

On linear differential operators: an application of Hermite polynomials

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October 2008



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Hermite polynomials

Hermite polynomials, $h_n(x)$

Are a classical orthogonal polynomial sequence that arise:

- In physics, as the eigenstates of the quantum harmonic oscillator.
- In probability, such as the Edgeworth series.
- In combinatorics, as an example of an Appell sequence.
- In the Brownian motion and the Schrödinger wave equation.

Hermite polynomials

Definition

They can be described in various ways, for example,

- as solutions to the differential equation

$$y'' - 2xy' + 2ny = 0$$

- by the generating function

$$\exp(2tx - t^2)$$

expanded about zero as a Taylor series in t .

Hermite polynomials

Definition

- by a differential recurrence solution, in fact

$$h'_n = 2nh_{n-1}$$

- by the formulae

$$h_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx} \right)^n e^{-x^2}$$

Hermite functions

Definition

The Hermite functions, intimately related to Hermite polynomials, are given by

$$H_n(x) = (2^n n! \sqrt{\pi})^{-\frac{1}{2}} e^{-\frac{x^2}{2}} h_n(x)$$

They form an orthonormal basis for the space $L^2(\mathbb{R})$ and this fact allows us to use a method that would help to solve, at least in some concrete examples, a problem which is described below.

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The Problem

Let A and B be two linear differential operators defined on a space Λ

Is there an isomorphism S from Λ onto Λ such that

$$AS = SB?$$

History

- Delsarte (1938): A and B to be two differential operators of order two and Λ a space of functions of one variable defined for $x \geq 0$.
- Lions (1956): operators of order greater than two with infinitely differentiable coefficients and the space $C^\infty(\mathbb{R})$.
- If A and B are of order greater than two, with infinitely differentiable coefficients, there are not, in general, transformation operators and the problem for spaces of functions with domain in the real line seems to be a difficult one.

History

- Delsarte-Lions (1957): two differential operators of the same order, without singularities in the complex plane and the space of entire functions.
- Viner (1965): transformations of differential operators in the space of holomorphic functions.
- Kushnirchuk, Nagnibida and Fishman (1974): equivalence of differential operators with a regular singular point.

History

- Nagnibida and Oliinyk (1977): equivalence of differential operators of infinite order on analytic spaces.
- M. Maldonado, J. Prada and M. J. Senosiain (2008): equivalence of differential operators with constant coefficients on the space $\mathcal{C}_{2\pi}^{\infty}(\mathbb{R})$.

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$\mathcal{C}_{2\pi}^{\infty}(\mathbb{R})$ 2π -periodic \mathcal{C}^{∞} -functions on \mathbb{R}

The space of rapidly decreasing sequences, \mathfrak{s}

$$\mathfrak{s} = \{x = (x_n) : \|x\|_k = \sum_{n=0}^{\infty} |x_n| n^k < \infty, \forall k \in \mathbb{N}\}$$

Isomorphism

$\mathcal{C}_{2\pi}^{\infty}(\mathbb{R})$ can be identified with \mathfrak{s} by means of the Fourier series:

$$F: \mathcal{C}_{2\pi}^{\infty}(\mathbb{R}) \longrightarrow \mathfrak{s}$$
$$f \longmapsto (\tilde{f}_0, \tilde{f}_1, \tilde{f}_{-1}, \tilde{f}_2, \tilde{f}_{-2}, \dots)$$

$f = \sum_{n \in \mathbb{Z}} \tilde{f}_n e^{inx}$ is the Fourier series of f .

The result

$$\mathbb{A}_2 = \begin{pmatrix} 0 & & & & & \\ & i & & & & \\ & & -i & & & \\ & & & 2i & & \\ & & & & -2i & \\ & & & & & \ddots \end{pmatrix}$$

The result

If A_1 and A_2 are “equivalent” in the sense indicated above, so they are \mathbb{A}_1 and \mathbb{A}_2 ($XA_1 = A_2X$ if and only if $\mathbb{X}\mathbb{A}_1 = \mathbb{A}_2\mathbb{X}$, $\mathbb{X} = F^{-1}XF$).

