

# HEIGHT-2 TODA SYSTEMS

**Hervé Sabourin**, Université de Poitiers-France

Joint work with **Pantelis Damianou** and **Pol Vanhaecke**

Let  $(\mathbb{R}^{2n}, \{, \})$  be the usual Poisson manifold, and  $(p_1, \dots, p_n, q_1, \dots, q_n)$  the corresponding canonical coordinates.

Define the Hamiltonian  $H$  by :

$$H = \frac{1}{2} \sum_i p_i^2 + \sum_i e^{q_i - q_{i+1}}$$

and Hamiltonian equations :

$$\dot{q}_i = \{q_i, H\}, = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q_i}$$

**Problem** : determine an Integrable System, relatively to  $H$ , i.e. a set of  $n$  independent functions in involution which are constants of motion for Hamiltonian equations.

Using Flaschka's variables :

$$\forall i, 1 \leq i \leq n-1, a_i = \frac{1}{2} e^{\frac{1}{2}(q_i - q_{i+1})}, \quad \forall i, 1 \leq i \leq n, b_i = -\frac{1}{2} p_i$$

Hamiltonian's equations are then equivalent to the following Lax equation

$$\dot{L} = [B, L]$$

where :

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \dots & \dots \\ a_1 & b_2 & a_2 & 0 & \dots \\ 0 & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & b_{n-1} & a_{n-1} \\ 0 & \dots & \dots & a_{n-1} & b_n \end{pmatrix}$$
$$B = \begin{pmatrix} 0 & a_1 & 0 & \dots & \dots \\ -a_1 & 0 & a_2 & 0 & \dots \\ 0 & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & a_{n-1} \\ 0 & \dots & \dots & -a_{n-1} & 0 \end{pmatrix}$$

The underlying Poisson structure on  $L = L(a_1, \dots, a_{n-1}, b_1, \dots, b_n)$  is defined, up to a constant multiple, by :

$$\forall i, \{a_i, b_i\}_L = -a_i, \quad \{a_i, b_{i+1}\}_L = a_i \quad (1)$$

The other brackets are zero.

The underlying Poisson structure on  $L = L(a_1, \dots, a_{n-1}, b_1, \dots, b_n)$  is defined, up to a constant multiple, by :

$$\forall i, \{a_i, b_i\}_L = -a_i, \quad \{a_i, b_{i+1}\}_L = a_i \quad (1)$$

The other brackets are zero.

All the entries of  $L$  vary over time, but the eigenvalues remain constant. It follows that the functions  $H_i = \frac{1}{i} \text{Tr} L^i$  are constants of motion and give us the right "Integrable system".

# Some facts in Lie theory

Given any semi-simple complex Lie algebra  $\mathfrak{g}$ , we introduce the following data :

# Some facts in Lie theory

Given any semi-simple complex Lie algebra  $\mathfrak{g}$ , we introduce the following data :

- A Cartan subalgebra  $\mathfrak{h}$  (i.e a maximal abelian subalgebra composed of semi-simple elements )(ex : Diagonal matrices in  $(sl_n)$ ).

# Some facts in Lie theory

Given any semi-simple complex Lie algebra  $\mathfrak{g}$ , we introduce the following data :

- A Cartan subalgebra  $\mathfrak{h}$  (i.e a maximal abelian subalgebra composed of semi-simple elements )(ex : Diagonal matrices in  $(sl_n)$ ).
- A root system  $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ .

# Some facts in Lie theory

Given any semi-simple complex Lie algebra  $\mathfrak{g}$ , we introduce the following data :

- A Cartan subalgebra  $\mathfrak{h}$  (i.e a maximal abelian subalgebra composed of semi-simple elements )(ex : Diagonal matrices in  $(sl_n)$ ).
- A root system  $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ .
- A basis of simple roots  $\Pi$  in  $\Delta$ ,  
 $\Pi = \{\alpha_1, \dots, \alpha_l\}$ ,  $l = \dim \mathfrak{h} = \text{rk } \mathfrak{g}$ .

# Some facts in Lie theory

Given any semi-simple complex Lie algebra  $\mathfrak{g}$ , we introduce the following datas :

- A Cartan subalgebra  $\mathfrak{h}$  (i.e a maximal abelian subalgebra composed of semi-simple elements )(ex : Diagonal matrices in  $(sl_n)$ ).
- A root system  $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ .
- A basis of simple roots  $\Pi$  in  $\Delta$ ,  
 $\Pi = \{\alpha_1, \dots, \alpha_l\}$ ,  $l = \dim \mathfrak{h} = \text{rk } \mathfrak{g}$ .
- $\forall \alpha \in \Delta$ ,  $\alpha = \sum_j n_j \alpha_j$ ,  $n_j \in \mathbb{Z}$ ,  $n_j \geq 0, \forall j$ , or  $n_j \leq 0, \forall j$ .
- $\Delta = \Delta^+ \cup \Delta^-$ .

