Yang-Baxter Maps,
Dynamical Yang-Baxter Maps
and
Integrable Lattice Equations

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The 4th International Workshop in
Group Analysis of Differential Equations and Integrable Systems

Protaras,Cyprus October 26-30, 2008

A trivial(!) map and its couplings

Consider the map

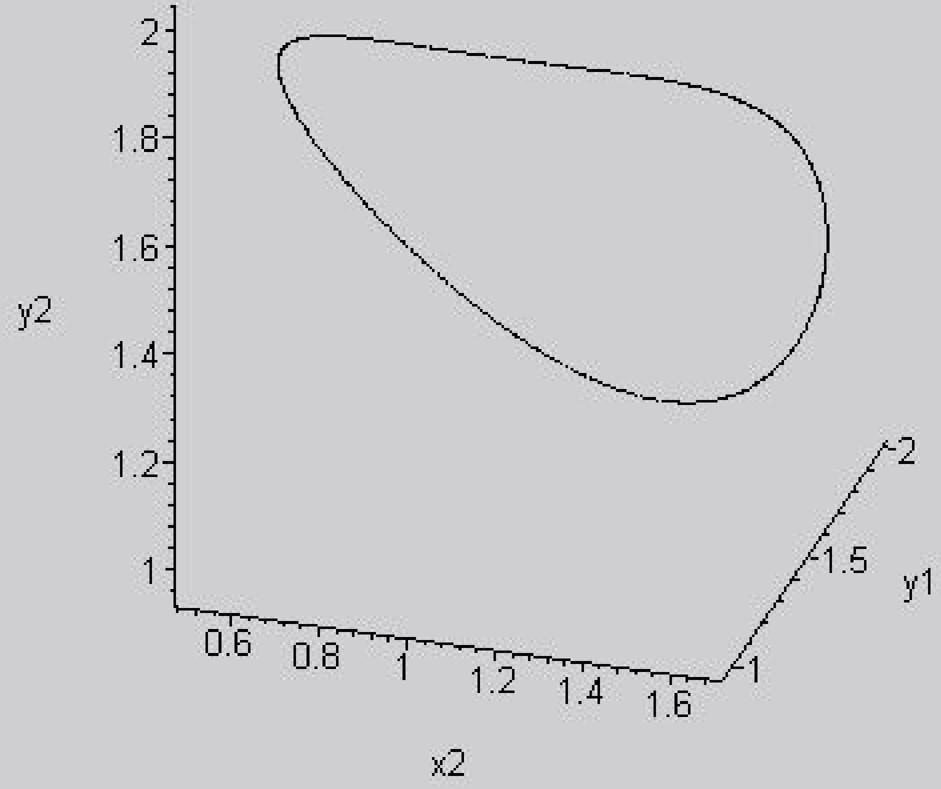
$R(x, y) = (u, v)$

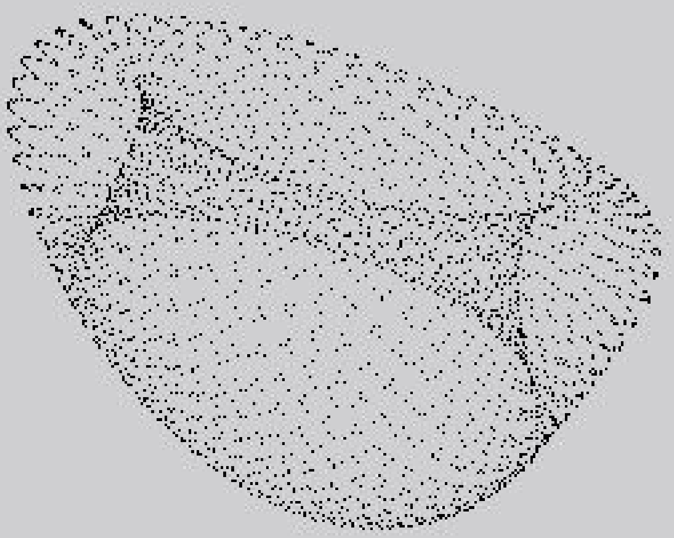
$$\boxed{u = y + \frac{\alpha_1 - \alpha_2}{x + y}, \qquad v = x - \frac{\alpha_1 - \alpha_2}{x + y}}$$

$$R^2 = Identity$$

$$u_1 = y_1 + \frac{\alpha_1 - \alpha_2}{x_1 + y_1}, \qquad v_1 = x_2 - \frac{\alpha_1 - \alpha_2}{x_2 + y_2}$$

$$u_2 = y_2 + \frac{\alpha_1 - \alpha_2}{x_2 + y_2}, \qquad v_2 = x_1 - \frac{\alpha_1 - \alpha_2}{x_1 + y_1}$$


