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## ORBIT FUNCTIONS AND CORRESPONDING FOURIER-LIKE TRANSFORMS

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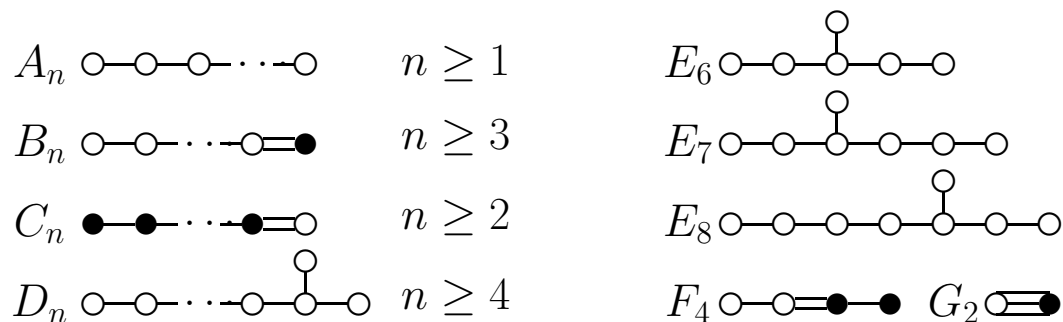
### Abstract

Recently introduced families of special functions are recalled and their most valuable properties are pointed out. Those are the following:

- the functions are defined for each compact semisimple Lie group  $G$ ; there are several infinite families of functions per  $G$ ;
- the number of continuous variables on which the functions depend, is equal to the rank of  $G$ ;
- the functions have well defined symmetries with respect to the affine Weyl group of  $G$ ;
- the functions within each family are orthogonal when integrated over the fundamental region of  $F$  of the maximal torus  $\mathbb{T}$  of  $G$ ;
- the functions can be sampled on the lattice fragment  $F_M = F \cap L/M$  of the weight lattice  $L$  refined by  $M \in \mathbb{N}$ ; there is a finite subset  $\Lambda_M$  of such ‘digital’ functions that are pairwise orthogonal when summed up over the points of  $F_M$ .

# SIMPLE LIE GROUPS/ALGEBRAS

Dynkin diagrams of simple Lie groups



Relative lengths of simple roots (= circles)

$$(\text{open circle}) : (\text{black circle}) = \begin{cases} 2 : 1 & \text{all but } G_2 \\ 3 : 1 & G_2 \end{cases}$$

$$\text{Relative angles} = \begin{cases} 90^\circ & \text{no directly connecting line} \\ 120^\circ & \text{single line} \\ 135^\circ & \text{double line} \\ 150^\circ & \text{triple line} \end{cases}$$

Standard additional convention:

$$\text{square length} = \langle \text{open circle}, \text{open circle} \rangle = 2$$

# WEYL GROUP AND AFFINE WEYL GROUP

Real Euclidean space  $\mathbb{R}^n$  of dimension  $n$

The finite reflection group  $\iff$  the Weyl group

Reflection  $r_\xi$  in mirror orthogonal to  $\xi$  and passing through the origin:

$$r_\xi x = x - \frac{2\langle \xi, x \rangle}{\langle \xi, \xi \rangle} \xi, \quad x \in \mathbb{R}^n, \quad \langle \cdot, \cdot \rangle = \text{scalar product in } \mathbb{R}^n$$

Affine reflection  $R_\xi$  in mirror orthogonal to  $\xi$  and passing through the point  $\frac{1}{2}\xi$ :

$$R_\xi x = \xi + r_\xi x, \quad x \in \mathbb{R}^n$$

Translations

$$r_\xi R_\xi x = x - \xi, \quad R_\xi r_\xi x = x + \xi$$

Examples

$$\begin{array}{lll} r_\xi \xi = -\xi & R_\xi \xi = 0 & r_\xi^2 x = x \\ r_\xi 0 = 0 & R_\xi 0 = \xi & R_\xi^2 x = x \end{array}$$