- If  $\alpha = \sum_j n_j \alpha_j$  is a positive root, then  $ht(\alpha) = \sum_j n_j$  is the height of  $\alpha$ .

- If  $\alpha = \sum_j n_j \alpha_j$  is a positive root, then  $ht(\alpha) = \sum_j n_j$  is the height of  $\alpha$ .
- If  $\alpha \in \Delta$ , let  $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{h}, [H, X] = \alpha(H)X\}$  be the corresponding one dimensional root-space.
- Then,  $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ .

- If  $\alpha = \sum_j n_j \alpha_j$  is a positive root, then  $ht(\alpha) = \sum_j n_j$  is the height of  $\alpha$ .
- If  $\alpha \in \Delta$ , let  $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{h}, [H, X] = \alpha(H)X\}$  be the corresponding one dimensional root-space.
- Then,  $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ .
- Each positive root  $\alpha$  leads to a triple  $(X_\alpha, X_{-\alpha}, H_\alpha)$  of root-vectors which generate a subalgebra isomorphic to  $sl_2$ . The set  $(X_\alpha, X_{-\alpha}, \alpha \in \Delta^+, H_\alpha, \alpha \in \Pi)$  is a basis of  $\mathfrak{g}$ .

- If  $\alpha = \sum_j n_j \alpha_j$  is a positive root, then  $ht(\alpha) = \sum_j n_j$  is the height of  $\alpha$ .
- If  $\alpha \in \Delta$ , let  $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{h}, [H, X] = \alpha(H)X\}$  be the corresponding one dimensional root-space.
- Then,  $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ .
- Each positive root  $\alpha$  leads to a triple  $(X_\alpha, X_{-\alpha}, H_\alpha)$  of root-vectors which generate a subalgebra isomorphic to  $sl_2$ . The set  $(X_\alpha, X_{-\alpha}, \alpha \in \Delta^+, H_\alpha, \alpha \in \Pi)$  is a basis of  $\mathfrak{g}$ .
- $\mathfrak{n}^+ = \sum_{\alpha \in \Delta^+} \mathbb{C}X_\alpha$ ,  $\mathfrak{n}^- = \sum_{\alpha \in \Delta^-} \mathbb{C}X_{-\alpha}$ .
- $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ ,  $\mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-$  are Borel subalgebras of  $\mathfrak{g}$ .

# Lie-Poisson Structure

- Let  $\mathfrak{g}$  be any complex Lie algebra. The Lie-Poisson structure on its dual  $\mathfrak{g}^*$  is defined as follows :

$$\forall F, G \in \mathcal{F}(\mathfrak{g}^*), \forall \mu \in \mathfrak{g}^*, \{F, G\}(\mu) = \mu([d_\mu F, d_\mu G])$$

# Lie-Poisson Structure

- Let  $\mathfrak{g}$  be any complex Lie algebra. The Lie-Poisson structure on its dual  $\mathfrak{g}^*$  is defined as follows :

$$\forall F, G \in \mathcal{F}(\mathfrak{g}^*), \forall \mu \in \mathfrak{g}^*, \{F, G\}(\mu) = \mu([d_\mu F, d_\mu G])$$

- The symplectic leaves are the co-adjoint orbits.

# Lie-Poisson Structure

- Let  $\mathfrak{g}$  be any complex Lie algebra. The Lie-Poisson structure on its dual  $\mathfrak{g}^*$  is defined as follows :

$$\forall F, G \in \mathcal{F}(\mathfrak{g}^*), \forall \mu \in \mathfrak{g}^*, \{F, G\}(\mu) = \mu([d_\mu F, d_\mu G])$$

- The symplectic leaves are the co-adjoint orbits.
- The Poisson-Rank of  $\mathfrak{g}^*$ ,  $Prk(\mathfrak{g}^*)$ , is then the maximal dimension of co-adjoint orbits.