2
1.8
1.6
y2
1.4
1.2
1
0.6
0.8
1
1.2
1.4
1.6
x2
-1
-1.5
-2
y1

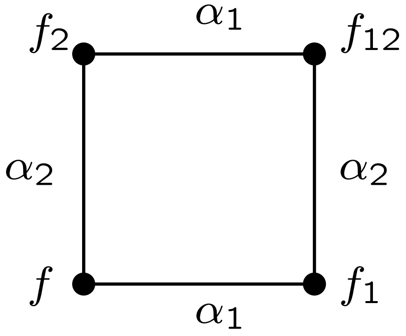


Integrable equations on quad-graphs

Equations associated to planar graphs with elementary quadrilateral faces

$f : \mathbb{Z}^2 \to \mathbb{C}$

$$\mathcal{E}(f, f_1, f_2, f_{12}; \alpha_1, \alpha_2) = 0\,,$$



$$f_1 := f(n_1+1, n_2)\,, \quad f_{12} := f(n_1+1, n_2+1)\,, e.t.c$$

$$(f_{12} - f)(f_1 - f_2) = \alpha_1 - \alpha_2\,,$$

$$a_1(f\,f_1 + f_2\,f_{12}) - a_2(f\,f_2 + f_1\,f_{12}) = \delta({a_1}^2 - {a_2}^2)\,,$$

$$\frac{f_1 - \alpha_1 f}{f_2 - \alpha_2 f}\,\frac{f_2 - \alpha_1 f_{12}}{f_1 - \alpha_2 f_{12}} = \frac{1 - {\alpha_1}^2}{1 - {\alpha_2}^2}\,.$$

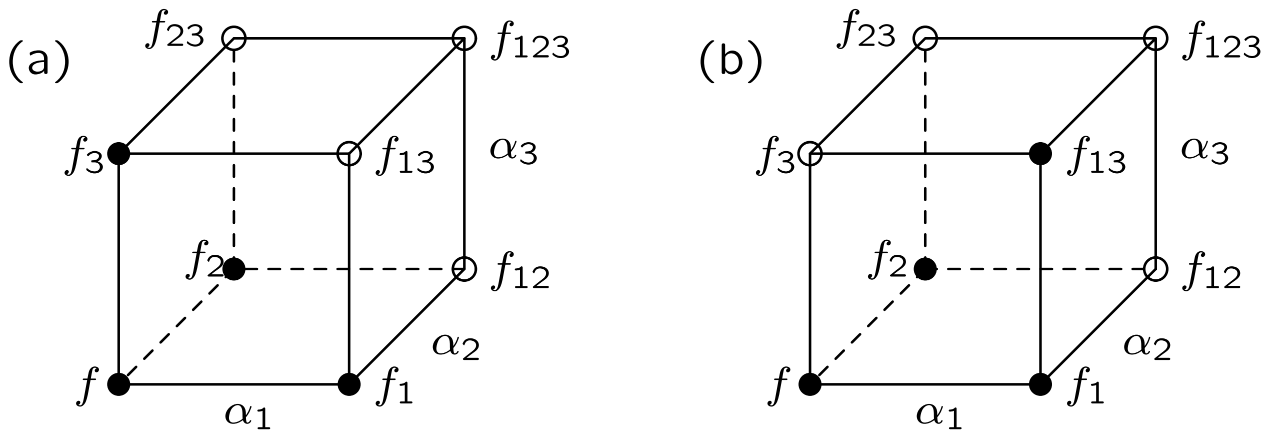
The consistency property

The overdetermined system of equations

$$\mathcal{E}(f, f_i, f_j, f_{ij}; a_i, a_j) = 0\,, \qquad 1 \leq i < j \leq n\,,$$

possesses a non empty set of solutions.

The property can be verified by considering an elementary initial value problem on the cube.



Example The discrete potential Korteweg-deVries (KdV)

$$(f_{12} - f)(f_1 - f_2) = \alpha_1 - \alpha_2\,,$$

with given initial data configuration as in figure (a). For this particular equation one finds that the value f_{123} is

$$f_{123} = \frac{(\alpha_1 - \alpha_2)f_1\,f_2 + (\alpha_3 - \alpha_1)f_1\,f_3 + (\alpha_2 - \alpha_3)f_2\,f_3}{(\alpha_2 - \alpha_1)f_3 + (\alpha_1 - \alpha_3)f_2 + (\alpha_3 - \alpha_2)f_1}\,.$$

