MYTHS of NONCLASSICAL ΣYMMETRY

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MYTHS of NONCLASSICAL SYMMETRY

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Introduction

The notion of nonclassical symmetry (called also Q-conditional or, simply, conditional symmetry) was introduced by Bluman and Cole in 1969. In the two last decades, the theoretical background of nonclassical symmetry was intensively investigated and the technique based on nonclassical symmetry was effectively applied to finding exact solutions of many partial differential equations arising in physics, biology, financial mathematics etc.

In spite of the long period of studying nonclassical symmetry and hopeful results in its applications, a number of basic problems of this theory are still open. Moreover, there exists a series of non-rigorous definitions of related key notions and heuristic results on fundamental properties of nonclassical symmetry in the literature, which are used up to now and form the mythology of nonclassical symmetry. These definitions and results need a special feeling in order to correctly apply them. Otherwise, certain contradictions and inaccurate statements may be obtained.

Mythology is an unavoidable step in the development of any subject.

Definition

Let $Q = \{Q^1, \dots, Q^l\}$ be an involutive family of l $(l \leqslant n)$ vector fields

$$Q^s = \xi^{si}(x,u)\partial_i + \eta^s(x,u)\partial_u$$

in the space (x, u), where rank $\|\xi^{si}(x,u)\| = l$. The involution means that

$$\forall\, s, s' \quad \exists\, \zeta^{ss'\sigma} = \zeta^{ss'\sigma}(x,u)\colon \quad [Q^s, Q^{s'}] = \zeta^{ss'\sigma}Q^\sigma.$$

Hereafter we use the summation convention for repeated indices.

x is the n-tuple of independent variables $(x_1, \dots, x_n)$. u is the unknown function.

$i, j = 1, \dots, n, \quad s, \sigma = 1, \dots, l, \quad \partial_i = \partial/\partial x_i, \quad \partial_u = \partial/\partial u.$

Consider $\mathcal{L}$: $L[u] := L(x, u_{(r)}) = 0$ and the characteristic system $Q[u] = 0$.

$$\mathcal{L} \sim \mathcal{L} = \{(x, u_{(r)}) \in J^r \mid L[u] = 0\},$$

$$\mathcal{Q}_{(r)} = \{(x, u_{(r)}) \in J^r \mid D_1^{\alpha_1} \dots D_n^{\alpha_n} Q^s[u] = 0, \;\; \alpha_i \in \mathbb{N} \cup \{0\}, \;\; |\alpha| < r\},$$

where D_i is the operator of total differentiation with respect to the variable x_i, $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $|\alpha| := \alpha_1 + \dots + \alpha_n$.

Definition 1. Q is called an *involutive family of nonclassical symmetry (or Q-conditional symmetry, conditional symmetry etc) operators of* $\mathcal{L}$ if

$$Q^s_{(r)} L(x, u_{(r)})\big|_{\mathcal{L} \cap \mathcal{Q}_{(r)}} = 0.$$

Myths on name and definition

Non-rigorous definition

Myth 1. *A nonclassical symmetry operator Q of an equation $\mathcal{L}$ is called a vector field Q which is a Lie symmetry operator of the united system of the equation $\mathcal{L}$ and the invariant surface condition $\mathcal{Q}$: $Q[u]=0$ corresponding to Q.*

This is the conventional non-rigorous way in order to quickly define nonclassical symmetry. It becomes rigorous only after a special interpretation of the notions of a system of differential equations and Lie symmetry. Otherwise, using the empiric definition leads to a number of inconsistencies.

Careful analysis shows that the above definition is a tautology. Indeed, the invariant surface condition $Q[u]=0$ means that the function u is a fixed point of the one-parametric local group G_Q of local transformations generated by the operator Q. Therefore, we can reformulate the definition in the following way.

Reformulation. If the set of solutions of the equation $\mathcal{L}$, which are fixed points of G_Q, is invariant with respect to G_Q then Q is called a nonclassical symmetry operator Q of an equation $\mathcal{L}$.

