

**Quasi-linear elliptic type equations
invariant under five-dimensional
solvable Lie algebras**

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1 Introduction

The problem of group classification of quasi-linear elliptic type equations

$$\Delta u = f(x, y, u, u_x, u_y) \quad (1)$$

in two-dimensional space is considered. The list of all equations of this type is obtained, which admit solvable Lie algebras of symmetry operators with dimension up to five.

Notice that the problem of classification of equations of form (1), which admit Lie algebras of symmetry operators with non-trivial Levi factor, is solved completely in [**Lahno V.I. and Spichak S.V., Group classification of quasi-linear elliptic type equations. I. Invariance under Lie algebras with nontrivial Levi decomposition, Ukrainian Mathematical Journal, 2007, V. 59, 11, 1532–1545 (in Ukrainian).**]. Furthermore, we are interested in essential nonlinear equations, that is which can not be reduced to linear ones with use of local or nonlocal transformations. For example, the first of the following two equations

$$\Delta u = f(u)(u_x^2 + u_y^2), \quad f \neq 0; \quad (2)$$

$$\Delta u = \lambda e^{\gamma u}, \quad \lambda, \gamma \in \mathbb{R}, \quad \lambda \cdot \gamma \neq 0. \quad (3)$$

is related to the linear equation $\Delta u = \lambda u$ by the local transformation, and the second one is related to the Laplace equation by the Bäcklund transformation [Bullaf R., Kodri F. Solitons. M.: Mir, 1983. – 408 p. (in Russian)].

Statement 1 *The invariance group of the equation (1) is generated by the infinitesimal operator*

$$v = a(x, y)\partial_x + b(x, y)\partial_y + c(x, y, u)\partial_u, \quad (4)$$

where functions a, b, c, F satisfy the following system of equations:

$$\begin{aligned} a_y + b_x &= 0, & a_x - b_y &= 0, \\ c_{xx} + c_{yy} + 2u_x c_{xu} + 2u_y c_{yu} + (u_x^2 + u_y^2)c_{uu} + (c_u - 2a_x)F &= \\ &= aF_x + bF_y + cF_u + [c_x + u_x(c_u - a_x) - u_y b_x]F_{u_x} & (5) \\ &+ [c_y + u_y(c_u - b_y) - u_x a_y]F_{u_y}. \end{aligned}$$

It is not difficult to see that the two first equations are Cauchy-Riemann conditions, that means that functions a and b are harmonic functions.

The equivalence group \mathcal{E} is formed by those transformations

$$\bar{x} = \alpha(x, y, u), \quad \bar{y} = \beta(x, y, u), \quad v = \gamma(x, y, u), \quad \frac{D(\bar{x}, \bar{y}, v)}{D(x, y, u)} \neq 0,$$

that preserve differential structure of the equation (1), that is transform it to an equation of the form

$$v_{\bar{x}\bar{x}} + v_{\bar{y}\bar{y}} = \Phi(\bar{x}, \bar{y}, v, v_{\bar{x}}, v_{\bar{y}}).$$

Statement 2 *The group \mathcal{E} of the equation (1) is formed by the following transformation:*

$$\bar{x} = \alpha(x, y), \quad \bar{y} = \beta(x, y), \quad v = \gamma(x, y, u), \quad (6)$$

$$\begin{aligned} \alpha_x &= \epsilon\beta_y, & \alpha_y &= -\epsilon\beta_x \quad (\epsilon = \pm 1), \\ \alpha_x^2 + \alpha_y^2 &= \beta_x^2 + \beta_y^2 \neq 0, & \gamma_u &\neq 0. \end{aligned}$$

Statement 3 *There exist such transformations from the group \mathcal{E} that reduce the operator (4) to one of the following operators:*

$$v = \partial_x; \quad v = \partial_u. \quad (7)$$

The corresponding classes of invariant equations are:

$$\begin{aligned} \Delta u &= F(y, u, u_x, u_y) : A_1^1 = \langle \partial_x \rangle, & (8) \\ \Delta u &= F(x, y, u_x, u_y) : A_1^2 = \langle \partial_u \rangle. \end{aligned}$$

2 The general method of classification

Let $A_n = \langle e_1, \dots, e_n \rangle$ be Lie algebra of dimension n , with basic elements

$$e_i = a_i(x, y)\partial_x + b_i(x, y)\partial_y + c_i(x, y, u)\partial_u, \quad i = \overline{1, n}, \quad (9)$$

where the couple of functions $a_i(x, y), b_i(x, y)$ satisfies of Cauchy-Riemann conditions (5) (nonclassification equations).

Definition 1. Representation (9) of the algebra A_n is called acceptable if there exist such functions $F(x, y, u, u_x, u_y)$ for which the three functions $(a_i(x, y), b_i(x, y), c_i(x, y, u))$ are solutions of determining equations (5) and corresponding invariant equations (1) are nonlinear and can not be linearized.

Definition 2. Two representations of the algebra A_n with functions $(a_i(x, y), b_i(x, y), c_i(x, y, u))$ and $(\tilde{a}_i(x, y), \tilde{b}_i(x, y), \tilde{c}_i(x, y, u))$ are called equivalent if the every operator (9) e_i can be reduced to the operator $\tilde{e}_i = \tilde{a}_i(x, y)\partial_x + \tilde{b}_i(x, y)\partial_y + \tilde{c}_i(x, y, u)\partial_u$ with use of equivalence group \mathcal{E} .

Definition 3. Two representations of an algebra is called isomorphic if the set of basic operators of one algebra can be obtained as linear combination of basic operators of another algebra.