# Lie-Poisson Structure

- Let  $\mathfrak{g}$  be any complex Lie algebra. The Lie-Poisson structure on its dual  $\mathfrak{g}^*$  is defined as follows :

$$\forall F, G \in \mathcal{F}(\mathfrak{g}^*), \forall \mu \in \mathfrak{g}^*, \{F, G\}(\mu) = \mu([d_\mu F, d_\mu G])$$

- The symplectic leaves are the co-adjoint orbits.
- The Poisson-Rank of  $\mathfrak{g}^*$ ,  $Prk(\mathfrak{g}^*)$ , is then the maximal dimension of co-adjoint orbits.
- Define the *Index* of  $\mathfrak{g}$  to be the lowest dimension of stabilizers in  $\mathfrak{g}$  of elements of  $\mathfrak{g}^*$ , denoted by  $Ind(\mathfrak{g})$ . Then,

$$Prk(\mathfrak{g}^*) = \dim \mathfrak{g} - Ind(\mathfrak{g})$$

# Lie-Poisson Structure

- Let  $\mathfrak{g}$  be any complex Lie algebra. The Lie-Poisson structure on its dual  $\mathfrak{g}^*$  is defined as follows :

$$\forall F, G \in \mathcal{F}(\mathfrak{g}^*), \forall \mu \in \mathfrak{g}^*, \{F, G\}(\mu) = \mu([d_\mu F, d_\mu G])$$

- The symplectic leaves are the co-adjoint orbits.
- The Poisson-Rank of  $\mathfrak{g}^*$ ,  $Prk(\mathfrak{g}^*)$ , is then the maximal dimension of co-adjoint orbits.
- Define the *Index* of  $\mathfrak{g}$  to be the lowest dimension of stabilizers in  $\mathfrak{g}$  of elements of  $\mathfrak{g}^*$ , denoted by  $Ind(\mathfrak{g})$ . Then,

$$Prk(\mathfrak{g}^*) = \dim \mathfrak{g} - Ind(\mathfrak{g})$$

- Suppose that  $\mathfrak{g}$  is semi-simple. We can identify  $\mathfrak{g}$  with its dual, via the Killing form  $\langle X, Y \rangle = Tr(adX.adY)$ . The Lie -Poisson structure on  $\mathfrak{g}$  is then given by :

$$\forall F, G \in \mathcal{F}(\mathfrak{g}), \forall X \in \mathfrak{g}, \{F, G\}(X) = \langle X, [d_X F, d_X G] \rangle$$

The Lax pair  $(L, B)$  in  $\mathfrak{g}$  can be described as follows, using the root-system  $\Delta$  :

$$L = \sum_{i=1}^{i=l} b_i H_{\alpha_i} + \sum_{i=1}^{i=l} a_i (X_{\alpha_i} + X_{-\alpha_i})$$

$$B = \sum_{i=1}^{i=l} a_i (X_{\alpha_i} - X_{-\alpha_i})$$

The Lax pair  $(L, B)$  in  $\mathfrak{g}$  can be described as follows, using the root-system  $\Delta$  :

$$L = \sum_{i=1}^{l} b_i H_{\alpha_i} + \sum_{i=1}^{l} a_i (X_{\alpha_i} + X_{-\alpha_i})$$

$$B = \sum_{i=1}^{l} a_i (X_{\alpha_i} - X_{-\alpha_i})$$

Generalizing this procedure, we can introduce the pair  $(L_\phi, B_\phi)$  :

$$L_\phi = \sum_{\alpha \in \phi} b_\alpha H_\alpha + \sum_{\alpha \in \phi} a_\alpha (X_\alpha + X_{-\alpha})$$

$$B_\phi = \sum_{\alpha \in \phi} a_\alpha (X_\alpha - X_{-\alpha})$$

Where  $\phi$  is any subset of  $\Delta^+$ , containing  $\Pi$ .

In order to get similar Lax equation, the bracket  $[L_\phi, B_\phi]$  should give an element of the same form than  $L_\phi$ . In that case, we say that  $\phi$  is *adapted*.

In order to get similar Lax equation, the bracket  $[L_\phi, B_\phi]$  should give an element of the same form than  $L_\phi$ . In that case, we say that  $\phi$  is *adapted*.

Let  $\mathbb{I}$  be the set of nilpotent ideals of  $\mathfrak{b}^+$ . We know that :

- If  $I \in \mathbb{I}$ ,  $I \subset \mathfrak{n}^+$ .
- $I$  is generated by  $\langle X_\alpha, \alpha \in \Delta_I \rangle$ , where  $\Delta_I$  is some subset of  $\Delta^+$ .

In order to get similar Lax equation, the bracket  $[L_\phi, B_\phi]$  should give an element of the same form than  $L_\phi$ . In that case, we say that  $\phi$  is *adapted*.

Let  $\mathbb{I}$  be the set of nilpotent ideals of  $\mathfrak{b}^+$ . We know that :

- If  $I \in \mathbb{I}$ ,  $I \subset \mathfrak{n}^+$ .
- $I$  is generated by  $\langle X_\alpha, \alpha \in \Delta_I \rangle$ , where  $\Delta_I$  is some subset of  $\Delta^+$ .