The tautology of the reformulation is obvious. If each element of the set is invariant then the whole set is necessarily invariant. The definition of nonclassical symmetry according to Myth 1 leads to the conclusion that

any differential equation is invariant, in nonclassical sense, with respect to any vector field in the corresponding space of dependent and independent variables.

The case when the equation $\mathcal{L}$ has no Q-invariant solutions well fits into the non-rigorous approach since the empty set is a very symmetric set. Therefore, uncritically following the non-rigorous approach, we would get

- no effective methods for construction of exact solutions
- no information on properties of differential equations

There exist different reformulations of Myth 1 in the literature.

Reformulation of Myth 1 in terms of conditional symmetry

Myth 2. *A nonclassical symmetry operator Q of an equation $\mathcal{L}$ is called a conditional symmetry operator of the equation $\mathcal{L}$ under the auxiliary condition* $Q[u] = 0$.

Here conditional symmetry is understood in the following sense.

Definition 2. A vector field Q is called a *conditional symmetry operator* of a system $\mathcal{S}$ of differential equations under an auxiliary condition $\mathcal{S}'$ (which is another system of differential equations in the same variables) if Q is a Lie symmetry operator of the united system of $\mathcal{S}$ and $\mathcal{S}'$.

Differences of conditional symmetry from nonclassical one:

- Auxiliary conditions do not involve associated conditional symmetry operators.
- The conditional symmetry operators of a system $\mathcal{S}$ under an auxiliary condition $\mathcal{S}'$ form a Lie algebra.
- Conditional symmetry really is a kind of symmetry and can applied to generate new solutions from known ones.
- Finding nontrivial auxiliary conditions is an art but not a regular procedure.

This is why sometimes nonclassical symmetries are called either

- *Q-conditional symmetries* where the prefix "Q" is used to emphasize differences between nonclassical and conditional symmetries or
- *conditional symmetries* without any connection with Definition 2.

Reformulation of Myth 1 in infinitesimal terms

Myth 3. *The nonclassical symmetry criterion for an equation $\mathcal{L}$ and an operator Q coincides with the infinitesimal Lie invariance criterion for the united system $\{\mathcal{L}, Q[u] = 0\}$ with respect to the same operator, i.e.,*

$$Q_{(r)}L[u] = 0 \quad \textit{if} \quad L[u] = 0 \quad \textit{and} \quad Q[u] = 0 \quad (\mathrm{ord}\, L = r).$$
$$(Q_{(r)}Q[u] = (\eta - \xi^j_u u_j)Q[u] \equiv 0 \quad \textit{if} \quad Q[u] = 0.)$$

But

$$Q_{(r)}L[u] = L_{u_\alpha}[u]D_1^{\alpha_1}\dots D_n^{\alpha_n}Q[u] + \xi^i D_i L[u] \equiv 0 \quad \text{if} \quad L[u] = 0 \text{ and } Q[u] = 0.$$

Under the local approach within group analysis of differential equations, a system of differential equations is associated with the infinite tuple of systems of algebraic equations defined by this system and its differential consequences in the infinite tower of the corresponding jet spaces.

The exclusion of the above differential consequence from the consideration is unnatural from the point of view of group analysis!

Myth on name (main Myth of nonclassical symmetry)

Myth 4. *Nonclassical symmetry is a kind of symmetry of differential equations.*

Any kind of symmetries of differential equations (Lie, contact, hidden, conditional, approximate, generalized, potential, nonlocal etc.) has the *invariance* property, i.e., symmetries transform solutions to solutions in certain sense.

Prerequisite of the definition of nonclassical symmetry is the consideration only of the set of solutions unchangeable by the associated finite transformations.

It is impossible to use nonclassical symmetries in order to generate new solutions from known ones.

A nonclassical symmetry operator Q of $\mathcal{L}$ represent only a symmetry of

- each of Q-invariant solutions of $\mathcal{L}$ (as a weak symmetry) and
- the manifold $\mathcal{L} \cap \mathcal{Q}_{(r)}$ in J^r, where $r = \operatorname{ord} \mathcal{L}$.