It is easily seen, from the algebraic equations given by $\mathbb{A}_1\mathbb{X} = \mathbb{X}\mathbb{A}_2$, that \mathbb{A}_1 and \mathbb{A}_2 are not equivalent.

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Assume that H is a complex vector space with a natural topology and a basis (h_n) . Then if $h \in H$

$$h = \sum a_n h_n$$

where (a_n) is a sequence of complex numbers.

If there is a sequence space Λ with a natural topology such that the mapping

$$\begin{aligned} F: H &\longrightarrow \Lambda \\ h &\longmapsto (a_n) \end{aligned}$$

is an isomorphism, that is, F is linear, bijective and bicontinuous, then we say that both spaces can be identified.

Let A and B be two linear differential operators mapping the space H on H . The following diagram

$$\begin{array}{ccc}
 H & \xrightarrow{A, B} & H \\
 F^{-1} \uparrow & & \downarrow F \\
 \Lambda & \xrightarrow{\mathbb{A}, \mathbb{B}} & \Lambda
 \end{array}$$

shows that the mappings A and B induce on Λ two mappings \mathbb{A} and \mathbb{B} such that $\mathbb{A} = F^{-1}AF$, $\mathbb{B} = F^{-1}BF$.

As F is an isomorphism, A and B are equivalent if and only if \mathbb{A} and \mathbb{B} are.

The transformation operator X between A and B can be obtained by determining the transformation operator between \mathbb{A} and \mathbb{B} .

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$\mathcal{S}(\mathbb{R})$, the space of rapidly decreasing functions

$$\mathcal{S}(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}) : \sum_{\alpha+\beta \leq k} \int |\mathbf{x}|^\alpha |f^{(\beta)}(\mathbf{x})|^2 dx < \infty, \forall k \in \mathbb{N} \right\}$$

All the functions in $\mathcal{S}(\mathbb{R})$, with all their derivatives, decrease faster than each polynomial.

$\mathcal{S}(\mathbb{R})$, the space of rapidly decreasing functions

Topology

The topological structure of $\mathcal{S}(\mathbb{R})$ is given by the countable sequence of seminorms

$$\|f\|_k^2 = \sum_{\alpha+\beta \leq k} \int |\mathbf{x}|^\alpha |f^{(\beta)}(\mathbf{x})|^2 dx, \quad k \in \mathbb{N}$$

or by the equivalent system of seminorms

$$|||f|||_k = \sup \left\{ \left| \mathbf{x}^\alpha f^{(\beta)}(\mathbf{x}) \right|, \mathbf{x} \in \mathbb{R}, \alpha + \beta \leq k \right\}, \quad k \in \mathbb{N}$$

The space of rapidly decreasing sequences, \mathfrak{s}

The space \mathfrak{s} is defined by

$$\mathfrak{s} = \left\{ \mathbf{x} \in \mathbb{C}^{\mathbb{N}} : \|\mathbf{x}\|_k^2 = \sum_{j \in \mathbb{N}} |x_j|^2 j^{2k} < \infty, \text{ for all } k \in \mathbb{N} \right\}$$

We have also

$$\mathfrak{s} = \left\{ \mathbf{x} \in \mathbb{C}^{\mathbb{N}} : \lim_{j \rightarrow \infty} |x_j| j^k = 0 \text{ for all } k \in \mathbb{N} \right\}$$

The space of rapidly decreasing sequences, \mathfrak{s}

Topology

The topological structure of \mathfrak{s} is given by the seminorms

$$\|x\|_k^2 = \sum_{j \in \mathbb{N}} |x_j|^2 j^{2k}, \quad k \in \mathbb{N}$$

or equivalently

$$\|x\|_k^2 = \sup_j \left\{ |x_j|^2 j^{2k} \right\}, \quad k \in \mathbb{N}$$

The Hermite functions

An orthonormal basis

The Hermite functions $H_n(x)$, $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ are elements of $\mathcal{S}(\mathbb{R})$ and form an orthonormal basis for $L^2(\mathbb{R})$.