## W-INVARIANT FUNCTIONS

Irreducible character  $\chi(\Lambda)$  of representation  $\Lambda$

$$\chi_\Lambda(x) = \sum_{\lambda} d_{\Lambda\lambda} C_\lambda(x)$$

$\lambda$  = dominant weight of  $\Lambda$ ,  $d_{\Lambda\lambda}$  = multiplicity of the dominant weight  $\lambda$  in  $\Lambda$   
( $d_{\Lambda\lambda}$ ) = triangular matrix,  $d_{\Lambda\Lambda} = 1$  in all cases

$C$ -orbit function

$$C_\lambda(x) = \sum_{\Lambda} (d_{\Lambda\lambda})^{-1} \chi_\Lambda(x)$$

Weyl character formula

$$\chi_\Lambda(x) = \frac{S_{\Lambda+\rho}(x)}{S_\rho(x)}$$

$S_\mu(x)$  is  $S$ -orbit function,  $\rho = (1, 1, \dots, 1)$  in an appropriate basis

$E$ -orbit function  $E_\mu(x)$

$$E_\mu(x) = \frac{1}{2}(C_\mu(x) + S_\mu(x))$$

## ORBIT FUNCTIONS

$W_\lambda$  = set of distinct points generated from  $\lambda$  by  $W$

Number of points  $|W_\lambda|$ :  $1 \leq |W_\lambda| \leq |W|$

$$C_\lambda(x) = \sum_{\mu \in W_\lambda} e^{2\pi i \langle \mu, x \rangle} \quad C\text{-function}$$

$$S_\lambda(x) = \sum_{\mu \in W_\lambda} (-1)^{l(\mu)} e^{2\pi i \langle \mu, x \rangle}, \quad l(\mu) = \text{number of reflections from } \lambda \text{ to } \mu \quad S\text{-function}$$

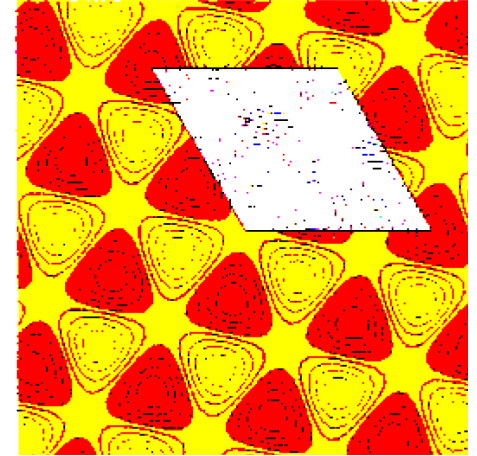
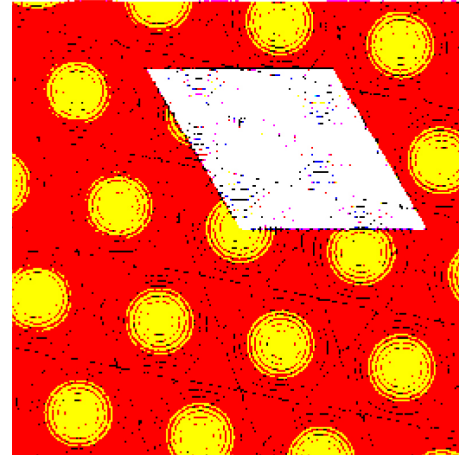
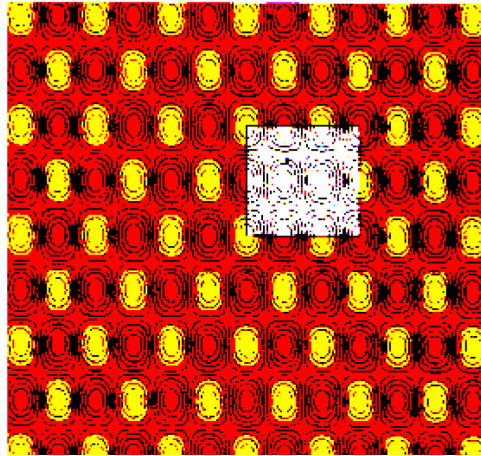
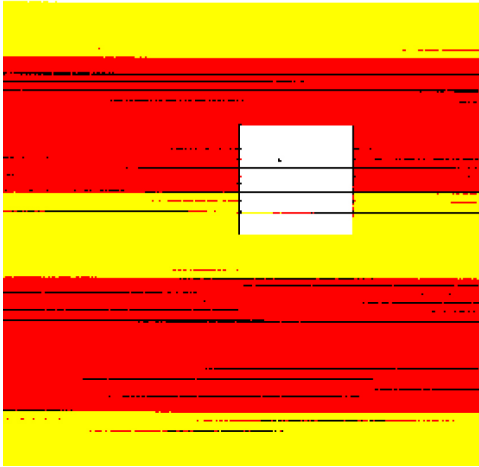
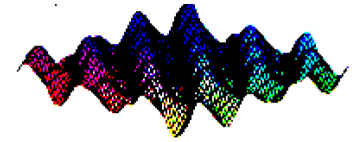
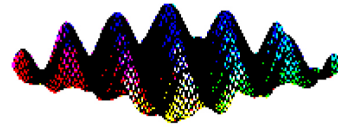
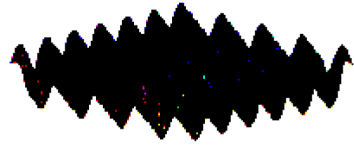
$$E_\lambda(x) = \sum_{\mu \in W_e \lambda} e^{2\pi i \langle \mu, x \rangle}, \quad W_e = \text{even subgroup of } W \quad E\text{-function}$$