Definition 4. Two representations of an algebra is called different if operators one of these can not be obtained with use of superposition of equivalence transformations \mathcal{E} and linear transformations of basic elements of algebra.

Examples for the algebra $A_{2.1} = \langle e_1, e_2 \rangle, [e_1, e_2] = 0$:

1. The representations $\langle \partial_x, \partial_y \rangle$ and $\langle \partial_y, \partial_x \rangle$ are isomorphic and equivalent.
2. The representations $\langle \partial_x, \partial_u \rangle$ and $\langle \partial_u, \partial_x \rangle$ are isomorphic but not equivalent.
3. The representations $\langle \partial_u, x\partial_u \rangle$ and $\langle \partial_u, y\partial_u \rangle$ are equivalent but not isomorphic.
4. The representations $\langle \partial_y, \partial_x \rangle$ and $\langle \partial_u, x\partial_u \rangle$ are different.

Let us denote the set of all non-equivalent and acceptable representations (maybe isomorphic) of an algebra $A_{k,l}$ of dimension k as $\mathfrak{N}A_{k,l}$, and the set of all different and acceptable representations as $\mathfrak{D}A_{k,l}$ (l is index number of algebra).

The general inductive algorithm of construction of representations $\mathfrak{D}A_{k,l}$, and of corresponding classes of invariant equations is following.

- In according to Statement 3. for 1-dimensional algebra $A_{1,1}$ we have $\mathfrak{N}A_{1,1} = \mathfrak{D}A_{1,1} = \{\partial_x, \partial_u\}$ and corresponding classes of invariant equations (8).
- Let us suppose that we construct the sets of representations $\mathfrak{N}A_{k,l}$ and $\mathfrak{D}A_{k,l}$ for all algebras $A_{k,l}$ of dimension k , and corresponding to them classes of invariant equations.
- Then, we consider algebras $A_{k+1,l}$ of dimension $k + 1$. Each of them includes some ideal $A_{k,m}$ of dimension k (this is right for solvable algebras). Then using sequentially a constructed representation from the set $\mathfrak{N}A_{k,m}$ for algebra $A_{k,m} = \langle e_1, \dots, e_k \rangle$ we obtain an additional operator e_{k+1} so that they satisfy commuting relationships of algebra $A_{k+1,l}$. Also we can simplify a view of the operators with use of equivalence transformations group \mathcal{E} (6).
- Verify if obtained representation be acceptable, that is if the corresponding class of nonlinear equations is exist. If it is so we include it to the set $\mathfrak{N}A_{k+1,l}$, but if it is not we exclude it from the further consideration. From the constructed set $\mathfrak{N}A_{k+1,l}$ we pick out the maximal set $\mathfrak{D}A_{k+1,l}$ of non-isomorphic representations.
- After constructing the set $\mathfrak{D}A_{4,l}$ of different representations of algebras of dimension 4 and corresponding classes of invariant equations, we take advantage of the straight Ovsjannikov method of classification [Ovsjannikov L.V. Group properties of nonlinear heat equations// Dokl. Acad of science of USSR. — 1959.— V. 125, N 3. — P.492–495.] for obtaining list of concrete equations invariant under 5-dimensional Lie algebra.

For illustration of above algorithm we give construction the set of different representations of the commutative 3-dimensional algebra $A_{3.1} = \langle e_1, e_2, e_3 \rangle$. So, the ideal of this algebra is $A_{2.1} = \langle e_1, e_2 \rangle$. We found that $\mathfrak{N}A_{2.1} = \{A_{2.1}^1, A_{2.1}^2, A_{2.1}^3, A_{2.1}^4\}$, where

$$\begin{aligned} A_{2.1}^1 &= \langle \partial_x, \lambda_1 \partial_x + \lambda_2 \partial_y \rangle \ (\lambda_2 \neq 0), \quad A_{2.1}^2 = \langle \partial_u, f(x, y) \partial_u \rangle \\ A_{2.1}^3 &= \langle \partial_x, \lambda_1 \partial_x + \partial_u \rangle, \quad A_{2.1}^4 = \langle \partial_u, \partial_x \rangle \end{aligned} \quad (10)$$

(λ_1, λ_2 are constants, $f(x, y)$ is arbitrary function). Then each of the representation (10) we expand by operator e_3 so that it commutes with another. And then, after simplification all three operator with use of equivalence transformations group, we get all non-equivalent representations:

$$\begin{aligned} A_{3.1}^1 &= \langle \partial_x, \lambda_1 \partial_x + \lambda_2 \partial_y, \lambda_3 \partial_x + \lambda_4 \partial_y + \partial_u \rangle, \ (\lambda_2 \neq 0), \\ A_{3.1}^2 &= \langle \partial_x, \lambda_1 \partial_x + \partial_u, \lambda_3 \partial_x + \lambda_4 \partial_y \rangle, \ (\lambda_4 \neq 0), \\ A_{3.1}^3 &= \langle \partial_x, \lambda_1 \partial_x + \partial_u, \lambda_3 \partial_x + f(y) \partial_u \rangle, \ (f'(y) \neq 0), \\ A_{3.1}^4 &= \langle \partial_u, \partial_x, \lambda_3 \partial_x + \lambda_4 \partial_y \rangle, \ (\lambda_4 \neq 0), \\ A_{3.1}^5 &= \langle \partial_u, \partial_x, \lambda_3 \partial_x + f(y) \partial_u \rangle, \ (f'(y) \neq 0), \\ A_{3.1}^6 &= \langle \partial_u, f(y) \partial_u, \partial_x \rangle, \ (f'(y) \neq 0), \\ A_{3.1}^7 &= \langle \partial_u, f(x, y) \partial_u, g(x, y) \partial_u \rangle. \end{aligned} \quad (11)$$

It is easily to see that for $A_{3.1}^3$ and $A_{3.1}^4$ representations of its 2-dimensional ideals $A_{2.1} = \langle e_1, e_2 \rangle$ are isomorphic, but they oneself are different. It can be an opposite case, that is when representations of ideals are different, but its expansions are isomorphic.

