^r/\Lambda_c$
- The holomorphic vector fields $\mathcal{X}_{F_i}$ are constant (translation invariant) on these $\mathbf{C}^r/\Lambda_c$.

Abelian varieties

An Abelian variety is a complex torus $\mathbf{C}^r/\Lambda$ which can be holomorphically embedded in some complex projective space $\mathbf{CP}^N$.

For $r = 1$ every $\mathbf{C}/\Lambda$ embeds in projective space (elliptic curve)

For $r > 1$ non-trivial conditions on Λ.

Main example: The Jacobian of an algebraic curve of genus g

$$\mathrm{Jac}(\Gamma) := \frac{H^0(\Gamma, \Omega^1)^*}{H_1(\Gamma, \mathbf{Z})}$$

Other descriptions:

$$\mathrm{Jac}(\Gamma) \cong \frac{\mathrm{Div}^g \Gamma}{\sim} \cong \mathrm{Pic}^0(\Gamma)$$

The periodic Kac-van Moerbeke system

Phase space $M = \mathbf{C}^5 \ni (a_1, \ldots, a_5)$ (indices mod 5)
Quadratic Poisson structure

$$\{a_i, a_j\} = a_i a_j (\delta_{i,j+1} - \delta_{i+1,j})$$

Fonctions in involution

$$H_1 = a_1 + a_2 + a_3 + a_4 + a_5$$

$$H_2 = a_1 a_3 + a_2 a_4 + a_3 a_5 + a_4 a_1 + a_5 a_2$$

$$H_3 = a_1 a_2 a_3 a_4 a_5$$

The first vector field:

$$\dot{a}_i = a_i (a_{i+1} - a_{i-1})$$

The group $\mathbf{Z}/\mathbf{Z}_5$ acts on all of this.

Algebraic integrability (Fernandes, V.)

For $(c_1, c_2) \in \mathbf{C}^2$ such that the affine curve

$$\Gamma_{c_1 c_2} : y^2 = (x^3 - c_1 x^2 + c_2 x)^2 - 4x$$

is smooth, the affine surface

$$\{(a_1, \ldots, a_5) \in \mathbf{C}^5 \mid H_0 = 1, H_1 = c_1, H_2 = c_2\}$$

is isomorphic to $\mathrm{Jac}(\Gamma_{c_1 c_2}) \setminus \mathcal{D}_{c_1 c_2}$, where $\mathcal{D}_{c_1 c_2}$ consists of 5 copies of $\Gamma_{c_1 c_2}$ in $\mathrm{Jac}(\Gamma_{c_1 c_2})$.

Moreover the flow of both vector fields $\mathcal{X}_{H_1}$ and $\mathcal{X}_{H_2}$ is linear on these Jacobians.

The Kowalevski-Painlevé Criterion (Adler, van Moerbeke, V.)

Let $(M, \{\cdot\,,\cdot\}\,, \mathbf{F})$ be an (irreducible) a.c.i. system.

Then each of the vector fields $\mathcal{X}_{F_i}$ admits Laurent solutions which depend on $\dim M - 1$ free parameters.

Periodic Toda lattices (Adler and van Moerbeke)

Let $e_0, e_1, \dots, e_\ell \in \mathbf{R}^{\ell+1}$ be such that

- $e_0, e_1, \dots, e_\ell$ are dependent
- $\forall i : e_0, \dots, \widehat{e_i}, \dots, e_\ell$ are independent

Let $A = (a_{ij})$ its Cartan matrix

$$a_{ij} := \frac{2\langle e_i | e_j \rangle}{\langle e_j | e_j \rangle}$$

On $\mathbf{C}^{2(l+1)}$ vector field $\mathcal{V}$:

$$\dot{x} = x \cdot y \qquad \dot{y} = Ax.$$

where $(x \cdot y)_i := x_i y_i$.

Theorem: If $\mathcal{V}$ is a.c.i., then A is the Cartan matrix of a (twisted) affine Lie algebra.