Example: Generic  $C$ -,  $S$ -, and  $E$ -functions of  $A_2$

$$C_\lambda(x) = e^{2\pi i \langle \lambda, x \rangle} + e^{2\pi i \langle r_1 \lambda, x \rangle} + e^{2\pi i \langle r_2 \lambda, x \rangle} + e^{2\pi i \langle r_2 r_1 \lambda, x \rangle} + e^{2\pi i \langle r_1 r_2 \lambda, x \rangle} + e^{2\pi i \langle r_1 r_2 r_1 \lambda, x \rangle}$$

Note that  $r_1 r_2 r_1 = r_2 r_1 r_2$

# EXAMPLES OF ORBIT FUNCTIONS



$C$  – functions of  $A_1 \times A_1$   
for  $\lambda = (1, 0)$  and  $\lambda = (2, 3)$

$E$ -function of  $A_2$   
for  $\lambda = (1, 2)$  real and imaginary parts

## BASES IN $\mathbb{R}^n$

$\alpha$  – basis      simple roots, see Dynkin diagrams

$$\omega \text{ – basis} \quad \langle \alpha_j, \omega_k \rangle = \delta_{kj} \frac{\langle \alpha_j, \alpha_j \rangle}{2}$$

$$\check{\alpha} \text{ – basis} \quad \check{\alpha}_j = \alpha \frac{2}{\langle \alpha_j, \alpha_j \rangle}$$

$$\check{\omega} \text{ – basis} \quad \check{\omega}_j = \omega \frac{2}{\langle \alpha_j, \alpha_j \rangle}$$

In particular

$$\langle \omega_k | \check{\alpha}_j \rangle = \langle \alpha_j | \check{\omega}_k \rangle = \delta_{jk}$$

In general

$$\alpha = C\omega, \quad \text{Cartan matrix} \quad \boxed{C = \frac{2\langle \alpha_j | \alpha_k \rangle}{\langle \alpha_k | \alpha_k \rangle}}$$

$$\omega = C^{-1}\alpha, \quad \text{inverse Cartan matrix}$$

## FUNDAMENTAL REGION AND LATTICE GRID

**Fundamental region**  $F = \text{simplex in } \mathbb{R}^n$  it is described by  $n + 1$  vertices

$$F = \left\{ 0, \frac{\check{\omega}_1}{m_1}, \dots, \frac{\check{\omega}_n}{m_n} \right\}.$$

Unique highest root and highest co-root

$$\zeta = \sum_{k=1}^n m_k \alpha_k, \quad \check{\zeta} = \sum_{k=1}^n \check{m}_k \check{\alpha}_k.$$

Unique extension root  $\alpha_0 = -\zeta$ .