**Proposition :**  $\phi$  is adapted if and only if the subspace  $I_\phi$  generated by  $\langle X_\alpha, \alpha \in \Delta^+ \setminus \phi \rangle$  is a nilpotent ideal of  $\mathfrak{b}^+$ .

In order to get similar Lax equation, the bracket  $[L_\phi, B_\phi]$  should give an element of the same form than  $L_\phi$ . In that case, we say that  $\phi$  is *adapted*.

Let  $\mathbb{I}$  be the set of nilpotent ideals of  $\mathfrak{b}^+$ . We know that :

- If  $I \in \mathbb{I}$ ,  $I \subset \mathfrak{n}^+$ .
- $I$  is generated by  $\langle X_\alpha, \alpha \in \Delta_I \rangle$ , where  $\Delta_I$  is some subset of  $\Delta^+$ .

**Proposition :**  $\phi$  is adapted if and only if the subspace  $I_\phi$  generated by  $\langle X_\alpha, \alpha \in \Delta^+ \setminus \phi \rangle$  is a nilpotent ideal of  $\mathfrak{b}^+$ .

Thus, for such  $\phi$ , we obtain an Hamiltonian system ; the problem is to study this new system and determine whether it is integrable.

- $\phi = \Pi$ ,  $I_\phi = \langle X_\alpha, \alpha \in \Delta^+ \setminus \Pi \rangle$ , Classical Toda lattice.
- $\phi = \Delta^+$ ,  $I_\phi = \{0\}$ , Full "symmetric" Toda lattice
- "Intermediate" Toda lattices.

# Kostant form

By conjugation,

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \dots & \dots \\ a_1 & b_2 & a_2 & 0 & \dots \\ 0 & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & b_{n-1} & a_{n-1} \\ 0 & \dots & \dots & a_{n-1} & b_n \end{pmatrix} \longrightarrow \begin{pmatrix} b_1 & 1 & 0 & \dots & \dots \\ a_1 & b_2 & 1 & 0 & \dots \\ 0 & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & b_{n-1} & 1 \\ 0 & \dots & \dots & a_{n-1} & b_n \end{pmatrix}$$

The latter matrix is the "Kostant form" of the phase space of the classical Toda.

# Kostant form

By conjugation,

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \dots & \dots \\ a_1 & b_2 & a_2 & 0 & \dots \\ 0 & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & b_{n-1} & a_{n-1} \\ 0 & \dots & \dots & a_{n-1} & b_n \end{pmatrix} \longrightarrow \begin{pmatrix} b_1 & 1 & 0 & \dots & \dots \\ a_1 & b_2 & 1 & 0 & \dots \\ 0 & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & b_{n-1} & 1 \\ 0 & \dots & \dots & a_{n-1} & b_n \end{pmatrix}$$

The latter matrix is the "Kostant form" of the phase space of the classical Toda.

- Lax equation :  $\dot{X} = [p_-(X), X]$ , where  $p_-$  is the projection on the strictly lower triangular part of  $X$ .

# Kostant form

By conjugation,

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \dots & \dots \\ a_1 & b_2 & a_2 & 0 & \dots \\ 0 & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & b_{n-1} & a_{n-1} \\ 0 & \dots & \dots & a_{n-1} & b_n \end{pmatrix} \longrightarrow \begin{pmatrix} b_1 & 1 & 0 & \dots & \dots \\ a_1 & b_2 & 1 & 0 & \dots \\ 0 & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & b_{n-1} & 1 \\ 0 & \dots & \dots & a_{n-1} & b_n \end{pmatrix}$$

The latter matrix is the "Kostant form" of the phase space of the classical Toda.

- Lax equation :  $\dot{X} = [p_-(X), X]$ , where  $p_-$  is the projection on the strictly lower triangular part of  $X$ .

- Hamiltonian :  $H = \frac{1}{2} \text{Tr}(X^2)$ .

- $\varepsilon = \sum_{i=1}^l X_{\alpha_i}$
- $M = \varepsilon + \mathfrak{b}^- \simeq (\mathfrak{b}^+)^*$   
( $\varepsilon + X \rightarrow \phi_X = \langle X, \cdot \rangle$ )
- $\forall l \in \mathbb{I}, M_l = \{X \in M \mid \langle X, l \rangle = 0\} = \varepsilon + l^\perp \simeq (\mathfrak{b}^+ / l)^*$ .

# Poisson structure on $M$ and $M_I$ .

- The Poisson structure on  $M$  is defined by using the theory of  $R$ -matrices.

# Poisson structure on $M$ and $M_I$ .

- The Poisson structure on  $M$  is defined by using the theory of  $R$ -matrices.