Consequently the linear map

$$H: \mathcal{S}(\mathbb{R}) \longrightarrow \mathfrak{s}$$

defined by

$$H(f) = (\langle f, H_n \rangle)_{n \in \mathbb{N}_0}$$

is bijective and bicontinuous (H and H^{-1} are continuous), that is an isomorphism.

The Hermite functions

The Hermite functions satisfy the following equations

$$H_{-1} = 0$$

$$H'_n = \frac{1}{\sqrt{2}} \left(\sqrt{n} H_{n-1} - \sqrt{n+1} H_{n+1} \right)$$

$$xH_n = \frac{1}{\sqrt{2}} \left(\sqrt{n} H_{n-1} + \sqrt{n+1} H_{n+1} \right)$$

The Hermite functions

The operator D (respectively, xI) from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$ can be identified with the operator \mathbb{D} (respectively, $x\mathbb{I}$) from \mathfrak{s} to \mathfrak{s} given by

$$\mathbb{D}\delta_n = \frac{1}{\sqrt{2}} \left(\sqrt{n}\delta_{n-1} - \sqrt{n+1}\delta_{n+1} \right)$$

$$(x\mathbb{I})\delta_n = \frac{1}{\sqrt{2}} \left(\sqrt{n}\delta_{n-1} + \sqrt{n+1}\delta_{n+1} \right)$$

where $\delta_n = \left(\delta_j^n \right)_{j=0}^{\infty}$.

Linear differential operator

By an induction process a linear differential operator of the form

$$A = p_0(x)I + p_1(x)D + \dots + p_{m-2}(x)D^{m-2} + p_{m-1}(x)D^{m-1} + D^m,$$

where $p_j(x)$ are polynomials, can be identified with a linear operator from \mathfrak{s} to \mathfrak{s} .

The formula for differential operators of order greater than two is really cumbersome but a simple example shows how the procedure works.

Example

Let A be a linear differential operator of second order, precisely

$$A = I + xD + x^2D^2.$$

Then

$$AH_n = H_n + xH'_n + x^2H''_n$$

Example

As

$$xH'_n = \frac{\sqrt{n(n-1)}}{2}H_{n-2} - \frac{1}{2}H_n - \frac{\sqrt{(n+1)(n+2)}}{2}H_{n+2}$$

$$x^2H''_n = x^2 \left(\frac{\sqrt{n(n-1)}}{2}H_{n-2} - \frac{2n+1}{2}H_n + \frac{\sqrt{(n+1)(n+2)}}{2}H_{n+2} \right)$$

$$x^2H_n = \frac{\sqrt{n(n-1)}}{2}H_{n-2} + \frac{2n+1}{2}H_n + \frac{\sqrt{(n+1)(n+2)}}{2}H_{n+2}$$

Example

It follows that

$$\begin{aligned} AH_n = & \frac{\sqrt{n(n-1)(n-2)(n-3)}}{4} H_{n-4} - \frac{\sqrt{n(n-1)}}{2} H_{n-2} \\ & - \frac{2n^2 + 2n - 3}{4} H_n + \frac{\sqrt{(n+1)(n+2)}}{2} H_{n+2} \\ & + \frac{\sqrt{(n+1)(n+2)(n+3)(n+4)}}{4} H_{n+4} \end{aligned}$$