Below we give the results of classification according with our algorithm.

3 Invariance with respect to 2-dimensional Lie algebras

Algebra $A_{2.1}$

Commutation relationships: $[e_1, e_2] = 0$

Representations and respective forms of functions:

1. $A_{2.1}^1 = \langle \partial_x, \partial_y \rangle : F = G(u, u_x, u_y);$
 2. $A_{2.1}^2 = \langle \partial_x, \partial_u \rangle : F = G(y, u_x, u_y);$
 3. $A_{2.1}^3 = \langle \partial_u, f(x, y)\partial_u \rangle : F = \frac{f_x u_x + f_y u_y}{f_x^2 + f_y^2} (f_{xx} + f_{yy}) + G(x, y, \omega),$
- (12)

$$\omega = f_y u_x - f_x u_y, \quad f_x^2 + f_y^2 \neq 0.$$

Algebra $A_{2.2}$

Commutation relationships: $[e_1, e_2] = e_2$

Representations and respective forms of functions:

$$\begin{aligned} A_{2.2}^2 &= \langle -x\partial_x - y\partial_y, \partial_x \rangle : F = (u_x^2 + u_y^2)G(u, \omega_1, \omega_2), \quad \omega_1 = yu_x, \quad \omega_2 = yu_y; \\ A_{2.2}^1 &= \langle \partial_x - u\partial_u, \partial_u \rangle : F = e^{-x}G(y, \omega_1, \omega_2), \quad \omega_1 = e^x u_x, \quad \omega_2 = e^x u_y; \\ A_{2.2}^3 &= \langle -u\partial_u, \partial_u \rangle : F = (u_x + u_y)G(x, y, \omega), \quad \omega = u_x u_y^{-1}. \end{aligned}$$

Let us note that for arbitrary forms of functions G the respective representations are maximal invariance algebras of equations.

4 Invariance with respect to 3-dimensional Lie algebras

The solvable algebras $A_{3.i} = \langle e_1, e_2, e_3 \rangle$, $i = \overline{1, 9}$ contain the 2-dimensional subalgebra $A_{2.1}$. Furthermore, it may be assumed for all algebras, except $A_{3.2}$, that $A_{2.1} = \langle e_1, e_2 \rangle$. For the algebra $A_{3.2}$ we assume that $A_{2.1} = \langle e_2, e_3 \rangle$.

The number of representations for algebras:

$A_{3.i}$ ($i = 1, 3, 4, 6, 7$) \rightarrow two different representations.

$A_{3.8}, A_{3.9}$ \rightarrow three representations.

$A_{3.2}, A_{3.5}$ \rightarrow four representations.

So, we have 24 representations for three-dimensional Lie algebras.

Algebra $A_{3.1}$

1. $A_{3.1}^1 = \langle \partial_x, \partial_y, \partial_u \rangle, F = G(u_x, u_y);$
 2. $A_{3.1}^2 = \langle \partial_x, \partial_u, f(y)\partial_u \rangle, f'(y) \neq 0, F = \frac{f''}{f'}u_y + G(y, u_x);$
 3. $A_{3.1}^3 = \langle \partial_u, f(x, y)\partial_u, g(x, y)\partial_u \rangle.$
- (13)

Let us note that the last representation is not acceptable, because either the corresponding invariant equation is linear or the infinitesimal operators are linearly dependent.

Algebra $A_{3.2}$

1. $A_{3.2}^1 = \langle -x\partial_x - y\partial_y, \partial_x, \partial_u \rangle, F = u_x^2 G(yu_x, yu_y);$
2. $A_{3.2}^2 = \langle -u\partial_u, \partial_u, \partial_x \rangle, F = (u_x + u_y)G\left(y, \frac{u_x}{u_y}\right);$
3. $A_{3.2}^3 = \langle \partial_y - u\partial_u, \partial_u, \partial_x \rangle, F = (u_x + u_y)G(e^y u_x, e^y u_y);$
4. $A_{3.2}^4 = \langle \partial_x - u\partial_u, \partial_u, f(y)e^{-x}\partial_u \rangle, f(y) \neq 0,$
 $F = -\frac{f + f''}{f}u_x + e^{-x}G(y, e^x f' u_x + e^x f u_y).$

Algebra $A_{3.3}$

1. $A_{3.3}^1 = \langle \partial_u, \partial_x, \partial_y + x\partial_u \rangle, F = G(u_x - y, u_y);$
2. $A_{3.3}^2 = \langle \partial_u, \partial_x, (f(y) + x)\partial_u \rangle, F = f''u_x + G(y, u_y - f'u_x).$

Algebra $A_{3.4}$

1. $A_{3.4}^1 = \langle \partial_u, \partial_x, x\partial_x + y\partial_y + (u + x)\partial_u \rangle, F = e^{-u_x}G(u_y, ye^{-u_x});$
2. $A_{3.4}^2 = \langle \partial_u, (f(y) - x)\partial_u, \partial_x + u\partial_u \rangle,$
 $F = \frac{f'u_y - u_x}{1 + (f')^2}f'' + e^x G(y, f'e^{-x}u_x + e^{-x}u_y).$