**Lattice grid**  $F_M \subset F \subset \mathbb{R}^n$ ,  $M$  positive integer

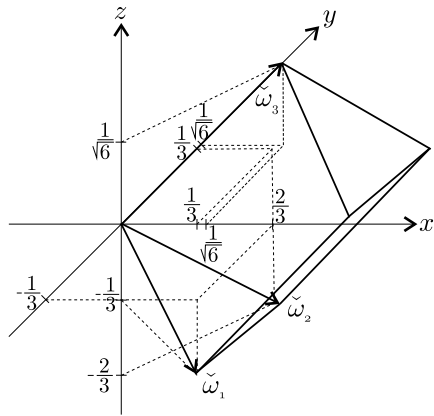
To find point  $x \in F_M$

$$x \in F_M \iff [s_1, \dots, s_n], \quad s_j \in \mathbb{Z}^{\geq 0}, \quad \sum_{i=1}^n m_i s_i \leq M \implies [s_1, \dots, s_n].$$

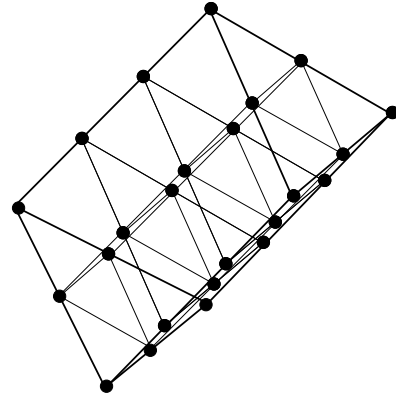
To draw the point  $x \in F_M$

$$x = \frac{1}{M} \sum_{i=1}^n s_i \check{\omega}_i = \left( \frac{s_1}{M}, \dots, \frac{s_n}{M} \right) = \frac{1}{M} (s_1, \dots, s_n).$$

# EXAMPLES OF FUNDAMENTAL REGIONS AND LATTICE GRIDS



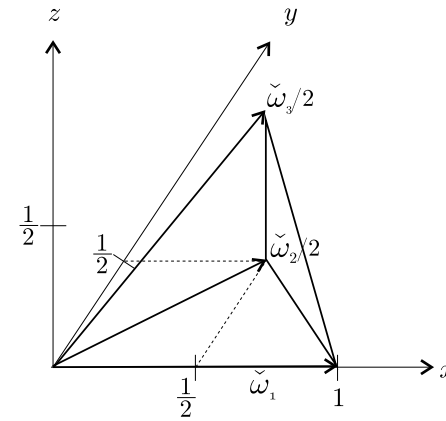
a



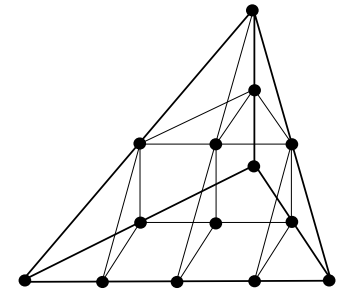
b

a) the fundamental region of the Lie algebra  $A_2 \times A_1$ ;

b) the grid  $F_{2,3}(A_2 \times A_1)$ .



a



b

a) the fundamental region of the Lie algebra  $B_3$ ;

b) the grid  $F_4(B_3)$ .

## CONTINUOUS ORTHOGONALITY OF ORBIT FUNCTIONS

Square integrable functions  $\phi(x)$  and  $\psi(x)$ ,

$$\text{Inner product } \langle \phi(x), \psi(x) \rangle = \int_F \phi(x) \bar{\psi}(x) dx.$$

Orbit functions are orthogonal with respect to the introduced inner product

$$\langle C_\lambda(x), C_{\lambda'}(x) \rangle = \int_F C_\lambda(x) \bar{C}_{\lambda'}(x) dx = |W_\lambda| \cdot |F| \cdot \delta_{\lambda\lambda'};$$

$$\langle S_\lambda(x), S_{\lambda'}(x) \rangle = \int_F S_\lambda(x) \bar{S}_{\lambda'}(x) dx = |W| \cdot |F| \cdot \delta_{\lambda\lambda'};$$

$$\langle E_\lambda(x), E_{\lambda'}(x) \rangle = \int_{F_e} E_\lambda(x) \bar{E}_{\lambda'}(x) dx = |W_{e\lambda}| \cdot |F_e| \cdot \delta_{\lambda\lambda'},$$