Example

\mathbb{A} can be identified with the operator \mathbb{A}

$$\begin{aligned}\mathbb{A}\delta_n &= \frac{\sqrt{n(n-1)(n-2)(n-3)}}{4}\delta_{n-4} - \frac{\sqrt{n(n-1)}}{2}\delta_{n-2} \\ &\quad - \frac{2n^2 + 2n - 3}{4}\delta_n + \frac{\sqrt{(n+1)(n+2)}}{2}\delta_{n+2} \\ &\quad + \frac{\sqrt{(n+1)(n+2)(n+3)(n+4)}}{4}\delta_{n+4}\end{aligned}$$

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$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), \quad A = I + pD, \quad B = I + qD, \quad p, q \in \mathbb{C}$$

We have two operators \mathbb{A} and \mathbb{B} from s to s given by

$$\mathbb{A}(\delta_n) = \begin{cases} \delta_0 - \frac{p}{\sqrt{2}}\delta_1, & n = 0 \\ \sqrt{\frac{n}{2}}p\delta_{n-1} + \delta_n - \sqrt{\frac{n+1}{2}}p\delta_{n+1}, & n \geq 1 \end{cases}$$

$$\mathbb{B}(\delta_n) = \begin{cases} \delta_0 - \frac{q}{\sqrt{2}}\delta_1, & n = 0 \\ \sqrt{\frac{n}{2}}q\delta_{n-1} + \delta_n - \sqrt{\frac{n+1}{2}}q\delta_{n+1}, & n \geq 1 \end{cases}$$

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), A = I + pD, B = I + qD, p, q \in \mathbb{C}$$

We are looking for a transformation operator \mathbb{X} (represented by a matrix $\mathbb{X} = (x_{j,n})_{j,n=0}^{\infty}$, that is $\mathbb{X}(\delta_n) = \sum_{j=0}^{\infty} x_{j,n} \delta_j$) such that

$$A\mathbb{X} = B\mathbb{X}$$

and $x_{0,0} = 1, x_{0,n} = 0, n \geq 1$.

From the two previous conditions (using the program Mathematica) we get the matrix \mathbb{X} whose elements are

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), \quad A = I + pD, \quad B = I + qD, \quad p, q \in \mathbb{C}$$

The matrix of \mathbb{X}

$$X_{j,n} = \begin{cases} 0 & j < n \\ 0 & j - n = 2m - 1, \\ & m \geq 0 \\ \sqrt{\binom{j}{n} \binom{j-n-1}{m} \frac{1}{2^{j-n-1}} \frac{p^n}{q^j} (q^2 - p^2)^m} & j - n = 2m, \\ & m \geq 1 \\ \frac{p^n}{q^n} & j = n. \end{cases}$$

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), \quad A = I + pD, \quad B = I + qD, \quad p, q \in \mathbb{C}$$

This matrix \mathbb{X} is a lower triangular matrix and so invertible. Therefore, from the algebraic point view, the transformation operator between \mathbb{A} and \mathbb{B} exists and is a linear operator from φ to φ where

$$\varphi = \{(x_n) : x_n \in \mathbb{C} \text{ and } x_n = 0 \text{ for all } n, \text{ except a finite number}\}$$

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), A = I + pD, B = I + qD, p, q \in \mathbb{C}$$

Continuity of \mathbb{X}

To ensure that \mathbb{X} is a linear continuous operator from s to s it is enough to prove the following condition:

$\forall k \in \mathbb{N}, \exists N(k) \in \mathbb{N}, \exists C(k) > 0$ such that

$$\sup_{j \geq n} \left\{ |x_{j,n}| j^k \right\} \leq C(k) n^{N(k)}, \text{ for all } n \in \mathbb{N}$$

or, equivalently

$$\sup_{j \geq n} \left\{ \frac{|x_{j,n}| j^k}{n^{N(k)}} \right\} \leq C(k), \text{ for all } n \in \mathbb{N}$$

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), A = I + pD, B = I + qD, p, q \in \mathbb{C}$$