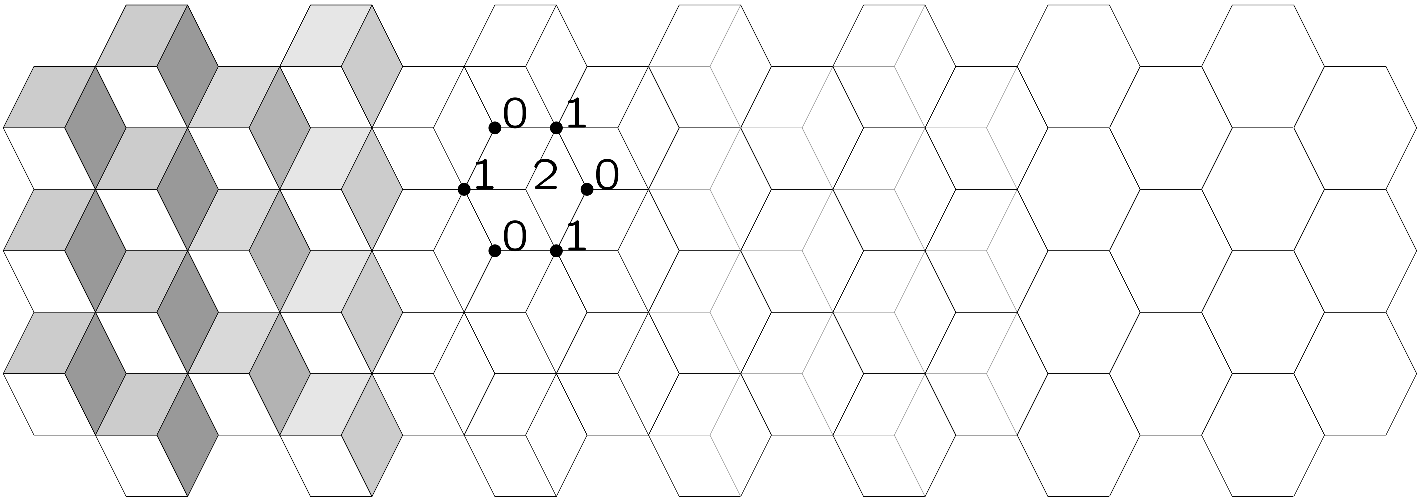
(Walquist-Estabrook formula)

From 3-d slices to non quadrilateral graphs

dpKdV on the ‘slice’ of the cubic lattice defined by

$$\left\{(n_1, n_2, n_3) \in \mathbb{Z}^3 \,, n_1 + n_2 + n_3 \in \{0\,,1\,,2\}\right\}$$

vertices are at levels 0 , 1 , 2



a version of the *dpKdV* equation on the hexagonal lattice (also considered using a zero-curvature approach).

The constant form of the Quantum Yang-Baxter Equation

It is a relation for a linear map

$$R : V \otimes V \to V \otimes V$$

where V is a vector space over a field k.

The Quantum Yang-Baxter equation (QYBE) is

$$R_{(1,2)}\, R_{(1,3)}\, R_{(2,3)} = R_{(2,3)}\, R_{(1,3)}\, R_{(1,2)}$$

in $\mathrm{End}(V \otimes V \otimes V)$

The maps $R_{(i,j)} \in \mathrm{End}(V \otimes V \otimes V)$, $1 \leq i < j \leq 3$ are defined as follows

$$R_{(1,2)} = R \otimes 1_V, \qquad R_{(2,3)} = 1_V \otimes R$$

$$R_{(1,3)} = (1_V \otimes T_{V,V})(R \otimes 1_V)(1_V \otimes T_{V,V})$$

where $T_{V,V} : V \otimes V \to V \otimes V$ is the twist map

$$T_{V,V}(m \otimes n) = n \otimes m$$

QYBE in coordinates

$B = \{m_1, \dots m_N\}$ a basis of V.

$$R(m_i \otimes m_j) = R_{i,j}^{k,l}\, m_k \otimes m_l$$

$$\boxed{R_{j,k}^{s_2,s_3}\, R_{i,s_3}^{s_1,c}\, R_{s_1,s_2}^{a,b} = R_{i,j}^{r_1,r_2}\, R_{r_1,k}^{a,r_3}\, R_{r_2,r_3}^{b,c}}$$

It is a coupled system of N^6 cubic polynomial equations in N^4 unknowns.

The one parameter form of QYBE

$$\begin{aligned} R_{(1,2)}(\alpha)\, R_{(1,3)}(\phi(\alpha,\beta))\, R_{(2,3)}(\beta) &= \\ R_{(2,3)}(\beta)\, R_{(1,3)}(\phi(\alpha,\beta))\, R_{(1,2)}(\alpha) & \end{aligned}$$

$\alpha, \beta \in S$, where S is a set.

The two parameter form of QYBE

$$\begin{aligned} R_{(1,2)}(\alpha_1,\alpha_2)\, R_{(1,3)}(\alpha_1,\alpha_3)\, R_{(2,3)}(\alpha_2,\alpha_3) &= \\ R_{(2,3)}(\alpha_2,\alpha_3)\, R_{(1,3)}(\alpha_1,\alpha_3)\, R_{(1,2)}(\alpha_1,\alpha_2) & \end{aligned}$$

$\alpha_i \in S$ where S is a set.