The geometry of Laurent solutions

The Laurent solutions also have a positive use

- Explicitly embedding the tori in projective space
- Determining and studying the divisor at infinity
- Proving algebraic integrability

The Laurent solutions of KM ($n = 5$)

Recall

$$\dot{a}_i = a_i(a_{i+1} - a_{i-1})$$

$$H_1 = a_1 + a_2 + a_3 + a_4 + a_5$$

$$H_2 = a_1a_3 + a_2a_4 + a_3a_5 + a_4a_1 + a_5a_2$$

$$H_3 = a_1a_2a_3a_4a_5$$

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$$H_2 = a_1a_3 + a_2a_4 + a_3a_5 + a_4a_1 + a_5a_2$$

$$H_3 = a_1a_2a_3a_4a_5$$

One of the families of Laurent solutions:

$$\begin{aligned}
a_1(t) &= ct + O(t^2) \\
a_2(t) &= d + O(t^2) \\
a_3(t) &= bt + O(t^2) \\
a_4(t) &= -\frac{1}{t} + a - \frac{1}{3}(a^2 + 2b + c)t + O(t^2) \\
a_5(t) &= \frac{1}{t} + a + \frac{1}{3}(a^2 - b - 2c)t + O(t^2)
\end{aligned}$$

Embedding the tori

Abstract theory: Take any positive line bundle $\mathcal{L}$ on the torus $\mathbf{T}^r$. For $k \geqslant 3$ the sections of $\mathcal{L}^k$ define an embedding of $\mathbf{T}^r$ in projective space:

$$\begin{array}{rcccl} \phi_{\mathcal{L}^k} & : & \mathbf{T}^r & \longrightarrow & \mathbf{P}H^0(\mathbf{T}^2, \mathcal{L}^k)^* \\ & & m & \longmapsto & \text{sections vanishing at } m \end{array}$$

Stated in terms of divisors: the functions with a pole of order at most 3 on a given divisor on $\mathbf{T}^r$ provide an embedding.

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Stated in terms of divisors: the functions with a pole of order at most 3 on a given divisor on $\mathbf{T}^r$ provide an embedding.

Concretely one uses the following theorem: let p be a polynomial on $\mathbf{C}^n$; the pole order in t of the Laurent series $p(t)$ equals the pole order of p, restricted to $\mathbf{T}^r$, at infinity.

Concrete embedding

$$
\begin{aligned}
a_1(t) &= ct + O(t^2), \\
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Four functions with a pole of order 2 at most on this one

- $z_0 := 1$
- $z_1 := a_3 a_4 = -t^{-2}(1 + \mathcal{O}(t))$
- $z_2 := a_1 a_3 a_4 = -dt^{-2}(1 + \mathcal{O}(t))$
- $z_3 := a_1 a_2 a_4 = -at^{-2}(1 + \mathcal{O}(t))$

The complete embedding

Nine functions with a pole of order 3 at most

- $z_0 := 1$
- $z_1 := a_3 a_4$
- $z_2 := a_1 a_3 a_4$
- $z_3 := a_1 a_2 a_4$
- $z_4 := a_1 a_3^2 a_4$
- $z_5 := a_1 a_3 a_4^2$
- $z_6 := a_1 a_3 a_4((a_1 + a_5)a_3 - (a_1 + a_2)a_4)$
- $z_7 := a_1 a_2 a_3^2 a_4^2$
- $z_8 := a_1 a_3 a_4^2((a_1 + a_2)^2 + a_1 a_5)$

This gives an embedding of each of the Abelian surfaces in $\mathbf{P}^8$

Embedding the divisor at infinity

Step 1: Equations for the divisor at infinity (5 curves)