## DISCRETE INNER PRODUCT

$$\text{Notation } \widetilde{F}_M = \begin{cases} F_M & \text{for } C\text{-functions,} \\ F_M \setminus \partial F & \text{for } S\text{-functions,} \\ F_e M & \text{for } E\text{-functions.} \end{cases}$$

Discrete inner product

$$\langle \phi(x), \psi(x) \rangle_M = \sum_{i=1}^N \varepsilon(x_i) \phi(x_i) \overline{\psi(x_i)}, \quad N = |\widetilde{F}_M|, \quad x_i \in F_M.$$

Here  $\varepsilon(x_i)$  is the number of elements in the maximal torus of the Lie group conjugate to  $x_i$ .

$$\varepsilon(x_i) = \begin{cases} |W|/|\text{Stab}_W(x_i)|, & \text{for } C\text{-functions,} \\ |W|, & \text{for } S\text{-functions,} \\ |W_e|/|\text{Stab}_{W_e}(x_i)|, & \text{for } E\text{-functions.} \end{cases}$$

Note that  $\varepsilon(x)$  does not depend on the density of the grid in  $F_M$ .

## DISCRETE ORTHOGONALITY OF $C$ -, $S$ -, AND $E$ -FUNCTIONS

In each family only finite set  $\Lambda_M$  of functions can be pairwise orthogonal on the points of  $\tilde{F}_M$

$$\langle C_\lambda(x), C_{\lambda'}(x) \rangle_M = \sum_{i=1}^N \varepsilon(x_i) C_\lambda(x_i) \overline{C_{\lambda'}(x_i)} = |W_\lambda| \cdot |A_M| \cdot \delta_{\lambda\lambda'},$$

$$\langle S_\lambda(x), S_{\lambda'}(x) \rangle_M = |W| \sum_{i=1}^N S_\lambda(x_i) \overline{S_{\lambda'}(x_i)} = |W| \cdot |A_M| \cdot \delta_{\lambda\lambda'},$$

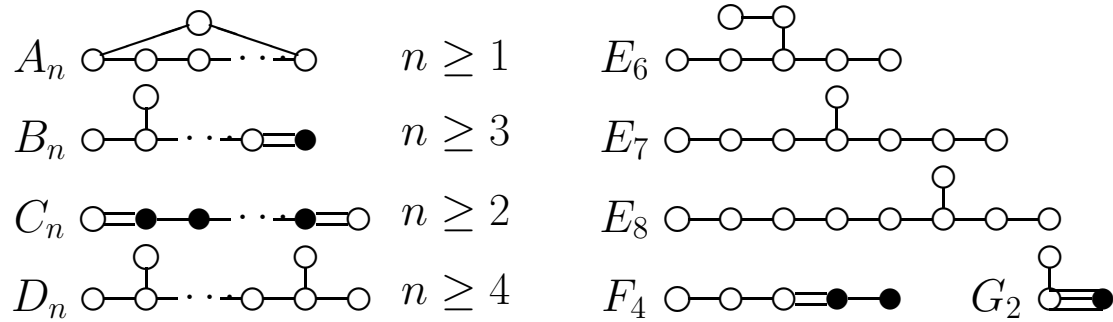
$$\langle E_\lambda(x), E_{\lambda'}(x) \rangle_M = \sum_{i=1}^N \varepsilon(x_i) E_\lambda(x_i) \overline{E_{\lambda'}(x_i)} = |W_{e\lambda}| \cdot |A_{eM}| \cdot \delta_{\lambda\lambda'},$$

where  $|A_M| = \sum_{i=1}^N |W_{x_i}|$  for  $C$ - and  $S$ -functions and  $|A_{eM}| = \sum_{i=1}^N |W_{e x_i}|$  for  $E$ -functions.

$|A_M|$  is the order of the Abelian group whose conjugacy classes are represented by the points of  $\tilde{F}_M$ .

# COEFFICIENTS $\varepsilon(x)$ FOR THE INNER PRODUCTS

Extended Dynkin diagrams of the simple Lie groups



1) For  $x \in F_M$  ( $x = (\frac{s_1}{M}, \frac{s_2}{M}, \frac{s_3}{M})$  in  $\omega$ -basis) we consider the set  $[s_0, s_1, s_2, s_3]$  such that

$$s_0, s_1, s_2, s_3 \in \mathbb{Z}^{\geq 0} \quad \text{and} \quad s_0 + s_1 + s_2 + s_3 = M$$

as attached to the corresponding nodes of the extended Dynkin diagram.