At the same time, properties of the set of nonclassical symmetries and properties of the set of Q-invariant solutions for each nonclassical symmetry operator Q characterize the equation $\mathcal{L}$.

Abbreviation?

non~~classical~~ symmetry $\longrightarrow$ non-symmetry

Another name?

Reality

Properties of Lie symmetries:

Invariance (transforming solutions to solutions)

Formal compatibility (attaching the invariant surface conditions to the initial system of differential equations gives no nontrivial differential consequences)

Reduction (each invariance algebra satisfying the infinitesimal transversality condition leads to an ansatz reducing the initial system to a system with a less number of independent variables)

Conditional compatibility (there exists a bijection between solutions of the initial system, which satisfy the invariant surface conditions, and solutions of the corresponding reduced system)

Properties of nonclassical symmetries:

Formal compatibility, Reduction, Conditional compatibility

Nonclassical symmetries of $\mathcal{L}$ $\simeq$ **first-order quasilinear differential constraints which are formally compatible with $\mathcal{L}$**

What property is the main and completely represent the essence of nonclassical symmetry?

"first-order quasilinear" $\Longrightarrow$ "integrable" $\Longrightarrow$ ansatz
Then "formally compatible" $\Longrightarrow$ reduction by this ansatz

There exist integrable differential constrains which are not formally compatible with the initial system.

There exist differential constrains which are formally compatible with the initial system but are not integrable.

"first-order" $\Longrightarrow$
the number of old dep. variables = the number of "invariant" dep. variables $\Longrightarrow$ reduction in the classical sense

REDUCTION!

A possible name for operators of nonclassical symmetry is **reduction operators**.

One more important property of Lie symmetries is broken for nonclassical symmetries.

Lie symmetries: $(r = \operatorname{ord}\mathcal{L})$

- Q is a Lie symmetry of $\mathcal{L}_{(r)} \quad \Longrightarrow \quad Q$ is a Lie symmetry of $\mathcal{L}_{(\rho)}$ for any ρ: $\rho > r$
- Q is a Lie symmetry of $\mathcal{L}_{(\rho)}$ for a ρ: $\rho > r \quad \Longrightarrow \quad Q$ is a Lie symmetry of $\mathcal{L}_{(r)}$

$\mathcal{L}_{(k)}$ denotes a maximal set of algebraically independent differential consequences of $\mathcal{L}$ that have, as differential equations, orders not greater than k. It is identified with the corresponding system of algebraic equations in $J^k(x|u)$ and the manifold determined by this system.

nonclassical symmetries:

Q is a Lie symmetry of $\mathcal{L}_{(r)} \cap \mathcal{Q}_{(r)} \quad \Longrightarrow$
Q is a Lie symmetry of $\mathcal{L}_{(\rho)} \cap \mathcal{Q}_{(\rho)}$ for any ρ: $\rho > r$

But

Q is a Lie symmetry of $\mathcal{L}_{(\rho)} \cap \mathcal{Q}_{(\rho)}$ for a ρ: $\rho > r \quad \not\Longrightarrow$
Q is a Lie symmetry of $\mathcal{L}_{(r)} \cap \mathcal{Q}_{(r)}$

Example.

$$L = u_t + u_{xx} + t u_x, \quad Q = \partial_t$$

$$\mathcal{L}_{(2)} \cap \mathcal{Q}_{(2)}: \quad u_t = u_{tt} = u_{tx} = 0, \quad u_{xx} = -t u_x$$

$$Q_{(2)} L \Big|_{\mathcal{L}_{(2)} \cap \mathcal{Q}_{(2)}} = u_x \neq 0$$

$$\text{Ansatz:} \quad u = \varphi(\omega), \quad \omega = x$$

$$\varphi_{\omega\omega} + t\varphi_\omega = 0 \qquad \text{No reduction!}$$

$$L, \ D_t L, \ D_x L, \quad Q = \partial_t$$

$$\mathcal{L}_{(3)} \cap \mathcal{Q}_{(3)}: \quad u_t = u_{tt} = u_{tx} = u_{ttt} = u_{ttx} = u_{txx} = 0, \quad u_x = u_{xx} = u_{xxx} = 0$$