Set theoretic solutions of QYBE (YB maps)

$\mathbb{X} \rightarrow$ complex affine algebraic variety

$R \rightarrow$ a birational isomorphism of $\mathbb{X} \times \mathbb{X}$ into itself.

For any $x, y \in \mathbb{X}$

$$R(x,y) = \big(f(x,y), g(x,y)\big).$$

$R_{(i,j)} : \mathbb{X}^n \rightarrow \mathbb{X}^n \qquad n \geq 2$ and $1 \leq i, j \leq n$, $i \neq j$

$R_{(i,j)}(x^1, x^2, \ldots x^n) =$

$$\begin{cases} (x^1, \ldots, x^{i-1}, f(x^i, x^j), x^{i+1}, \ldots, x^{j-1}, g(x^i, x^j), x^{j+1}, \ldots x^n) & i < j, \\ (x^1, \ldots, x^{j-1}, g(x^i, x^j), x^{j+1}, \ldots, x^{i-1}, f(x^i, x^j), x^{i+1}, \ldots x^n) & i > j \end{cases}$$

E.g.

$$R_{(1,2)} = R$$

$$R_{(2,1)}(x,y) = \big(g(y,x), f(y,x)\big) = T\,R\,T$$

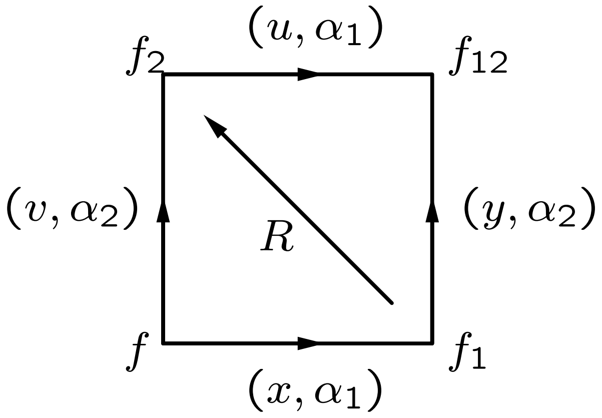
- *A map R is called a* YB map *if it satisfies*

$$R_{(1,2)}\,R_{(1,3)}\,R_{(2,3)} = R_{(2,3)}\,R_{(1,3)}\,R_{(1,2)}$$

regarded as an equality of maps in $\mathbb{X} \times \mathbb{X} \times \mathbb{X}$.

From discrete KdV to YB maps

$$(f_{12} - f)(f_1 - f_2) = \alpha_1 - \alpha_2\,,$$



Variables assigned to the edges:

$$x = f_1 - f, \quad y = f_{12} - f_1, \quad u = f_{12} - f_2, \quad v = f_2 - f,$$

- $(x + y)(x - v) = \alpha_1 - \alpha_2$
- $x + y = u + v$

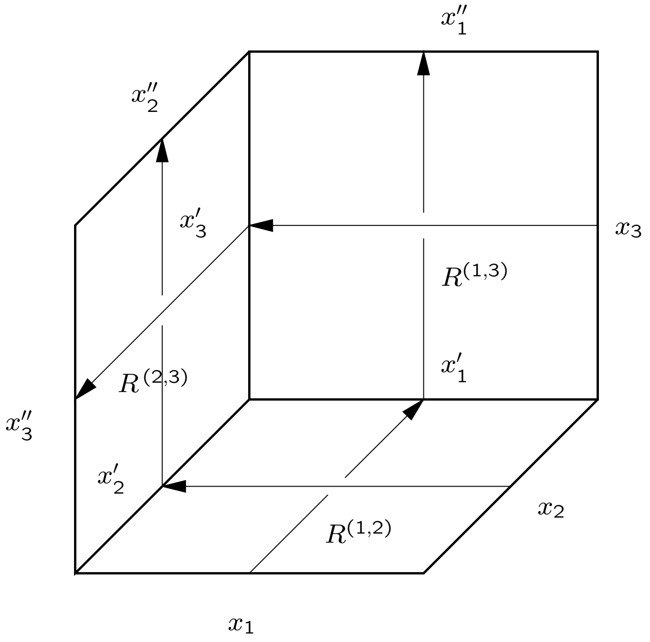
YB map $R(x, y) = (u, v)$

$$\boxed{u = y + \frac{\alpha_1 - \alpha_2}{x + y}, \qquad v = x - \frac{\alpha_1 - \alpha_2}{x + y}}$$