$\mathfrak{g} = \mathfrak{b}^+ \oplus \mathfrak{n}^-$ ,  $P^+ : \mathfrak{g} \longrightarrow \mathfrak{b}^+$  canonical projection.

$\forall F, G \in \mathcal{F}(M), \forall X \in M$ ,

$$\{F, G\}(X) = \langle X, [P^+(d_X \tilde{F}), P^+(d_X \tilde{G})] \rangle$$

where  $\tilde{F}, \tilde{G}$  are arbitrary extensions of  $F, G$  to  $\mathfrak{g}$ .

# Poisson structure on $M$ and $M_I$ .

- The Poisson structure on  $M$  is defined by using the theory of  $R$ -matrices.

$\mathfrak{g} = \mathfrak{b}^+ \oplus \mathfrak{n}^-$ ,  $P^+ : \mathfrak{g} \longrightarrow \mathfrak{b}^+$  canonical projection.

$\forall F, G \in \mathcal{F}(M), \forall X \in M$ ,

$$\{F, G\}(X) = \langle X, [P^+(d_X \tilde{F}), P^+(d_X \tilde{G})] \rangle$$

where  $\tilde{F}, \tilde{G}$  are arbitrary extensions of  $F, G$  to  $\mathfrak{g}$ .

- $\forall I \in \mathbb{I}, M_I$  is a Poisson submanifold of  $M$ .

$$Prk(M_I) = \dim \mathfrak{b}^+ / I - \text{Ind}(\mathfrak{b}^+ / I)$$

$$\text{Prk}(M_I) = \dim \mathfrak{b}^+ / I - \text{Ind}(\mathfrak{b}^+ / I)$$

We compute the index, using *stable forms*.

**Definition :** Let  $\mathfrak{a}$  be a complex algebraic Lie algebra,  $A$  its corresponding adjoint Lie group. A linear form  $f \in \mathfrak{a}^*$  is said to be *stable* if there exists a neighborhood  $V$  of  $f$  in  $\mathfrak{a}^*$  such that,  $\forall \varphi \in V$ , the stabilizers of  $f$  and  $\varphi$  in  $\mathfrak{a}$  are  $A$ -conjugate.

$$\text{Prk}(M_I) = \dim \mathfrak{b}^+ / I - \text{Ind}(\mathfrak{b}^+ / I)$$

We compute the index, using *stable forms*.

**Definition :** Let  $\mathfrak{a}$  be a complex algebraic Lie algebra,  $A$  its corresponding adjoint Lie group. A linear form  $f \in \mathfrak{a}^*$  is said to be *stable* if there exists a neighborhood  $V$  of  $f$  in  $\mathfrak{a}^*$  such that,  $\forall \varphi \in V$ , the stabilizers of  $f$  and  $\varphi$  in  $\mathfrak{a}$  are  $A$ -conjugate.

$$f \text{ stable} \implies f \text{ regular}$$

$$\text{Prk}(M_I) = \dim \mathfrak{b}^+ / I - \text{Ind}(\mathfrak{b}^+ / I)$$

We compute the index, using *stable forms*.

**Definition :** Let  $\mathfrak{a}$  be a complex algebraic Lie algebra,  $A$  its corresponding adjoint Lie group. A linear form  $f \in \mathfrak{a}^*$  is said to be *stable* if there exists a neighborhood  $V$  of  $f$  in  $\mathfrak{a}^*$  such that,  $\forall \varphi \in V$ , the stabilizers of  $f$  and  $\varphi$  in  $\mathfrak{a}$  are  $A$ -conjugate.

$$f \text{ stable} \implies f \text{ regular}$$

$$\text{Ind}(\mathfrak{a}) = \dim \mathfrak{a} - \dim A.f, f \text{ stable}$$

Using Kostant's "Cascade Method", **P.Tauvel** and **R.Yu** (2005) get an explicit stable form on  $\mathfrak{b}^+$ , which leads to the formula :

$$\text{Ind}(\mathfrak{b}^+) = \text{rk}(\mathfrak{g}) - \kappa_{\mathfrak{g}}$$

**Question** : Stable forms on  $\mathfrak{b}^+/I, I \in \mathbb{I}$ ?

# Height-2-Toda systems

# Height-2-Toda systems

From now, assume that each positive root of height 2 is of the form  $\alpha_k + \alpha_{k+1}$ ,  $1 \leq k \leq l - 1$ .

# Height-2-Toda systems

From now, assume that each positive root of height 2 is of the form  $\alpha_k + \alpha_{k+1}$ ,  $1 \leq k \leq l-1$ .

- Let  $\mathcal{S} = \{\Pi_1, \Pi'_1, \Pi_2, \Pi'_2, \dots, \Pi_k, \Pi'_k\}$  be an "ordered" partition of  $\Pi$ , composed of subsets of consecutive simple roots.