Algebra $A_{3.5}$

1. $A_{3.5}^1 = \langle \partial_x, \partial_y, x\partial_x + y\partial_y \rangle$, $F = (u_x + u_y)^2 G\left(u, \frac{u_x}{u_y}\right)$;
2. $A_{3.5}^2 = \langle \partial_x, \partial_y, x\partial_x + y\partial_y + \partial_u \rangle$, $F = e^{-2u} G\left(e^u u_x, \frac{u_x}{u_y}\right)$;
3. $A_{3.5}^3 = \langle \partial_x, \partial_u, x\partial_x + y\partial_y + u\partial_u \rangle$, $F = y^{-1} G(u_x, u_y)$;
4. $A_{3.5}^4 = \langle \partial_u, f(y)\partial_u, \partial_x + u\partial_u \rangle$, $f' \neq 0$, $F = \frac{u_y}{f'} f'' + e^x G(y, f' e^{-x} u_x)$;
5. $A_{3.5}^5 = \langle \partial_u, f(x, y)\partial_u, u\partial_u \rangle$.

The further analysis show that for the last representation the corresponding invariant equation is linear.

Algebra $A_{3.6}$

1. $A_{3.6}^1 = \langle \partial_x, \partial_u, x\partial_x + y\partial_y - u\partial_u \rangle$, $F = y^{-3} G(y^2 u_x, y^2 u_y)$;
2. $A_{3.6}^2 = \langle \partial_u, e^{2x} f(y)\partial_u, \partial_x + u\partial_u \rangle$, $f(y) \neq 0$,

$$F = \frac{2f u_x + f' u_y}{4f^2 + (f')^2} (f'' + 4f) + e^x G(y, f' e^{-x} u_x - 2f e^{-x} u_y).$$

Algebra $A_{3.7}^q$

1. $A_{3.7}^1 = \langle \partial_x, \partial_u, x\partial_x + y\partial_y + qu\partial_u \rangle$, $q \neq 0, \pm 1$,
 $F = y^{q-2} G(y^{1-q} u_x, y^{1-q} u_y)$;
2. $A_{3.7}^2 = \langle \partial_u, e^{(1-q)x} f(y)\partial_u, \partial_x + u\partial_u \rangle$, $0 < |q| < 1$,

$$F = \frac{(1-q)f u_x + f' u_y}{(1-q)^2 f^2 + (f')^2} (f'' + f(1-q)^2) + e^x G(y, f' e^{-x} u_x + (q-1)f e^{-x} u_y).$$

Algebra $A_{3.8}$

1. $A_{3.8}^1 = \langle \partial_x, \partial_y, y\partial_x - x\partial_y \rangle$, $F = G(u, u_x^2 + u_y^2)$;
2. $A_{3.8}^2 = \langle \partial_x, \partial_y, y\partial_x - x\partial_y + \partial_u \rangle$, $F = G\left(u_x^2 + u_y^2, \arctan\left(\frac{u_x}{u_y}\right) - u\right)$;
3. $A_{3.8}^3 = \langle \partial_u, \tan(f(y) - x)\partial_u, \partial_x - u \tan(f(y) - x)\partial_u \rangle$,

$$F = \frac{f' u_y - u_x}{(f')^2 + 1} f'' + 2(f' u_y - u_x) \tan(f - x) + (f' u_x + u_y) G(y, \cos(f - x)(f' u_x + u_y)).$$

Algebra $A_{3.9}^q$

$$1. A_{3.9}^1 = \langle \partial_x, \partial_y, (qx + y)\partial_x + (qy - x)\partial_y \rangle, \quad q > 0,$$

$$F = (u_x^2 + u_y^2)G \left(u, \ln(u_x^2 + u_y^2) + 2q \arctan \frac{u_x}{u_y} \right);$$

$$2. A_{3.9}^2 = \langle \partial_x, \partial_y, (qx + y)\partial_x + (qy - x)\partial_y + \partial_u \rangle, \quad q > 0,$$

$$F = e^{-2qu}G \left(\arctan \left(\frac{u_x}{u_y} \right) - u, \ln(u_x^2 + u_y^2) + 2q \arctan \frac{u_x}{u_y} \right);$$

$$3. A_{3.9}^3 = \langle \partial_u, \tan(f(y) - x)\partial_u, \partial_x + (q - \tan(f(y) - x))u\partial_u \rangle, \quad q > 0,$$

$$F = \frac{f'u_y - u_x}{(f')^2 + 1} f'' + 2(f'u_y - u_x) \tan(f - x) +$$

$$(f'u_x + u_y)G(y, \cos(f - x)(f'u_x + u_y)e^{-qx}).$$

Annotation:

1. Operators of certain representations contain arbitrary functions $f(y)$, which can not be reduced with use of equivalence transformations group.
2. The all representations of the algebra $A_{3.7}$ when $q = -1$ coincide with the representations of the algebra $A_{3.6}$, as well as corresponding classes of invariant equations.
3. The representation $A_{3.7}^1$ when $q = 1$ coincides with the representation $A_{3.5}^3$.
4. The representations of the algebra $A_{3.9}$ when $q = 0$ coincide with the representations of the algebra $A_{3.8}$.