Substitute Laurent series in the constants of motion:

$$H_1 = a_1 + a_2 + a_3 + a_4 + a_5 = 2a + d = c_1$$

$$H_2 = a_1 a_3 + a_2 a_4 + a_3 a_5 + a_4 a_1 + a_5 a_2 = b - c + 2ad = c_2$$

$$H_3 = a_1 a_2 a_3 a_4 a_5 = bcd = c_3$$

Eliminating a and c gives the equation of an algebraic curve:

$$bd(b - c_2 + d(c_1 - d)) + c_3 = 0$$

Step 2: Embedding each curve

Substitute each of the five Laurent series in the embedding and let $t \longrightarrow 0$. For the first two for example:

$$\varphi^{(1)} : (b, d) \mapsto (0 : 0 : 0 : 1 : 0 : d : -d : 2d^2 : bd : -d^3)$$

$$\varphi^{(2)} : (b, d) \mapsto (1 : 0 : bd : 0 : bd^2 : 0 : bd(2ad - b) : 0 : 2ab^2 d)$$

$$\vdots$$

One sees that the image curves are *disjoint*.

Step 3: Compactifying the curves

Each of the curves

$$bd(b - c_2 + d(c_1 - d)) + c_3 = 0$$

is smoothly compactified by adding three points at infinity:

$$\begin{array}{lrclrcl}
\infty : & d & = & \varsigma^{-1} & b & = & \varsigma^{-2} - c_1\varsigma^{-1} + c_2 - c_3\varsigma^3 + O(\varsigma^4) \\
\infty' : & b & = & \varsigma^{-1} & d & = & -c_3\varsigma^2 - c_2c_3\varsigma^3 - c_2^2c_3\varsigma^4 + O(\varsigma^5) \\
\infty'' : & d & = & \varsigma^{-1} & b & = & c_3\varsigma^3 + c_1c_3\varsigma^4 + O(\varsigma^5)
\end{array}$$

Step 4: Points at infinity in $\mathbf{P}^8$

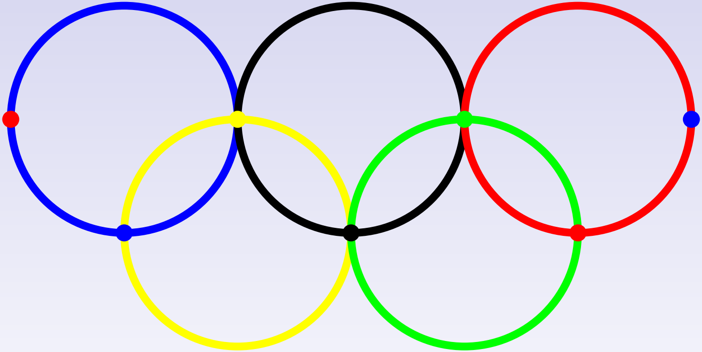
	$\varphi^{(1)}$	$\varphi^{(2)}$	$\varphi^{(3)}$	$\varphi^{(4)}$	$\varphi^{(5)}$
∞	P_5	P_1	P_2	P_3	P_4
∞'	P_1	P_2	P_3	P_4	P_5
∞''	P_2	P_3	P_4	P_5	P_1

Step 4: Points at infinity in $\mathbf{P}^8$

	$\varphi^{(1)}$	$\varphi^{(2)}$	$\varphi^{(3)}$	$\varphi^{(4)}$	$\varphi^{(5)}$
∞	P_5	P_1	P_2	P_3	P_4
∞'	P_1	P_2	P_3	P_4	P_5
∞''	P_2	P_3	P_4	P_5	P_1

$$\begin{aligned}
P_1 &= (0:0:0:1:0:0:0:0:0)\\
P_2 &= (0:0:0:0:0:0:0:0:1)\\
P_3 &= (1:0:0:0:0:0:c_3:0:-c_1c_3)\\
P_4 &= (1:0:0:0:0:0:-c_3:0:0)\\
P_5 &= (0:0:0:0:0:0:0:1:-1)
\end{aligned}$$

The divisor at infinity



Each of the circles represents a copy of the compactified genus two curve $\Gamma_{c_1 c_2}$.