- $\Pi_j = \{\alpha_{i_j}, \dots, \alpha_{i_j+l_j-1}\}$



$$\mathbb{E}_{\mathcal{S}} = \{\alpha = \alpha_i + \alpha_{i+1}, \alpha_i, \alpha_{i+1} \in \Pi_j, 1 \leq j \leq k\}$$

- Let  $I_{\mathcal{S}}$  be the nilpotent ideal defined by :

$$I_{\mathcal{S}} = \langle X_{\alpha} \mid ht(\alpha) \geq 3 \text{ or } ht(\alpha) = 2 \text{ and } \alpha \notin \mathbb{E}_{\mathcal{S}} \rangle$$

- $M_{\mathcal{S}} = M_{I_{\mathcal{S}}}$  is the corresponding *Height-2-Toda* manifold.

# Poisson Rank

The Poisson rank of a Height-2-Toda manifold is obtained by computing an explicit stable form :

# Poisson Rank

The Poisson rank of a Height-2-Toda manifold is obtained by computing an explicit stable form :

Let  $X_j, X'_j \in \mathfrak{b}^-$ ,  $f_j, f'_j \in (\mathfrak{b}^+)^*$  defined by

$$X_j = \varepsilon_j X_{-\alpha_j} + \sum_{i=j}^{i_j+l_j-2} X_{-\alpha_i-\alpha_{i+1}}, \text{ where } \varepsilon_j = \frac{1 - (-1)^{l_j}}{2}$$

$$\forall Z \in \mathfrak{b}^+, f_j(Z) = \langle X_j, Z \rangle$$

$$X'_j = \sum_{\alpha \in \Pi'_j} X_{-\alpha}$$

$$\forall Z \in \mathfrak{b}^+, f'_j(Z) = \langle X'_j, Z \rangle$$

Let  $f_S = \sum_{j=1}^k (f_j + f'_j)$ .

**Proposition :**

-  $f_S$  is a stable form on  $\mathfrak{b}^+ / I_S$ .

-  $Ind(\mathfrak{b}^+ / I_S) = \sum_{j=1}^k \frac{1 + (-1)^{l_j}}{2}$

-  $Prk(M_S) = 2l + \sum_{j=1}^k l_j - 1 - \frac{1 + (-1)^{l_j}}{2}$ .

# Integrability for the Full-Toda ( $A_n$ -case)

$\mathfrak{g} = \mathfrak{sl}_n$ ,  $G = SL_n$ ,  $\mathfrak{b}^+ = n \times n$  upper triangular matrices

$$\dim M = \frac{n(n+1)}{2}, \text{Ind}(\mathfrak{b}) = \left[ \frac{n-1}{2} \right], \text{Prk}(M) = \frac{n(n+1)}{2} - \left[ \frac{n-1}{2} \right]$$

# Integrability for the Full-Toda ( $A_n$ -case)

$\mathfrak{g} = \mathfrak{sl}_n$ ,  $G = SL_n$ ,  $\mathfrak{b}^+ = n \times n$  upper triangular matrices

$$\dim M = \frac{n(n+1)}{2}, \text{Ind}(\mathfrak{b}) = \left[ \frac{n-1}{2} \right], \text{Prk}(M) = \frac{n(n+1)}{2} - \left[ \frac{n-1}{2} \right]$$

Number of functions required :  $\dim M - 1/2 \text{Prk}(M) = \left[ \left( \frac{n+1}{2} \right)^2 \right]$ .

# Integrability for the Full-Toda ( $A_n$ -case)

$\mathfrak{g} = \mathfrak{sl}_n$ ,  $G = SL_n$ ,  $\mathfrak{b}^+ = n \times n$  upper triangular matrices

$$\dim M = \frac{n(n+1)}{2}, \text{Ind}(\mathfrak{b}) = \left[ \frac{n-1}{2} \right], \text{Prk}(M) = \frac{n(n+1)}{2} - \left[ \frac{n-1}{2} \right]$$

Number of functions required :  $\dim M - 1/2 \text{Prk}(M) = \left[ \left( \frac{n+1}{2} \right)^2 \right]$ .

The functions  $H_i = 1/i \text{Tr} X^i$  are still in involution, but not enough to ensure integrability. We need more functions.

let  $k$  be an integer,  $0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$ . Let  $X \in \mathfrak{g}$ . Define  $X_k$  to be the matrix obtained by removing the first  $k$  rows and the last  $k$  columns from  $X$ .