5 Invariance with respect to 4-dimensional Lie algebras

Algebraic structure:

1. $A_{3.i} \subset \tilde{A}_{4.i} = A_{3.i} \oplus A_1, (i = 1, 2, \dots, 9);$
2. $A_{2.2} \subset \tilde{A}_{4.10} = A_{2.2} \oplus A_{2.2};$
3. $A_{3.1} \subset A_{4.i}, (i = 1, 2, \dots, 6);$
4. $A_{3.3} \subset A_{4.i}, (i = 7, 8, 9);$
5. $A_{3.5} \subset A_{4.10};$

The number of representations for algebras:

$A_{4.i}$ ($i = 1, 4, 7, 9$); $\tilde{A}_{4.3} \rightarrow$ one representation.

$\tilde{A}_{4.i}$ ($i = 2, 4, \dots, 10$); $A_{4.i}$ ($i = 3, 6$) \rightarrow two different representations.

$A_{4.i}$ ($i = 2, 5, 10$) \rightarrow three representations.

$A_{4.8} \rightarrow$ four representations.

So, we have 38 representations for four-dimensional Lie algebras.

Decomposable algebras

Algebra $\tilde{A}_{4.2} = A_{3.2} \oplus A_1$

1. $\tilde{A}_{4.2}^1 = \langle -u\partial_u, \partial_u, \partial_x \rangle \oplus \langle \partial_y \rangle, F = (u_x + u_y)G\left(\frac{u_x}{u_y}\right);$
2. $\tilde{A}_{4.2}^2 = \langle \partial_y - u\partial_u, \partial_u, \partial_x \rangle \oplus \langle e^{-y}\partial_u \rangle, F = e^{-y}G(e^y u_x) - (u_x + u_y).$

Algebra $\tilde{A}_{4.3} = A_{3.3} \oplus A_1$

1. $\tilde{A}_{4.3}^1 = \langle \partial_u, \partial_x, x\partial_u \rangle \oplus \langle \partial_y \rangle, F = G(u_y);$
2. $\tilde{A}_{4.3}^2 = \langle \partial_u, \partial_x, \partial_y + x\partial_u \rangle \oplus \langle \partial_x + y\partial_u \rangle, F = G(u_x - y).$

With use of the change of variables $u = \tilde{u} + xy, \tilde{x} = y, \tilde{y} = -x$ this representation reduced to the case 1. (It is given for illustration that under expansion of different representations we can get not obligatory different ones).

Algebra $\tilde{A}_{4.4} = A_{3.4} \oplus A_1$

1. $\tilde{A}_{4.4}^1 = \langle \partial_u, \partial_x, x\partial_x + y\partial_y + (u+x)\partial_u \rangle \oplus \langle y\partial_u \rangle, F = e^{-ux}G(ye^{-ux});$
2. $\tilde{A}_{4.4}^2 = \langle \partial_u, -x\partial_u, \partial_x + u\partial_u \rangle \oplus \langle \partial_y \rangle, F = e^x G(e^{-x}u_y).$

Algebra $\tilde{A}_{4.5} = A_{3.5} \oplus A_1$

1. $\tilde{A}_{4.5}^1 = \langle \partial_x, \partial_y, x\partial_x + y\partial_y \rangle \oplus \langle \partial_u \rangle, F = (u_x + u_y)^2 G\left(\frac{u_x}{u_y}\right);$
2. $\tilde{A}_{4.5}^2 = \langle \partial_x, \partial_u, x\partial_x + y\partial_y + u\partial_u \rangle \oplus \langle y\partial_u \rangle, F = y^{-1}G(u_x);$

Algebra $\tilde{A}_{4.6} = A_{3.6} \oplus A_1$

1. $\tilde{A}_{4.6}^1 = \langle \partial_x, \partial_u, x\partial_x + y\partial_y - u\partial_u \rangle \oplus \langle y^{-1}\partial_u \rangle,$
 $F = -2y^{-1}u_y + y^{-3}G(y^2u_x);$
2. $\tilde{A}_{4.6}^2 = \langle \partial_u, e^{2x}\partial_u, \partial_x + u\partial_u \rangle \oplus \langle \partial_y \rangle, F = 2u_x + e^xG(e^{-x}u_y).$

Algebra $\tilde{A}_{4.7}^q = A_{3.7}^q \oplus A_1$

1. $\tilde{A}_{4.7}^1 = \langle \partial_x, \partial_u, x\partial_x + y\partial_y + qu\partial_u \rangle \oplus \langle y^q\partial_u \rangle, q \neq 0, \pm 1,$
 $F = (q - 1)y^{-1}u_y + y^{q-2}G(y^{1-q}u_x);$
2. $\tilde{A}_{4.7}^2 = \langle \partial_u, e^{(1-q)x}\partial_u, \partial_x + u\partial_u \rangle \oplus \langle \partial_y \rangle, 0 < |q| < 1,$
 $F = (1 - q)u_x + e^xG(e^{-x}u_y).$

Algebra $\tilde{A}_{4.8} = A_{3.8} \oplus A_1$

1. $\tilde{A}_{4.8}^1 = \langle \partial_x, \partial_y, y\partial_x - x\partial_y \rangle \oplus \langle \partial_u \rangle, F = G(u_x^2 + u_y^2);$
2. $\tilde{A}_{4.8}^2 = \langle \partial_u, -\tan x\partial_u, \partial_x + u \tan x\partial_u \rangle \oplus \langle \partial_y \rangle,$
 $F = 2u_x \tan x + u_y G(u_y \cos x).$