2) Then we take the subdiagram with attached zeros only.

3) It gives a Weyl group of order  $|\text{Stab}_W(x)|$ .

## THE VALUES OF $\varepsilon(x)$ FOR THE RANK 3 SIMPLE LIE GROUPS

$x$	$A_3$		$B_3$		$C_3$	
	$ \text{Stab}_W(x) $	$\varepsilon(x)$	$ \text{Stab}_W(x) $	$\varepsilon(x)$	$ \text{Stab}_W(x) $	$\varepsilon(x)$
$[s_0, s_1, s_2, s_3]$	1	24	1	48	1	48
$[s_0, s_1, s_2, 0]$	2	12	2	24	2	24
$[s_0, s_1, 0, s_3]$	2	12	2	24	2	24
$[s_0, 0, s_2, s_3]$	2	12	2	24	2	24
$[0, s_1, s_2, s_3]$	2	12	2	24	2	24
$[s_0, s_1, 0, 0]$	6	4	8	6	8	6
$[s_0, 0, s_2, 0]$	4	6	4	12	4	12
$[0, s_1, s_2, 0]$	6	4	4	12	4	12
$[s_0, 0, 0, s_3]$	6	4	6	8	6	8
$[0, s_1, 0, s_3]$	4	6	6	8	4	12
$[0, 0, s_2, s_3]$	6	4	4	12	8	6
$[s_0, 0, 0, 0]$	24	1	48	1	48	1
$[0, 0, s_2, 0]$	24	1	8	6	16	3
$[0, s_1, 0, 0]$	24	1	48	1	16	3
$[0, 0, 0, s_3]$	24	1	24	2	48	1

Assuming  $s_0, s_1, s_2, s_3 > 0$

## CONTINUOUS AND DISCRETE TRANSFORMS

$C$ -transforms:

$$f(x) = \sum_{\lambda \in P^+} c_\lambda C_\lambda(x), \quad c_\lambda = \frac{\langle f, C_\lambda \rangle}{\langle C_\lambda, C_\lambda \rangle} = \frac{1}{|W_\lambda||F|} \int_F f(x) \overline{C}_\lambda(x) dx,$$

$$f(x_i) = \sum_{\lambda \in \Lambda_M} c_\lambda C_\lambda(x_i), \quad c_\lambda = \frac{\langle f, C_\lambda \rangle_M}{\langle C_\lambda, C_\lambda \rangle_M} = \frac{1}{|W_\lambda||A_M|} \sum_{i=1}^N \varepsilon(x_i) f(x_i) \overline{C}_\lambda(x_i).$$

$S$ -transforms:

$$f(x) = \sum_{\lambda \in P^{++}} c_\lambda S_\lambda(x), \quad c_\lambda = \frac{\langle f, S_\lambda \rangle}{\langle S_\lambda, S_\lambda \rangle} = \frac{1}{|W_\lambda||F|} \int_F f(x) \overline{S}_\lambda(x) dx,$$

$$f(x_i) = \sum_{\lambda \in \Lambda_M} c_\lambda S_\lambda(x_i), \quad c_\lambda = \frac{\langle f, S_\lambda \rangle_M}{\langle S_\lambda, S_\lambda \rangle_M} = \frac{|W|}{|W_\lambda||A_M|} \sum_{i=1}^N f(x_i) \overline{S}_\lambda(x_i).$$

$E$ -transforms:

$$f(x) = \sum_{\lambda \in P_e^+} c_\lambda E_\lambda(x), \quad c_\lambda = \frac{\langle f, E_\lambda \rangle}{\langle E_\lambda, E_\lambda \rangle} = \frac{1}{|W_\lambda||F_e|} \int_{F_e} f(x) \overline{E}_\lambda(x) dx,$$

$$f(x_i) = \sum_{\lambda \in \Lambda_M} c_\lambda E_\lambda(x_i), \quad c_\lambda = \frac{\langle f, E_\lambda \rangle_M}{\langle E_\lambda, E_\lambda \rangle_M} = \frac{1}{|W_{e\lambda}||A_{eM}|} \sum_{i=1}^N \varepsilon(x_i) f(x_i) \overline{E}_\lambda(x_i).$$

## CONCLUSIONS

- A number of other properties of the orbit functions may prove to be useful. In particular each function is an eigenfunction of the Laplace operator appropriate for  $G$ . Their eigenvalues are known explicitly.