Continuity of \mathbb{X}

Let write the formula for $j = n + 2m, m = 0, 1, 2, \dots, n = 0, 1, 2, \dots$. Then

$$\begin{aligned} & \sup_{n, m \in \mathbb{N}} \left\{ |x_{n+2m, n}| \frac{(n+2m)^k}{n^{N(k)}} \right\} \\ &= \sup_{n, m \in \mathbb{N}} \left\{ \frac{1}{2^{\frac{2m-1}{2}}} (2m)! \binom{2m-1}{m} \left| \frac{(q^2 - p^2)^m}{q^{2m}} \right| \right. \\ & \quad \left. \left[(n+1) \dots (n+2m) \frac{(n+2m)^k}{n^{N(k)}} \left| \frac{p}{q} \right|^n \right] \right\} \end{aligned}$$

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), \quad A = I + pD, \quad B = I + qD, \quad p, q \in \mathbb{C}$$

Continuity of \mathbb{X}

Assume that given k , $\exists N(k)$ such that

$$\sup_{n, m \in \mathbb{N}} \left\{ \frac{1}{2^{\frac{2m-1}{2}} (2m)!} \binom{2m-1}{m} \left| \frac{(q^2 - p^2)^m}{q^{2m}} \right| \left[(n+1) \dots (n+2m) \frac{(n+2m)^k}{n^{N(k)}} \left| \frac{p}{q} \right|^n \right] \right\} < \infty$$

and take $m = N(k) + 1 - k$.

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), A = I + pD, B = I + qD, p, q \in \mathbb{C}$$

Continuity of \mathbb{X}

As

$$(n+1) \dots (n+2m) \frac{(n+2m)^k}{n^{N(k)}} \geq \frac{n^{m+k}}{n^{N(k)}}$$

it follows that for such an m

$$(n+1) \dots (n+2m) \frac{(n+2m)^k}{n^{N(k)}} \rightarrow \infty \text{ with } n$$

and there is a contradiction unless

$$q^2 - p^2 = 0 \quad \text{or} \quad \left| \frac{p}{q} \right| < 1.$$

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), \quad A = I + pD, \quad B = I + qD, \quad p, q \in \mathbb{C}$$

Continuity of \mathbb{X}

Suppose $q^2 - p^2 = 0$.

- When $p = q$, $\mathbb{A} = \mathbb{B}$ ($A = B$) and the matrix $\mathbb{X} = I$.
- If $p = -q$, \mathbb{X} is

$$x_{jn} = \begin{cases} 0 & j \neq n \\ 1 & j = n, n \text{ even} \\ -1 & j = n, n \text{ odd} \end{cases} \quad \mathbb{X}\delta_n = \begin{cases} \delta_n & n \text{ even} \\ -\delta_n & n \text{ odd} \end{cases}$$

Obviously \mathbb{A} and \mathbb{B} (so A and B) are equivalent.

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), A = I + pD, B = I + qD, p, q \in \mathbb{C}$$

Continuity of \mathbb{X}

Assume now that $\left| \frac{p}{q} \right| < 1$ and $p \neq \pm q$.

If \mathbb{A} and \mathbb{B} were equivalent the operator \mathbb{X} would be an isomorphism.

So \mathbb{X} and \mathbb{X}^{-1} are continuous.

The matrix \mathbb{X}^{-1} is given by

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), \quad A = I + pD, \quad B = I + qD, \quad p, q \in \mathbb{C}$$

Continuity of \mathbb{X}

$$x_{j,n} = \begin{cases} 0 & j < n \\ 0 & j - n = 2m - 1, \\ & m \geq 0 \\ \sqrt{\binom{j}{n} \binom{j-n-1}{m} \frac{1}{2^{j-n-1}} \frac{q^n}{p^j} (p^2 - q^2)^m} & j - n = 2m, \\ & m \geq 1 \\ \frac{q^n}{p^n} & j = n. \end{cases}$$

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), A = I + pD, B = I + qD, p, q \in \mathbb{C}$$

Continuity of \mathbb{X}

The continuity of \mathbb{X}^{-1} implies that $\left| \frac{q}{p} \right| < 1$ (because $p \neq \pm q$).
Therefore $|p| = |q|$ and if $p \neq \pm q$ neither \mathbb{X} nor \mathbb{X}^{-1} are continuous.