Algebra $\tilde{A}_{4.9}^q = A_{3.9}^q \oplus A_1$

1. $\tilde{A}_{4.9}^1 = \langle \partial_x, \partial_y, (qx + y)\partial_x + (qy - x)\partial_y \rangle \oplus \langle \partial_u \rangle, q > 0,$
 $F = (u_x^2 + u_y^2)G \left((u_x^2 + u_y^2) \exp \left(2q \arctan \frac{u_x}{u_y} \right) \right);$
2. $\tilde{A}_{4.9}^2 = \langle \partial_u, -\tan x\partial_u, \partial_x + u(q + \tan x)\partial_u \rangle \oplus \langle \partial_y \rangle, q > 0,$
 $F = 2u_x \tan x + u_y G(u_y e^{-qx} \cos x)$

Algebra $\tilde{A}_{4.10} = A_{2.2} \oplus A_{2.2}$

1. $\tilde{A}_{4.10}^1 = \langle -x\partial_x - y\partial_y, \partial_x \rangle \oplus \langle -u\partial_u, \partial_u \rangle, F = \frac{(u_x^2 + u_y^2)}{yu_x} G \left(\frac{u_x}{u_y} \right);$
2. $\tilde{A}_{4.10}^2 = \langle \partial_x - u\partial_u, \partial_u \rangle \oplus \langle \lambda_1\partial_x + \lambda_2\partial_y, e^{-x}e^{\frac{(1+\lambda_1)y}{\lambda_2}}\partial_u \rangle, \lambda_2 \neq 0,$
 $F = - \left(1 + \left(\frac{1 + \lambda_1}{\lambda_2} \right)^2 \right) u_x + e^{-x} e^{\frac{\lambda_1 y}{\lambda_2}} G \left(e^x e^{-\frac{\lambda_1 y}{\lambda_2}} \left(\frac{1 + \lambda_1}{\lambda_2} u_x + u_y \right) \right).$

In the last case the arbitrary parameters λ_1, λ_2 can not be reduced with use of equivalence transformations group

Non-decomposable algebras

Algebra $A_{4.1}$

$$A_{4.1}^1 = \langle \partial_y - xy\partial_u \rangle \oplus \langle \partial_u, -y\partial_u, \partial_x \rangle, \quad F = G(y^2 + 2u_x).$$

Algebra $A_{4.2}^q$

1. $A_{4.2}^1 = \langle x\partial_x + y\partial_y + (u + y)\partial_u \rangle \oplus \langle \partial_x, \partial_u, \partial_y \rangle, \quad q = 1, \quad F = e^{-uy}G(u_x);$
2. $A_{4.2}^2 = \langle qx\partial_x + qy\partial_y + u\partial_u \rangle \oplus \langle \partial_x, \partial_u, -q^{-1} \ln y \partial_u \rangle, \quad q \neq 0,$

$$F = -\frac{u_y}{y} + \frac{u_x}{y} G\left(y^{\frac{q-1}{q}} u_x\right);$$
3. $A_{4.2}^3 = \langle x\partial_x + y\partial_y + (qu + y^{q-1}x)\partial_u \rangle \oplus \langle \partial_u, y^{q-1}\partial_u, \partial_x \rangle \quad (q \neq 0; 1),$

$$F = \frac{q-2}{y} u_y + y^{q-2} G(u_x y^{1-q} - \ln y).$$

Algebra $A_{4.3}$

1. $A_{4.3}^1 = \langle x\partial_x + y\partial_y \rangle \oplus \langle \partial_x, \partial_u, -\ln y \partial_u \rangle, \quad F = -\frac{u_y}{y} + u_x^2 G(yu_x);$
2. $A_{4.3}^2 = \langle \partial_y + (u + xe^y)\partial_u \rangle \oplus \langle \partial_u, e^y \partial_u, \partial_x \rangle, \quad F = u_y + e^y G(e^{-y}u_x - y).$

Algebra $A_{4.4}$

$$A_{4.4}^1 = \langle x\partial_x + y\partial_y + (u - x \ln y)\partial_u \rangle \oplus \langle \partial_u, -\ln y \partial_u, \partial_x \rangle,$$

$$F = -\frac{u_y}{y} + \frac{1}{y} G(2u_x + \ln^2 y).$$

Algebra $A_{4.5}^{p,q}$

1. $A_{4.5}^1 = \langle x\partial_x + y\partial_y + pu\partial_u \rangle \oplus \langle \partial_x, \partial_y, \partial_u \rangle, \quad p \neq 0; 1, \quad q = 1,$

$$F = (u_x u_y)^r G\left(\frac{u_x}{u_y}\right), \quad r = \frac{p-2}{2(p-1)};$$
2. $A_{4.5}^2 = \langle x\partial_x + y\partial_y + qu\partial_u \rangle \oplus \langle \partial_x, \partial_u, y^{q-p}\partial_u \rangle, \quad -1 \leq p < q \leq 1, \quad pq \neq 0,$

$$F = \frac{q-p-1}{y} u_y + y^{q-2} G(u_x y^{1-q}); \tag{14}$$
3. $A_{4.5}^3 = \langle qx\partial_x + qy\partial_y + u\partial_u \rangle \oplus \langle \partial_u, \partial_x, y^{(1-p)/q}\partial_u \rangle, \tag{15}$

$$-1 \leq p < 1, \quad -1 \leq q \leq 1, \quad pq \neq 0,$$

$$F = \frac{1-q-p}{qy} u_y + y^{(1-2q)/q} G(u_x y^{(q-1)/q}).$$

Algebra $A_{4.6}^{p,q}$

1. $A_{4.6}^1 = \langle (px + y)\partial_x + (py - x)\partial_y + qu\partial_u \rangle \oplus \langle \partial_u, \partial_x, \partial_y \rangle, \quad q \neq 0, \quad p \geq 0,$