$$Q_{(2)} L \Big|_{\mathcal{L}_{(3)} \cap \mathcal{Q}_{(3)}} = Q_{(3)} D_t L \Big|_{\mathcal{L}_{(3)} \cap \mathcal{Q}_{(3)}} = Q_{(3)} D_x L \Big|_{\mathcal{L}_{(3)} \cap \mathcal{Q}_{(3)}} = 0$$

$$\begin{pmatrix} \varphi_{\omega\omega} + t\varphi_\omega \\ \varphi_\omega \\ \varphi_{\omega\omega\omega} + t\varphi_{\omega\omega} \end{pmatrix} = \begin{pmatrix} t & 1 & 0 \\ 1 & 0 & 0 \\ 0 & t & 1 \end{pmatrix} \begin{pmatrix} \varphi_{\omega\omega} \\ \varphi_{\omega\omega} \\ \varphi_{\omega\omega\omega} \end{pmatrix} = 0$$

Definition of nonclassical symmetries for systems

Myth 5. *The definition of nonclassical symmetry for systems of differential equations is a simple extension of the definition of nonclassical symmetry for single partial differential equations to the case of systems.*

$\mathcal{L}$: $L^\mu[u] := L^\mu(x, u_{(r)}) = 0$, $\mu = 1, \dots, l$, $u = (u^1, \dots, u^m)$, $r = \operatorname{ord}\mathcal{L}$

$Q_{(r)}L^\mu\big|_{\mathcal{L}_{(r)}\cap\mathcal{Q}_{(r)}} = 0$ or $Q_{(r)}L^\mu\big|_{\mathcal{L}\cap\mathcal{Q}_{(r)}} = 0$?

Example. The Navier–Stokes equations

$$\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} - \Delta\vec{u} + \nabla p + \vec{x}\times\nabla(\operatorname{div}\vec{u}) = \vec{0}, \quad \operatorname{div}\vec{u} = 0.$$

Myths on possible applications

Myth 6. *One-parametric transformation groups associated with nonclassical symmetry operators can be used for generating new exact solutions from known ones.*

The generation is impossible since the corresponding transformations can be applied, by definition, only to solutions which are fixed points of them.

Useful applications for such groups are not found up to now.

An additional problem is that nonclassical symmetry operators are partitioned, by the equivalence up to nonvanishing functional multiplier, into classes of indistinguishable operators. Therefore, the association has no sense.

Myth 7. *Finding exact solutions of differential equations by reduction is a unique application of nonclassical symmetries.*

Possible applications:

- deriving inequalities for solutions
- studying characteristic directions

Specific myths of multi-dimensional case

Myth 8. *Any reduction in a number of independent variables can be decomposed in a step-by-step sequence of reductions in a single independent variable.*

Example. $u_t = (x^2 + y^3)u_{xx}^2 + (x^5 + y^6)u_{yyy}, \quad Q^1 = \partial_x, \quad Q^2 = \partial_y$

Myth 9. *No-go results on nonclassical symmetries are trivially extendable to the multidimensional case.*

Example.

$u_t = u_{xx}$

$Q = \tau\partial_t + \xi\partial_x + \eta\partial_u$

The case $\tau = 0$ is singular and "no-go"

The case $\tau \neq 0$ is regular and "no-go"

$u_t = u_{xx} + u_{yy}$

$Q = \tau\partial_t + \xi^x\partial_x + \xi^y\partial_y + \eta\partial_u$

All cases are regular and admit closed answers.