Let  $G_k$  be the (Parabolic) subgroup of  $G$  consisting of matrices of the form :

$$\begin{pmatrix} \Delta & A & B \\ 0 & D & C \\ 0 & 0 & \Delta' \end{pmatrix}$$

where  $\Delta, \Delta'$  are upper triangular matrices of size  $k \times k$ , and  $A, B, C, D$  arbitrary.

let  $k$  be an integer,  $0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$ . Let  $X \in \mathfrak{g}$ . Define  $X_k$  to be the matrix obtained by removing the first  $k$  rows and the last  $k$  columns from  $X$ .

Let  $G_k$  be the (Parabolic) subgroup of  $G$  consisting of matrices of the form :

$$\begin{pmatrix} \Delta & A & B \\ 0 & D & C \\ 0 & 0 & \Delta' \end{pmatrix}$$

where  $\Delta, \Delta'$  are upper triangular matrices of size  $k \times k$ , and  $A, B, C, D$  arbitrary.

**Proposition** :  $\forall X \in \mathfrak{g}, \forall g \in G_k$ ,

$$\det(gXg^{-1})_k = \frac{\det \Delta}{\det \Delta'} \det X_k$$

Define,  $\forall X \in \mathfrak{g}, \forall \lambda \in \mathbb{C}$ ,

$$Q_k(X, \lambda) = \det(X - \lambda Id_n)_k = \sum_{i=0}^{i=n-2k} E_{i,k} \lambda^{n-2k-i}$$

Define,  $\forall X \in \mathfrak{g}, \forall \lambda \in \mathbb{C}$ ,

$$Q_k(X, \lambda) = \det(X - \lambda Id_n)_k = \sum_{i=0}^{i=n-2k} E_{i,k} \lambda^{n-2k-i}$$

All  $E_{i,k}$ 's are  $G_k$ -semi-invariant, with the same character.

Define,  $\forall X \in \mathfrak{g}, \forall \lambda \in \mathbb{C}$ ,

$$Q_k(X, \lambda) = \det(X - \lambda Id_n)_k = \sum_{i=0}^{i=n-2k} E_{i,k} \lambda^{n-2k-i}$$

All  $E_{i,k}$ 's are  $G_k$ -semi-invariant, with the same character.

Set

$$\forall (i, k), 0 \leq i \leq n - 2k, 0 \leq k \leq \left[ \frac{n-1}{2} \right], H_{i,k} = \frac{E_{i,k}}{E_{0,k}}$$

The  $(H_{i,k})$ 's are  $G_k$ -invariant functions :=  $k$ -chop integrals.

$$H = 1/2(H_{1,0}^2 - 2H_{2,0})$$

Define,  $\forall X \in \mathfrak{g}, \forall \lambda \in \mathbb{C}$ ,

$$Q_k(X, \lambda) = \det(X - \lambda Id_n)_k = \sum_{i=0}^{i=n-2k} E_{i,k} \lambda^{n-2k-i}$$

All  $E_{i,k}$ 's are  $G_k$ -semi-invariant, with the same character.

Set

$$\forall (i, k), 0 \leq i \leq n - 2k, 0 \leq k \leq \left[ \frac{n-1}{2} \right], H_{i,k} = \frac{E_{i,k}}{E_{0,k}}$$

The  $(H_{i,k})$ 's are  $G_k$ -invariant functions :=  $k$ -chop integrals.

$$H = 1/2(H_{1,0}^2 - 2H_{2,0})$$

Number of  $k$ -chop integrals :  $\left[ \left( \frac{n+1}{2} \right)^2 \right]$ .

Define,  $\forall X \in \mathfrak{g}, \forall \lambda \in \mathbb{C}$ ,

$$Q_k(X, \lambda) = \det(X - \lambda Id_n)_k = \sum_{i=0}^{i=n-2k} E_{i,k} \lambda^{n-2k-i}$$

All  $E_{i,k}$ 's are  $G_k$ -semi-invariant, with the same character.

Set

$$\forall (i, k), 0 \leq i \leq n - 2k, 0 \leq k \leq \left[ \frac{n-1}{2} \right], H_{i,k} = \frac{E_{i,k}}{E_{0,k}}$$

The  $(H_{i,k})$ 's are  $G_k$ -invariant functions :=  $k$ -chop integrals.

$$H = 1/2(H_{1,0}^2 - 2H_{2,0})$$

Number of  $k$ -chop integrals :  $\left[ \left( \frac{n+1}{2} \right)^2 \right]$ .