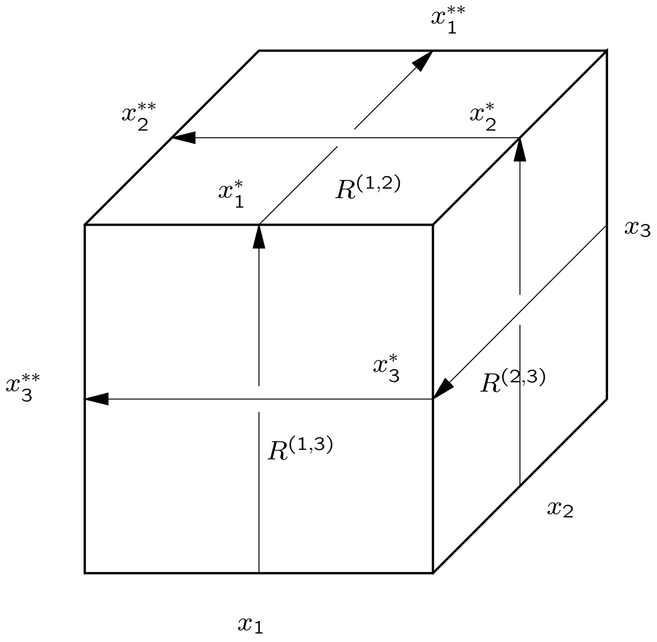
The consistency property and the QYBE

$$\text{(a)}:(x_1,x_2,x_3)\ \xrightarrow{R^{(1,2)}}\ (x_1',x_2',x_3)\ \xrightarrow{R^{(1,3)}}\ (x_1'',x_2',x_3')\ \xrightarrow{R^{(2,3)}}\ (x_1'',x_2'',x_3'') \tag{1}$$

$$\text{(b)}:(x_1,x_2,x_3)\ \xrightarrow{R^{(2,3)}}\ (x_1,x_2^*,x_3^*)\ \xrightarrow{R^{(1,3)}}\ (x_1^*,x_2^*,x_3^{**})\ \xrightarrow{R^{(1,2)}}\ (x_1^{**},x_2^{**},x_3^{**})$$

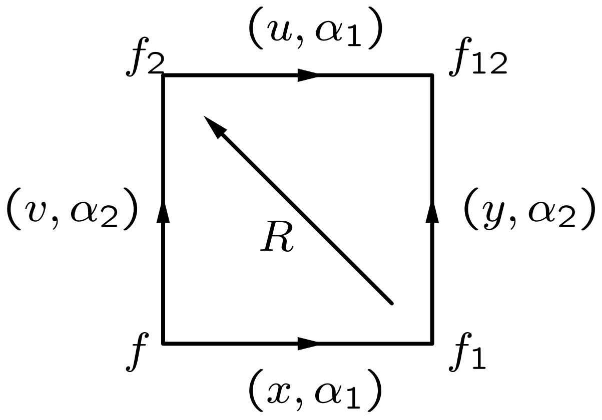


(a)



(b)

Method for constructing YB maps from consistent discrete equations



$$\mathcal{E}(f, f_1, f_2, f_{12}; \alpha_1, \alpha_2) = 0\,,$$

$$\begin{aligned} x &= G(f, f_1; a_1), \quad & y &= G(f_1, f_{12}; a_2), \\ u &= G(f_2, f_{12}; a_1), \quad & v &= G(f, f_2; a_2), \end{aligned}$$

- Find the geometrical (Lie point) symmetries of lattice equation

- Construct the corresponding symmetry invariants (x, y, u, v) (YB variables) assigned on the edges

- There is a functional relation between them since

$$\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}u \wedge \mathrm{d}v = 0$$

- Write lattice equation in terms of the edge invariants.

- Solve the resulting system for u, v in terms of x, y

- An orientation of the quadrilateral edges to those of the 3-cube, adapted to the Yang-Baxter relation.

The Harrison map

$$\frac{f_1-\alpha_1 f}{f_2-\alpha_2 f}\,\frac{f_2-\alpha_1 f_{12}}{f_1-\alpha_2 f_{12}}=\frac{1-{\alpha_1}^2}{1-{\alpha_2}^2}$$

Scaling invariance

$$f\mapsto e^{\varepsilon} f$$

YB variables by using the scaling symmetry

$$x=\frac{f_1}{\alpha_1\, f},\quad y=\frac{f_{12}}{\alpha_2\, f_1}\quad u=\frac{f_{12}}{\alpha_1 f_2}\quad v=\frac{f_2}{\alpha_2 f}$$

Functional dependence

$$x\,y=u\,v,$$

Lattice equation written in terms of the invariants

$$\frac{1-x^{-1}}{1-v^{-1}}=\frac{1-\gamma_1}{1-\gamma_2}\,\frac{1-\gamma_2\,y}{1-\gamma_1\,u}$$