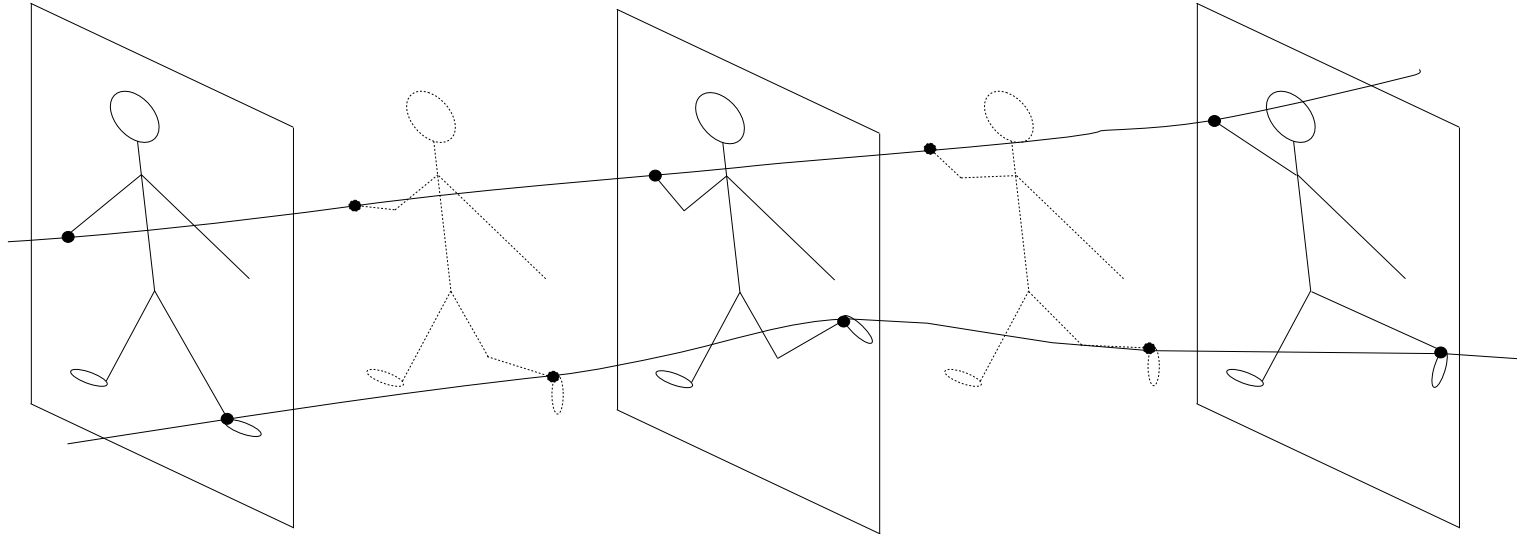
Let us mention that each of the three types of functions split into mutually exclusive congruence classes. For a given semisimple Lie group, the number of congruence classes equals the order of the center of the Lie group.

- The price paid for uniformity of our approach is the exploitation of non-orthogonal bases,  $\alpha$ ,  $\omega$ -bases and their duals whenever necessary.

It should be pointed out that, at least for some types of transform, it is possible to avoid paying the price i.e. of having to work with orthonormal bases. The Weyl group of  $SU(n)$  is isomorphic to the permutation group  $S_n$  of  $n$  elements. Recently introduced transforms based on  $S_n$  and on its alternating subgroup, exploit orthonormal bases in  $\mathbb{R}^n$ , although even there, the corresponding fundamental regions do not have orthogonal adjacent faces, in general.

## APPLICATIONS

One of the possible applications of orbit functions is the construction of unknown transitional data (smooth interpolation).



This figure shows how additional frames could be added to a film. In much the same way, the continuous deformation of the picture can be proceed and a 3-dimensional image can be created from corresponding 2-dimensional layers or cuts.