**Theorem :** *The chop integrals  $(H_{i,k})$  form an integrable system.*

# Integrability of Height-2-Toda : $A_n$ case.

- $\mathcal{S} = \{\Pi_1, \Pi'_1, \Pi_2, \Pi'_2, \dots, \Pi_k, \Pi'_k\}$
- $\#\Pi_j = l_j, \#\Pi'_j = l'_j.$

$$X_{\mathcal{S}} = \begin{pmatrix} b_1 & 1 & 0 & \dots & \dots \\ a_1 & b_2 & 1 & 0 & \dots \\ c_1 & a_2 & b_3 & \dots & \dots \\ \dots & \dots & \dots & \dots & 1 \\ 0 & \dots & c_{n-2} & a_{n-1} & b_n \end{pmatrix}$$

# $k$ -chops

There are only 0-chops and 1-chops.

# $k$ -chops

There are only 0-chops and 1-chops.

- 0-chop integrals :  $H_i = \frac{1}{i} \text{Tr}(X^i), 2 \leq i \leq n.$

There are only 0-chops and 1-chops.

- 0-chop integrals :  $H_i = \frac{1}{i} \text{Tr}(X^i), 2 \leq i \leq n.$

- 1-chop integrals.

Define  $Q_S(X, \lambda) = \det(X_S - \lambda I)_1.$

$$(X_S - \lambda I)_1 = \begin{pmatrix} M_1(\lambda) & \dots & \dots & \dots & \dots \\ 0 & M'_1(\lambda) & \dots & \dots & \dots \\ 0 & 0 & M_2(\lambda) & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & M'_k(\lambda) \end{pmatrix}$$

$$D_i(\lambda) = \det M_i(\lambda), D'_i = \det M'_i(\lambda)$$

- $M_i(\lambda)$  (resp.  $M'_i(\lambda)$ ) is a square matrix of size  $l_i$  (resp.  $l'_i$ ).
- $D_i(\lambda)$  is a polynomial in  $\lambda$ ,  $D'_i(\lambda)$  is independent of  $\lambda$ .
- $Q_S(X, \lambda) = \prod_{i=1}^k D_i(\lambda)D'_i(\lambda)$ .

- $M_i(\lambda)$  (resp.  $M'_i(\lambda)$ ) is a square matrix of size  $l_i$  (resp.  $l'_i$ ).
- $D_i(\lambda)$  is a polynomial in  $\lambda$ ,  $D'_i(\lambda)$  is independent of  $\lambda$ .
- $Q_S(X, \lambda) = \prod_{i=1}^k D_i(\lambda) D'_i(\lambda)$ .

**Proposition :**

(i)  $\forall i, D_i(\lambda)$  is a polynomial of degree  $d_i = \lfloor \frac{l_i}{2} \rfloor$ .

(ii) Let  $D_i(\lambda) = \sum_{j=1}^{d_i} l_{i,j} \lambda^j, H_{i,j} = \frac{l_{i,j}}{l_{i,d_i}}, 0 \leq j \leq d_i - 1$

Then, the functions  $(H_{i,j}, 1 \leq i \leq k, 0 \leq j \leq d_i - 1, H_i, 1 \leq i \leq n)$  are constants of motion in involution.

(iii) If  $l_i$  is odd,  $(H_{i,d_i-1})$  is a Casimir.

- $M_i(\lambda)$  (resp.  $M'_i(\lambda)$ ) is a square matrix of size  $l_i$  (resp.  $l'_i$ ).
- $D_i(\lambda)$  is a polynomial in  $\lambda$ ,  $D'_i(\lambda)$  is independent of  $\lambda$ .
- $Q_S(X, \lambda) = \prod_{i=1}^k D_i(\lambda)D'_i(\lambda)$ .

**Proposition :**

(i)  $\forall i, D_i(\lambda)$  is a polynomial of degree  $d_i = \lfloor \frac{l_i}{2} \rfloor$ .

(ii) Let  $D_i(\lambda) = \sum_{j=1}^{d_i} l_{i,j} \lambda^j, H_{i,j} = \frac{l_{i,j}}{l_{i,d_i}}, 0 \leq j \leq d_i - 1$

Then, the functions  $(H_{i,j}, 1 \leq i \leq k, 0 \leq j \leq d_i - 1, H_i, 1 \leq i \leq n)$  are constants of motion in involution.

(iii) If  $l_i$  is odd,  $(H_{i,d_i-1})$  is a Casimir.

This gives the right number of functions in involution and the exact number of Casimirs.

**Next step** (work in progress) : Generalize this procedure to other classical Lie algebras ( $B_n, C_n, D_n$ ).

**Last step** : All the functions ( $H_{i,j}, H_i$ ) are independent.

**Next step** (work in progress) : Generalize this procedure to other classical Lie algebras ( $B_n, C_n, D_n$ ).

**Last step** : All the functions ( $H_{i,j}, H_i$ ) are independent.

**Conjecture** : *For any classical Lie algebra, the corresponding set of functions ( $H_{i,j}, H_i$ ) form an integrable system.*