$$F = \exp\left((q - 2p) \arctan \frac{u_x}{u_y}\right) G\left((u_x^2 + u_y^2) \exp\left(2(p - q) \arctan \frac{u_x}{u_y}\right)\right);$$
2. $A_{4.6}^2 = \langle qx\partial_x + qy\partial_y + (p + \tan(q^{-1} \ln y)u)\partial_u \rangle \oplus$
 $\langle \partial_x, \partial_u, -\tan(q^{-1} \ln y)\partial_u \rangle, \quad q \neq 0, \quad p \geq 0,$

$$F = \left(\frac{2 \tan(q^{-1} \ln y)}{qy} - \frac{1}{y}\right) u_y + \frac{1}{\cos(q^{-1} \ln y) y^{\frac{2q-p}{q}}} G\left(u_x \cos(q^{-1} \ln y) y^{\frac{q-p}{q}}\right).$$

Algebra $A_{4.7}$

- $$A_{4.7}^1 = \langle x\partial_x + y\partial_y + (2u - \frac{x^2}{2} + \lambda xy)\partial_u \rangle \oplus \langle \partial_u, (\lambda y - x)\partial_u, \partial_x \rangle,$$
- $$F = -\ln y + G\left(\frac{\lambda u_x + u_y}{y} - \lambda^2 \ln y\right).$$

Algebra $A_{4.8}^q$

1. $A_{4.8}^1 = \langle x\partial_x + y\partial_y + 2u\partial_u \rangle \oplus \langle \partial_u, \partial_x, \partial_y + x\partial_u \rangle \quad (q = 1),$

$$F = G\left(\frac{u_x - y}{u_y}\right);$$
2. $A_{4.8}^2 = \langle x\partial_x + y\partial_y + (1 + q)u\partial_u \rangle \oplus \langle \partial_u, \partial_x, (\lambda y + x)\partial_u \rangle, \quad |q| \leq 1,$

$$F = y^{q-1} G(y^{-q}(u_y - \lambda u_x));$$
3. $A_{4.8}^3 = \langle \partial_y + u\partial_u \rangle \oplus \langle \partial_u, -x\partial_u, \partial_x + \lambda\partial_y \rangle, \quad (q = 0),$

$$F = \exp(y - \lambda x) G(\exp(\lambda x - y) u_y);$$
4. $A_{4.8}^4 = \langle qx\partial_x + qy\partial_y + (1 + q)u\partial_u \rangle \oplus \langle \partial_u, (\lambda y - x)\partial_u, \partial_x \rangle, \quad 0 < |q| \leq 1,$

$$F = y^{(1-q)/q} G\left(\frac{\lambda u_x + u_y}{y^{1/q}}\right).$$

Algebra $A_{4.9}^q$

- $$A_{4.9}^1 = \langle (qx + y)\partial_x + (qy - x)\partial_y + \left(2qu + \frac{y^2}{2} - \frac{x^2}{2}\right) \partial_u \rangle \oplus$$
- $$\langle \partial_u, \partial_x, \partial_y + x\partial_u \rangle, \quad q \geq 0,$$
- $$F = G\left(\left((u_x - y)^2 + u_y^2\right) \exp\left(-2q \arctan\left(\frac{u_x - y}{u_y}\right)\right)\right).$$

Algebra $A_{4.10}$

1. $A_{4.10}^1 = \langle y\partial_x - x\partial_y + \epsilon\partial_u \rangle \oplus \langle \partial_x, \partial_y, x\partial_x + y\partial_y \rangle$, $\epsilon = 0, 1$,

$$F = (u_x^2 + u_y^2)G \left(\epsilon \arctan \frac{u_x}{u_y} - u \right);$$
2. $A_{4.10}^2 = \langle \lambda_1\partial_x + \lambda_2\partial_y + u \tan(\lambda_2^{-1}y)\partial_u \rangle \oplus \langle \partial_u, -\tan(\lambda_2^{-1}y)\partial_u, \partial_x + u\partial_u \rangle$,

$$F = 2\lambda_2^{-1} \tan(\lambda_2^{-1}y)u_y + u_x G(\exp(\lambda_1 y/\lambda_2 - x) \cos(\lambda_2^{-1}y)u_x)$$
, $\lambda_2 \neq 0$;
3. $A_{4.10}^3 = \langle y\partial_x - x\partial_y + \lambda\partial_u \rangle \oplus \langle \partial_x, \partial_y, x\partial_x + y\partial_y + \partial_u \rangle$,

$$F = (u_x^2 + u_y^2)G \left((u_x^2 + u_y^2)e^{2u} \exp \left(-2\lambda \arctan \left(\frac{u_x}{u_y} \right) \right) \right).$$

6 Invariance with respect to 5-dimensional Lie algebras – straight method

Further, we apply the direct method (but essentially modified) to above obtained 38 classes of the equations. As result, the completed list of the equations were obtained invariant under five-dimensional Lie algebras.