Myths of singular cases, factorization and number

Myth 10. *Number of nonclassical symmetries is essentially greater than number of nonclassical symmetries.*

Yes, but partial "yes"

- the usual equivalence of nonclassical symmetries
- nonclassical symmetries equivalent to Lie ones
- the equivalence of nonclassical symmetries with respect Lie symmetry groups and equivalence groups
- no-go cases

Myth 11. *The factorization of the set of nonclassical symmetry operators with respect to the equivalence of nonclassical symmetries is a trivial step.*

Singular vector fields of differential functions

Definition. $Q \in \mathfrak{Q}$ is *singular* for $L = L[u]$ ($\operatorname{ord} L = r$) if $\exists \tilde{L} = \tilde{L}[u]$ ($\operatorname{ord} \tilde{L} < r$): $L|_{\mathcal{Q}_{(r)}} = \tilde{L}|_{\mathcal{Q}_{(r)}}$. Otherwise Q is *regular* for L.

the *singularity co-order* of Q for L = the minimal order of differential functions whose restrictions on $\mathcal{Q}_{(r)}$ coincide with $L|_{\mathcal{Q}_{(r)}}$

Q is *ultra-singular* for L if $L|_{\mathcal{Q}_{(r)}} \equiv 0$.

For convenience, the singularity co-order of ultra-singular vector fields and the order of identically vanishing differential functions are assumed to equal -1. Regular vector fields for the differential function L are assumed to have the singularity co-order $r = \operatorname{ord} L$.

If Q is a singular for L then any vector field equivalent to Q is singular for L with the same co-order of singularity.

$\tilde{L}$ can be constructively found!

$\xi^2 \neq 0$, $\mathcal{Q}_{(r)}$:

$$u_2 = \hat\eta - \hat\xi u_1,$$
$$u_{12} = \hat\eta_1 - \hat\xi_1 u_1 + \hat\eta_u u_1 - \hat\xi_u u_1^2 - \hat\xi u_{11},$$
$$u_{22} = \hat\eta_2 - \hat\xi_2 u_1 + (\hat\eta_u - \hat\xi_u u_1)(\hat\eta - \hat\xi u_1) - \hat\xi(\hat\eta_1 - \hat\xi_1 u_1 + \hat\eta_u u_1 - \hat\xi_u u_1^2 - \hat\xi u_{11}),$$
$$\dots$$

where $\hat\xi = \xi^1/\xi^2$ and $\hat\eta = \eta/\xi^2$.

Substituting the expressions for derivatives into L gives $\hat L$ depending only on x, u and derivatives of u with respect to x_1.

$\hat L$ is a *differential function associated with* L *on the manifold* $\mathcal{Q}_{(r)}$.

Q is singular L iff $\operatorname{ord}\hat L < r$. The singularity co-order of Q equals $\hat L$.

Q is ultra-singular iff $\hat L \equiv 0$.

Therefore, testing that a vector field is singular for a differential function with two independent variables is realized via the completely algorithmic procedure.

Consider the two-dimensional module $\{Q^\theta = \theta^i Q^i\}$ of vector fields over the ring of smooth functions of (x,u). $Q^i = \xi^{ij}(x,u)\partial_j + \eta^i(x,u)\partial_u$, $\operatorname{rank}(\xi^{i1},\xi^{i2},\eta^i) = 2$. $\theta = (\theta^1,\theta^2)(x,u)$. $i,j = 1,2$.

Definition. $\{Q^\theta\}$ is *singular* for L if $\forall\theta$ Q^θ is singular for $\mathcal{L}$. The singularity co-order of the module $\{Q^\theta\}$ coincides with the maximum of the singularity co-orders of its elements.

It is enough, up to point transformations, to study only singular sets of vector fields of the *reduced* form $\{Q^\zeta = \xi\partial_1 + \partial_2 + \zeta\partial_u\}$, where ξ is a fixed smooth function of (x,u) and ζ runs through the set of such functions.

Further simplification of the representation for elements from the module depends on whether the module is closed under the Lie bracket.