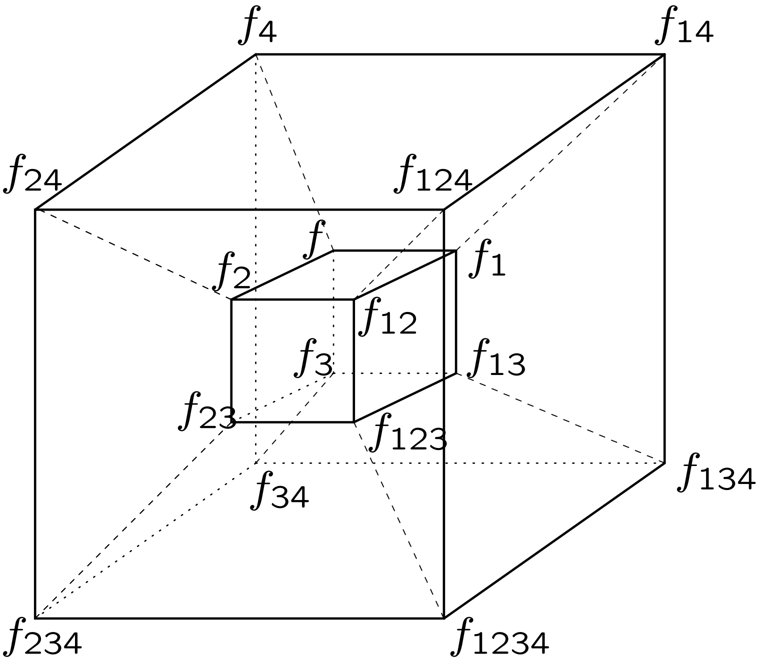
where $\gamma_i={\alpha_i}^2$

YB map

$$\boxed{u=y\,Q,\qquad v=x\,Q^{-1}}$$

$$Q=\frac{(1-\gamma_2)+(\gamma_2-\gamma_1)\,x+\gamma_2(\gamma_1-1)\,x\,y}{(1-\gamma_1)+(\gamma_1-\gamma_2)\,y+\gamma_1(\gamma_2-1)\,x\,y}$$

The Harrison map from KdV on a 4-cube



$$(f_{ij}-f)(f_i-f_j)=\alpha_i-\alpha_j \qquad 1\le i<j\le 4$$

$$x=\frac{f_1-f_3}{f_2-f_3}, \qquad v=\frac{f_1-f_4}{f_2-f_4},$$

$$u=x_4, \qquad y=v_3$$

$$\gamma_1=\frac{\alpha_2-\alpha_3}{\alpha_1-\alpha_3}, \qquad \gamma_2=\frac{\alpha_2-\alpha_4}{\alpha_1-\alpha_4}$$

$$\boxed{u=y\,Q, \qquad v=x\,Q^{-1}}$$

$$Q=\frac{(1-\gamma_2)+(\gamma_2-\gamma_1)\,x+\gamma_2(\gamma_1-1)\,x\,y}{(1-\gamma_1)+(\gamma_1-\gamma_2)\,y+\gamma_1(\gamma_2-1)\,x\,y}$$

Q_4 - **the master equation**

$$Q(w,x,y,z;(a,A),(b,B))=0$$

where the polynomial Q is given by

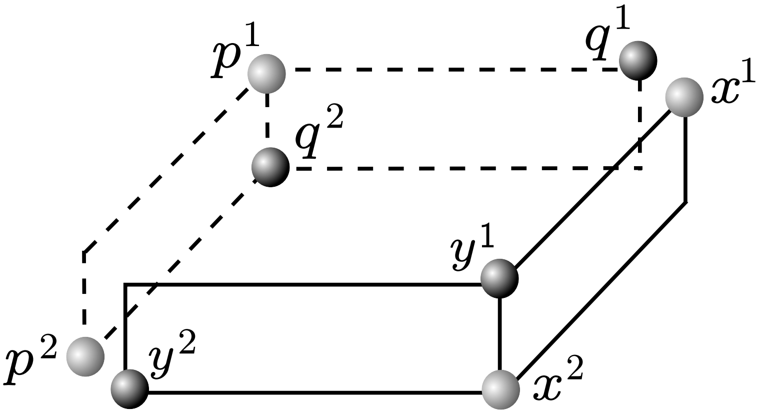
$Q(w,x,y,z;(a,A),(b,B)) =$
$a(wx+yz)-b(wy+xz)-\frac{aB-bA}{1-a^2b^2}\big(wz+xy-ab(1+wxyz)\big)$

The parameters $(a,A),(b,B)$ lay on the elliptic curve

$$\mathcal{E}=\left\{(\chi,\mathcal{X})\in\mathbb{C}^2\,:\,\mathcal{X}^2=\chi^4+k\,\chi^2+1\right\}$$

where k is the modulus of $\mathcal{E}$.