1. Expansion $\tilde{A}_{4.2}^1 \subset A_{2.2} \oplus A_{3.9}^q$, $q > 0$

$$\Delta u = \sqrt{u_x^2 + u_y^2} \exp \left(-q \arctan \frac{u_x}{u_y} \right),$$

$$\langle -u\partial_u, \partial_u \rangle \oplus \langle \partial_x, \partial_y, (qx + y)\partial_x + (qy - x)\partial_y \rangle$$

2. Expansion $\tilde{A}_{4.3}^1 \subset A_{5.20}^p$, $p = 0$

$$\Delta u = \exp(u_y), \langle -\partial_u, \partial_x, -x\partial_u, \partial_y, x\partial_x + y\partial_y + (u - y)\partial_u \rangle$$

3. Expansion $\tilde{A}_{4.3}^1 \subset A_{5.23}^p$, $p = 1$

$$\Delta u = \ln(u_y), \langle \partial_u, \partial_x, x\partial_u, \partial_y, x\partial_x + y\partial_y + \left(2u + \frac{x^2}{2} \right) \partial_u \rangle$$

4. Expansion $\tilde{A}_{4.3}^1 \subset A_{5.19}^{p,q}$, $p \neq 0; 1$, $q = 1$

$$\Delta u = u_y^{\frac{p-1}{p}}, \langle \partial_u, \partial_x, x\partial_u, \partial_y, x\partial_x + y\partial_y + (p+1)u\partial_u \rangle$$

5. Expansion $\tilde{A}_{4.5}^1 \subset A_{5.35}^{p,q}$, $p = 0$, $q \neq 0$

$$\Delta u = (u_x^2 + u_y^2) \exp\left(-q \arctan \frac{u_x}{u_y}\right),$$

$$\langle \partial_u, \partial_x, \partial_y, x\partial_x + y\partial_y, y\partial_x - x\partial_y + qu\partial_u \rangle$$

6. Expansion $\tilde{A}_{4.6}^1 \subset A_{5.19}^{p,q}$, $p = q = -1$

$$\Delta u = yu_x^2 - 2y^{-1}u_y, \langle -y^{-1}\partial_u, \partial_u, \partial_y - \frac{u}{y}\partial_u, \partial_x, -x\partial_x - y\partial_y + u\partial_u \rangle$$

7. Expansion $\tilde{A}_{4.8}^1 \subset A_{2.2} \oplus A_{3.8}$

$$\Delta u = \sqrt{u_x^2 + u_y^2}, \langle -u\partial_u, \partial_u \rangle \oplus \langle \partial_x, \partial_y, y\partial_x - x\partial_y \rangle$$

8. Expansion $\tilde{A}_{4.8}^1 \subset A_{5.35}^{p,q}$, $p \neq 0; 1$, $q = 0$

$$\Delta u = (u_x^2 + u_y^2)^{(2-p)/2(1-p)}, \langle \partial_u, \partial_x, \partial_y, x\partial_x + y\partial_y + pu\partial_u, y\partial_x - x\partial_y, \rangle$$

9. Expansion $\tilde{A}_{4.9}^1 \subset A_{5.35}^{p,\tilde{q}}$, $\tilde{q} = -pq$, $p \neq 0; 1$, $q > 0$

$$\Delta u = (u_x^2 + u_y^2)^{(2-p)/2(1-p)} \exp\left(\frac{qp}{1-p} \arctan \frac{u_x}{u_y}\right),$$

$$\langle \partial_u, \partial_x, \partial_y, x\partial_x + y\partial_y + pu\partial_u, y\partial_x - x\partial_y - qpu\partial_u, \rangle$$

10. Expansion $\tilde{A}_{4.10}^1 \subset A_{2.2} \oplus sl(2, \mathbb{R})$

$$\Delta u = \lambda y^{-1} \sqrt{u_x^2 + u_y^2},$$

$$\langle -u\partial_u, \partial_u \rangle \oplus \langle 2x\partial_x + 2y\partial_y, -(x^2 - y^2)\partial_x - 2xy\partial_y + y\partial_u, \partial_x \rangle$$

11. Expansion $A_{4.5}^3 \subset A_{5.30}^p$, $p = 1$

$$\Delta u = \lambda \sqrt{2u_x \pm y^2}, \langle \partial_u, -y\partial_u, \pm\partial_x, \partial_y \mp xy\partial_u, x\partial_x + y\partial_y + 3u\partial_u \rangle$$

12. Expansion $A_{4.8}^4 \subset A_{5.30}^p$, $p = 1$

$$\Delta u = (\lambda u_x + u_y)^{1/2} + \mu y, \langle \partial_u, (\lambda y - x)\partial_u,$$

$$-\frac{1 + \lambda^2}{\mu} \left(\partial_y + \frac{\mu}{2(1 + \lambda^2)} (\lambda y - x)^2 \partial_u \right), \partial_x, x\partial_x + y\partial_y + 3u\partial_u \rangle$$

At that, some of them depend from parameters λ, μ which can not be reduced to concrete number with the help of the equivalence transformations.

References

- [1] Ovsjannikov L.V. Group analysis of differential equations. — M.: Nauka, 1978. — 400 p. (in Russian)
- [2] Lahno V.I. and Spichak S.V., Group classification of quasi-linear elliptic type equations. I. Invariance under Lie algebras with non-trivial Levi decomposition, Ukrainian Mathematical Journal, 2007, V. 59, N 11, 1532–1545 (in Ukrainian).
- [3] Zhdanov R.Z. and Lahno V.I. Group classification of heat conductivity equations with a nonlinear source, J. Phys. A: Math. Gen., 1999, V. 32, 7405–7418.
- [4] Basarab–Horwath P., Lahno V. and Zhdanov R., The structure of Lie algebras and the classification problem for partial differential equations, Acta Appl. Math., 2001, V. 69, N 1, 43–94.
- [5] Mubarakzjanov G.M. On solvable Lie algebras // Izvestija vysh. uchebn. zavedenije. Matematika. — 1963. — N 1(32). — P. 114–123. (in Russian)