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), \quad A = pI + D, \quad B = qI + D, \quad p, q \in \mathbb{C}$$

We have two operators \mathbb{A} and \mathbb{B} from s to s given by

$$\mathbb{A}(\delta_n) = \begin{cases} p\delta_0 - \frac{1}{\sqrt{2}}\delta_1, & n = 0 \\ \sqrt{\frac{n}{2}}\delta_{n-1} + p\delta_n - \sqrt{\frac{n+1}{2}}\delta_{n+1}, & n \geq 1 \end{cases}$$

$$\mathbb{B}(\delta_n) = \begin{cases} q\delta_0 - \frac{1}{\sqrt{2}}\delta_1, & n = 0 \\ \sqrt{\frac{n}{2}}\delta_{n-1} + q\delta_n - \sqrt{\frac{n+1}{2}}\delta_{n+1}, & n \geq 1 \end{cases}$$

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), A = pl + D, B = ql + D, p, q \in \mathbb{C}$$

The matrix of \mathbb{X}

The matrix $\mathbb{X} = (x_{j,n})$ such that $\mathbb{X}A = B\mathbb{X}$ and $x_{0,0} = 1, x_{0,n} = 0, n \geq 1$ is

$$x_{j,n} = \begin{cases} 0 & j < n \\ 1 & j = n \\ \sqrt{\binom{j}{n} \frac{2^{j-n}}{(j-n)!}} (p-q)^{j-n} & j > n. \end{cases}$$

Introduction

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The Schwartz space $\mathcal{S}(\mathbb{R})$ and the sequence space s

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Operators of the form $pI + D$

Operators of the form $p_0(x) + p_1(x)D$

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), A = pI + D, B = qI + D, p, q \in \mathbb{C}$$

\mathbb{X} is a lower triangular matrix and so invertible. Therefore, from the algebraic point view, the transformation operator between \mathbb{A} and \mathbb{B} exists and is a linear operator from φ to φ .

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), \quad A = pl + D, \quad B = ql + D, \quad p, q \in \mathbb{C}$$

Continuity of \mathbb{X}

To ensure that \mathbb{X} is a linear continuous operator from s to s it is enough to prove the following condition:

$\forall k \in \mathbb{N}, \exists N(k) \in \mathbb{N}, \exists C(k) > 0$ such that

$$\sup_{j \geq n} \left\{ \frac{|x_{j,n}| j^k}{n^{N(k)}} \right\} \leq C(k), \quad \text{for all } n \in \mathbb{N}$$

which, in this case, reads

$$\sup_{j \geq n} \left\{ \sqrt{\binom{j}{n} \frac{2^{j-n}}{(j-n)!}} \left| (p-q)^{j-n} \right| \frac{j^k}{n^{N(k)}} \right\} \leq C(k), \quad \text{for all } n \in \mathbb{N}$$

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), A = pI + D, B = qI + D, p, q \in \mathbb{C}$$

Continuity of \mathbb{X}

Writing the formula for $j = n + m$, $m = 1, 2, \dots$, $n = 0, 1, 2, \dots$ we have

$$\sup_{n, m \in \mathbb{N}} \left\{ \sqrt{\binom{n+m}{n} \frac{2^m}{m!}} |(p-q)^m| \frac{(n+m)^k}{n^{N(k)}} \right\} \leq C(k)$$

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), A = pI + D, B = qI + D, p, q \in \mathbb{C}$$

Continuity of \mathbb{X}

It is easy to see that

$$\sup_{n, m \in \mathbb{N}} \left\{ \sqrt{\binom{n+m}{n} \frac{2^m}{m!}} |(p-q)^m| \frac{(n+m)^k}{n^{N(k)}} \right\}$$

$$\geq \sup_{n, m \in \mathbb{N}} \left\{ \frac{2^{\frac{m}{2}}}{m!} |p-q|^m \frac{(n+1)^{\frac{m}{2}+k}}{n^{N(k)}} \right\}$$