In any two-dimensional module of vector fields in the space of three variables (x_1,x_2,u), basis vector fields Q^1 and Q^2 can be locally reduced, by point transformations, to the form $Q^1 = \partial_2$ (resp. $Q^1 = u\partial_1 + \partial_2$) and $Q^2 = \partial_u$ if the module is closed (resp. not closed) with respect to the Lie bracket of vector fields.

Theorem. A differential function L with one dependent and two independent variables possesses a kth co-order singular two-dimensional module of vector fields iff it is represented, up to point transformations, in the form

$$L = \check{L}(x, \Omega_{r,k}),$$

where $\Omega_{r,k} = \big(\omega_\alpha = D_1^{\alpha_1}(\xi D_1 + D_2)^{\alpha_2}u, \alpha_1 \leqslant k, \alpha_1+\alpha_2 \leqslant r\big)$, $\xi \in \{0, u\}$, and $\check{L}_{\omega_\alpha} \neq 0$ for some ω_α with $\alpha_1 = k$.

Corollary. A differential function with one dependent and two independent variables admits a kth co-order singular two-dimensional module generated by commuting vector fields if and only if it is reduced by a point transformation of the variables to a differential function in which all differentiation with respect to one of independent variables are only up to order k.

Corollary. Any differential function with one dependent and two independent variables, which does not identically vanish, admits no ultra-singular two-dimensional module of singular vector fields.

Singular vector fields of differential equations

Q is *(strongly) singular for* $\mathcal{L}$: $L[u]=0$ if it is singular for $L[u]$.

Definition. Definition. $Q\in\mathfrak{Q}$ is *weakly singular* for $\mathcal{L}$: $L[u]=0$ $(\operatorname{ord}L=r)$ if $\exists\tilde L=\tilde L[u]$ $(\operatorname{ord}\tilde L<r)$ and $\exists\lambda=\lambda[u]$ $(\operatorname{ord}\lambda\leqslant r)$: $L|_{\mathcal{Q}_{(r)}}=\lambda\tilde L|_{\mathcal{Q}_{(r)}}$. Otherwise Q is *weakly regular* for L.

The notions of ultra-singularity in weak and strong senses coincide.

Strong singularity implies weak singularity.

Example: evolution equations

$u_t = H(t, x, u_{(r,x)}),$

$r > 1$, $u_0 := u$, $u_k = \partial^k u/\partial x^k$, $u_{(r,x)} = (u_0, u_1, \ldots, u_r)$ and $H_{u_r} \neq 0$.

Proposition. Q is singular for $L = u_t - H$ iff $\tau = 0$. The singularity co-order equals 1.

Corollary. $L = u_t - H$ possesses exactly one set of singular vector fields in the reduced form, $S = \{\partial_x + \zeta(x,u)\partial_u\}$. The singularity co-order of S equals 1.

$H_{u_r} \neq 0 \Longrightarrow$ weak singularity $\sim$ strong singularity

$Q \in \mathfrak{Q}_0(\mathcal{L}) = \mathfrak{Q}(\mathcal{L}) \cap S \Longrightarrow \mathrm{DE}_0(\mathcal{L})$:

$$\zeta_t + \zeta_u \tilde{H} = \tilde{H}_x + \zeta \tilde{H}_u, \quad \tilde{H} := H(t, x, u, \zeta, \zeta_x + \zeta\zeta_u, \ldots (\partial_x + \zeta\partial_u)^{r-1}\zeta),$$

$\Longleftrightarrow$ the compatibility condition of $u_x = \zeta$ and $\mathcal{L}$.

Theorem. Up to the equivalences of operators and solution families, for any evolution equation there exists a bijection between one-parametric families of its solutions and reduction operators with zero coefficients of ∂_t. Namely, each operator of such kind corresponds to the family of solutions which are invariant with respect to this operator. The problems of the construction of all one-parametric solution families of an evolution equation and the description of its reduction operators with zero coefficients of ∂_t are completely equivalent.