Lifting Q_4 to a two-field Yang-Baxter map



$$p^1 = F(y^1, x^1, y^2; \boldsymbol{a}, \boldsymbol{b}) \qquad q^2 = F(x^2, x^1, y^2; \boldsymbol{a}, \boldsymbol{b})$$

$$F(x, y, z; \boldsymbol{a}, \boldsymbol{b}) = \frac{K(y, z; \boldsymbol{a}, \boldsymbol{b})x + L(y, z; \boldsymbol{a}, \boldsymbol{b})}{M(y, z; \boldsymbol{a}, \boldsymbol{b})x + N(y, z; \boldsymbol{a}, \boldsymbol{b})}$$

$$K(y, z; \boldsymbol{a}, \boldsymbol{b}) = (1 - a^2 b^2)(b\,z - a\,y)$$

$$N(y, z; \boldsymbol{a}, \boldsymbol{b}) = (1 - a^2 b^2)(a\,z - b\,y)$$

$$L(y, z; \boldsymbol{a}, \boldsymbol{b}) = (a\,B - b\,A)(y\,z - a\,b)$$

$$M(y, z; \boldsymbol{a}, \boldsymbol{b}) = (a\,B - b\,A)(a\,b\,y\,z - 1)$$

Dynamical YB maps

H, X non-empty sets

$$\phi : H \times X \to X$$

$$R\,(\lambda) : X \times X \to X \times X,\ \lambda \in H$$

$$\begin{aligned} & R_{23}(\lambda)R_{13}(\phi(\lambda, X^{(2)}))R_{12}(\lambda) \\ & \quad = R_{12}(\phi(\lambda, X^{(3)}))R_{13}(\lambda)R_{23}(\phi(\lambda, X^{(1)})) \end{aligned}$$

on $X \times X \times X$

$$R_{12}(\lambda)(u, v, w) := (R(\lambda)(u, v), w)$$

$$R_{12}(\phi(\lambda, X^{(3)})) := R_{12}(\phi(\lambda, X^{(3)}))(u, v, w)$$

$$R_{23}(\phi(\lambda, X^{(1)})) := (u, R_{12}(\phi(\lambda, u))(v, w))$$

Dynamical YB maps & Ternary Systems

$$(M, \mu)$$

$$\mu : M \times M \times M \to M$$

$$\mu(a, \mu(a, b, c), \mu(\mu(a, b, c), c, d)) = \mu(a, , b, \mu(b, c, d))$$

$$\mu(\mu(a, b, c), c, d) = \mu(\mu(a, b, \mu(b, c, d)), \mu(b, c, d), d)$$

Discrete potential Kadomtsev-Petviashvili equation (*dpKP*)

$$\frac{p_1 - p_3 + u_{23} - u_{12}}{p_1 - p_3 + u_3 - u_1} = \frac{p_2 - p_3 + u_{13} - u_{12}}{p_2 - p_3 + u_3 - u_2}$$

It represents the Bianchi identity for the partial differential equation of Kadomtsev and Petviashvili. We are going to use it in the form

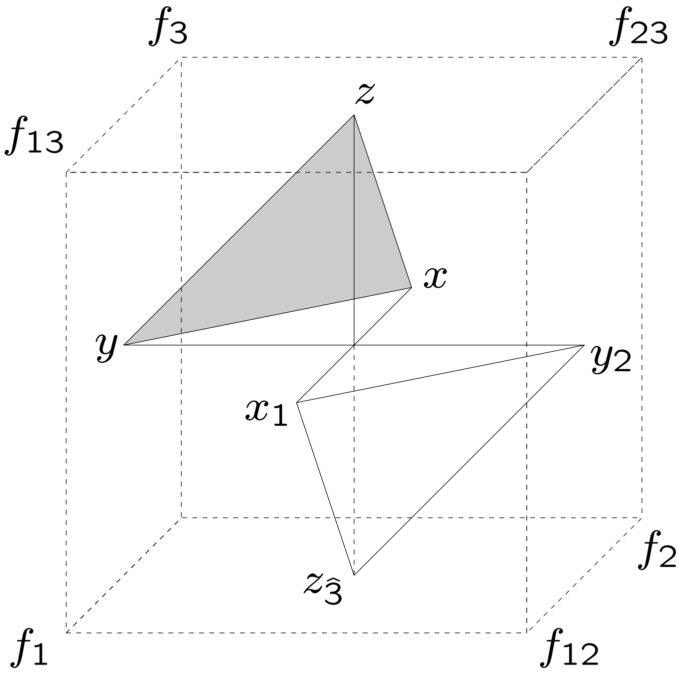
$$f_{12}(f_1 - f_2) + f_{23}(f_2 - f_3) + f_{13}(f_3 - f_1) = 0$$

which results by the substitution $u_{n_1,n_2,n_3} = f_{n_1,n_2,n_3} + \sum_{i=1}^{3} n_i p_i$.

dpKP is a concequence of the *dpKdV* on the $\mathbb{Z}^3$ lattice and it is concistent of on the hypercube in $\mathbb{Z}^4$.