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), \quad A = pI + D, \quad B = qI + D, \quad p, q \in \mathbb{C}$$

Continuity of \mathbb{X}

Assume that $\forall k, \exists N(k), \exists C(k)$ such that the previous condition is true. Then taking $m \in \mathbb{N}$ such that $\frac{m}{2} + k = N(k) + 2$ it follows that

$$\frac{2^{\frac{m}{2}}}{m!} |p - q|^m \frac{(n+1)^{\frac{m}{2}+k}}{n^{N(k)}} \geq \frac{2^{\frac{m}{2}}}{m!} |p - q|^m n^2 \xrightarrow{n \rightarrow \infty} \infty$$

and a contradiction appears unless $p = q$.

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), \quad A = p_0(x) + p_1(x)D, \\ B = q_0(x) + q_1(x)D$$

Consider linear operators of order one with coefficients polynomial functions,

$$A = p_0(x) + p_1(x)D$$

$$B = q_0(x) + q_1(x)D$$

A simple case is $p_0(x) = q_0(x) = 0$, $p_1(x) = x$, $q_1(x) = 1$, that is $A = xD$, $B = D$.

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), \quad A = p_0(x) + p_1(x)D,$$

$$B = q_0(x) + q_1(x)D$$

The induced operators are

$$\mathbb{A}(\delta_n) = \begin{cases} -\frac{1}{2}\delta_0 - \frac{\sqrt{2}}{2}\delta_2, & n = 0 \\ -\frac{1}{2}\delta_1 - \frac{\sqrt{2.3}}{2}\delta_3, & n = 1 \\ \frac{\sqrt{n(n-1)}}{2}\delta_{n-2} - \frac{1}{2}\delta_n - \frac{\sqrt{(n+1)(n+2)}}{2}\delta_{n+2}, & n \geq 2 \end{cases}$$

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), \quad A = p_0(x) + p_1(x)D, \\ B = q_0(x) + q_1(x)D$$

$$\mathbb{B}(\delta_n) = \begin{cases} -\sqrt{\frac{1}{2}}\delta_1, & n = 0 \\ \sqrt{\frac{n}{2}}\delta_{n-1} - \sqrt{\frac{n+1}{2}}\delta_{n+1}, & n \geq 1 \end{cases}$$

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), \quad A = p_0(x) + p_1(x)D,$$

$$B = q_0(x) + q_1(x)D$$

If $\mathbb{X} = (x_{jn})$ is a matrix such that $\mathbb{X}A = B\mathbb{X}$ and

$$x_{0,0} = x_{1,1} = 1$$

$$x_{0,n} = 0, \quad n \geq 1$$

$$x_{1,0} = 0$$

$$x_{1,n} = 0, \quad n > 1$$

then its elements are given by the recurrence formula

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), \quad A = p_0(x) + p_1(x)D,$$

$$B = q_0(x) + q_1(x)D$$

$$x_{j+2,n} = \begin{cases} \frac{1}{\sqrt{(j+1)(j+2)}} \left[\sqrt{(j-1)j} x_{j-2,n} + x_{j,n} \right. \\ \quad \left. + \sqrt{2n} x_{j,n-1} - \sqrt{2(n+1)} x_{j,n+1} \right] & n \leq j+2, \\ & j \geq 0 \\ 0 & n > j+2, \\ & j \geq 0 \end{cases}$$

$$A, B: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), \quad A = p_0(x) + p_1(x)D,$$

$$B = q_0(x) + q_1(x)D$$

A careful look to the formula gives immediately that $x_{j+2,n} = 0$,
when $j + 2 = n, j \geq 0$.

\mathbb{X} is not invertible and there is not an algebraic solution.