Corollary. The nonlinear $(1+2)$-dimensional equation $\mathrm{DE}_0(\mathcal{L})$ is reduced by composition of the nonlocal substitution $\zeta = -\Phi_x/\Phi_u$, where Φ is a function of (t, x, u), and the hodograph transformation

$$\text{the new independent variables:} \qquad \tilde{t} = t, \quad \tilde{x} = x, \quad \varkappa = \Phi,$$

$$\text{the new dependent variable:} \qquad \tilde{u} = u$$

to the initial equation $\mathcal{L}$ in $\tilde{u} = \tilde{u}(\tilde{t}, \tilde{x}, \varkappa)$ with $\varkappa$ playing the role of a parameter.

$\mathfrak{Q}_1(\mathcal{L}) = \{Q \in \mathfrak{Q}_1(\mathcal{L}) \mid \tau = 1\}$

$\mathcal{L}$: $u_t = (f(u)u_x)_x, \quad \mathfrak{Q}_1(\mathcal{L}) = ?$

$\mathcal{L}$: $u_t = u_{xx} + ue^{u_x} + xe^{2u_x} + te^{3u_x} + e^{4u_x} + e^{5u_x}, \quad \mathfrak{Q}_1(\mathcal{L}) = \varnothing$

linear evolution equations, Burgers equation: no-go

Example: nonlinear wave equations

$\mathcal{L}$: $u_{12} = F(u)$

$Q = \xi^i(x,u)\partial_i + \eta(x,u)\partial_u$ is singular for $L = u_{12} - F(u)$ iff $\xi^1\xi^2 = 0$.

weak singularity $\sim$ strong singularity

$L = u_{12} - F(u)$ possesses exactly two sets of singular vector fields in the reduced form,

$S = \{\partial_2 + \zeta(x,u)\partial_u\}$ and $S^* = \{\partial_1 + \zeta^*(x,u)\partial_u\}$.

$x_1 \leftrightarrow x_2 \Longleftrightarrow S \leftrightarrow S^*$

$Q = \partial_2 + \zeta\partial_u \in \mathfrak{Q}(\mathcal{L}) \Longrightarrow$

$(\zeta_{12} + \zeta_{1u}u_2 + \zeta_{2u}u_1 + \zeta_{uu}u_1u_2 + \zeta_u u_{12})|_{\mathcal{L}\cap\mathcal{Q}_{(r)}} = \zeta F_u$

$\mathcal{L} \cap \mathcal{Q}_{(2)}$: $\quad u_2 = \zeta$, $u_{12} = F$ and $\zeta_t + \zeta_u u_t = F$.

The further consideration depends on values of ζ_u and F_u.

$\zeta_u = 0,\ F_u = 0$

Q is ultra-singular for L.

the conditional invariance criterion: $\zeta_1 = F$

An ansatz constructed with Q reduces $\mathcal{L}$ to an identity.

The family of Q-invariant solutions of $\mathcal{L}$ is parameterized by a function of a single argument.

$\zeta_u = 0,\ F_u \neq 0$

The singularity co-order of Q for L equals 0.
The conditional invariance criterion: $\zeta_{12} = \zeta F_u(F^{-1}(\zeta_1))$.
The ansatz constructed with Q reduces $\mathcal{L}$ to an algebraic equation.
The family of Q-invariant solutions of $\mathcal{L}$ consists of a single solution.

Theorem. Up to the equivalence of solution families, there exists a bijection between solutions of $\mathcal{L}$ with $F_u \neq 0$ and reduction operators of the form $Q = \partial_2 + \zeta(x)\partial_u$ (resp. $Q^* = \partial_1 + \zeta^*(x)\partial_u$) whose singularity co-orders equal 0. Namely, each operator of such kind corresponds to the solution which is invariant with respect to this operator. The problems of solving the equation $\mathcal{L}$ with $F_u \neq 0$ and the description of its reduction operators of the above form are completely equivalent.