In terms of reducing variables *dKP* is equivalent to coupled copies of a mapping on the kagome lattice. This mapping is related to solutions of the *functional tetrahedron equation*.

dpKP as coupled FT mappings



dpKP can be written in terms of the following variables

$$x = f_2 - f_3\,,\ y = f_3 - f_1\,,\ z = f_{23} - f_{13}$$

and their shifted versions

$$x_1 = f_{12} - f_{13}\,,\ y_2 = f_{23} - f_{12}\,,\ z_{\hat{3}} = f_2 - f_1$$

as follows

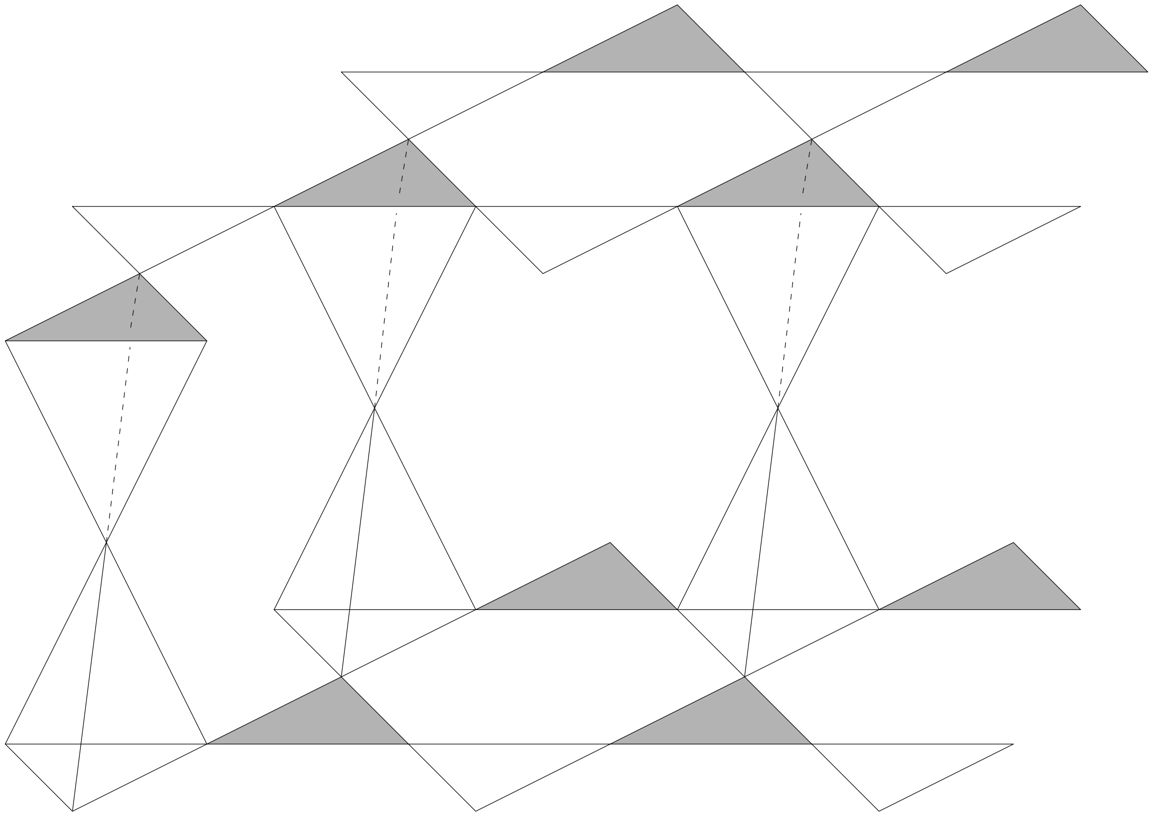
$$x_1 = \frac{xz}{x+y}\,,\ y_2 = \frac{yz}{x+y}\,,\ z_{\hat{3}} = x + y\ .$$

$$(x,y,z) \xrightarrow{R} (\frac{xz}{x+y}, \frac{yz}{x+y}, x+y)$$

"'maps" the three adjacent faces of the cube (whose centers form the 'black' triangle) to the three opposite ones (whose centers form the 'white' triangle). R up to conjugation with the transposition $(x,y,z) \longrightarrow (x,z,y)$ is a solution of the *Functional Tetrahedron equation*

$$R_{123}R_{124}R_{134}R_{234} = R_{234}R_{134}R_{124}R_{123}$$

Copies of this map are coupled on the kagome lattice in order to produce the evolution of an associated initial value problem for the *dKP* equation.



References

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