Corollary. Any solution $u = f(x)$ of $\mathcal{L}$ with $F_u \neq 0$ is invariant w.r.t. two reduction operators $Q = \partial_2 + \zeta(x)\partial_u$ and $Q^* = \partial_1 + \zeta^*(x)\partial_u$ of singularity co-order 0. Here $\zeta = f_2$ and $\zeta^* = f_1$. The property of possessing the same invariant solution of $\mathcal{L}$ establishes the canonical bijection $Q \leftrightarrow Q^*$ between the sets of reduction operators of singularity co-order 0. The adjoint values of ζ and ζ^* are connected by the formulas

$$\zeta^* = \frac{\zeta_{11}}{F_u(\check{F}(\zeta_1))}, \qquad \zeta = \frac{\zeta^*_{22}}{F_u(\check{F}(\zeta^*_2))}.$$

$\zeta_u \neq 0,\ F_u \neq 0$

The singularity co-order of Q for L equals 1.

The conditional invariance criterion: $\zeta_{12} + \zeta\zeta_{1u} + (\zeta_{2u} + \zeta\zeta_{uu})\frac{F-\zeta_1}{\zeta_u} + \zeta_u F = \zeta F_u$

The ansatz constructed with Q reduces $\mathcal{L}$ to a first order ODE.

Proposition. There exists a canonical bijection $Q \leftrightarrow Q^*$ between sets of singular reduction operators of $\mathcal{L}$ of the forms $Q = \partial_2 + \zeta(x,u)\partial_u$ and $Q^* = \partial_1 + \zeta^*(x,u)\partial_u$, where $\zeta_u \neq 0$ and $\zeta^*_u \neq 0$. This bijection is defined by the formulas

$$Q \to Q^*\colon \quad \zeta^* = \frac{F - \zeta_1}{\zeta_u}, \qquad Q^* \to Q\colon \quad \zeta = \frac{F - \zeta^*_2}{\zeta^*_u}.$$

A solution of $\mathcal{L}$ is invariant with respect to the operator Q if and only if it is invariant with respect to the operator Q^*.

Theorem. Up to the equivalence of solution families, for $\mathcal{L}$ with $F_u \neq 0$ there exists a bijection between one-parametric families of its solutions and reduction operators of the form $Q = \partial_2 + \zeta(x,u)\partial_u$, where $\zeta_u \neq 0$ (resp. $Q^* = \partial_1 + \zeta^*(x,u)\partial_u$, where $\zeta^*_u \neq 0$). Namely, each operator of such kind corresponds to the family of solutions which are invariant with respect to this operator. The problems of the construction of all one-parametric solution families of $\mathcal{L}$ with $F_u \neq 0$ and the description of its reduction operators of the above form are completely equivalent.

Corollary. The nonlinear three-dimensional determining equation for ζ is reduced by composition of the Bäcklund transformation $\zeta=-\Phi_2/\Phi_u$, $\zeta^*=-\Phi_1/\Phi_u$, where Φ is a function of (x,u), and the hodograph transformation

$$\begin{array}{ll}\text{the new independent variables:} & \tilde x_1=x_1,\quad \tilde x_2=x_2,\quad \varkappa=\Phi,\\ \text{the new dependent variable:} & \tilde u=u\end{array}$$

to the equation $\mathcal{L}$ for the function $\tilde u=\tilde u(\tilde x,\varkappa)$ with $\varkappa$ playing the role of a parameter.

The **natural partition** of the set of reduction operators of $\mathcal{L}$:

$\xi^1=0;\quad \xi^2=0;\quad \xi^1\xi^2\neq 0$

After factorization: $\xi^1=0,\ \xi^2=1;\quad \xi^2=0,\ \xi^1=1;\quad \xi^1\neq 0,\ \xi^2=1$

After extended factorization with symmetry: $\xi^1=0,\ \xi^2=1;\quad \xi^1\neq 0,\ \xi^2=1$

$u_{11}-u_{22}=F(u)$: $\quad \xi^1=-\xi^2;\quad \xi^1=\xi^2;\quad \xi^1\neq\pm\xi^2$

$u_{11}-(G(u)u_2)_2=F(u)$: $\quad \dots$