Galilean massless fields

A.G. Nikitin

Institute of Mathematics, National Academy of Sciences of Ukraine,
3 Tereshchenkivs'ka Street, Kyiv-4, Ukraine, 01601

Galilei-invariant equations for massless vector fields are obtained with using of two approaches. First, we obtain them via contractions of relativistic wave equations. Secondly, we deduce such equations directly, starting with completed list of finite dimensional indecomposable representations of the homogeneous Galilei group for scalar and vector fields. It is shown that the collection of non-equivalent Galilei-invariant wave equations for massless fields with spin equal 1 and 0 is very reach. It describes many physically consistent systems, e.g., those of electromagnetic fields in various media or Galilean Chern-Simon models. Finally, classification of all linear and a big group of non-linear Galilei-invariant equations for massless fields is presented.

In physics there are specific symmetries and fundamental ones. The most important examples of fundamental symmetries are relativistic invariance and Galilei invariance. It is the invariance w.r.t. Lorentz transformations or Galilei ones which is *a priori* required in a consistent physical theory.

Relativistic invariance is treated as a more fundamental one, since Galilei-invariant theories can be obtained as limiting case of relativistic ones. But there are reasons to study just Galilei-invariant theories:

- The majority of physical effects are non-relativistic (i.e., are characterized by velocities much smaller than the velocity of light). In fact, we newer observe a macroscopic body whose velocity is compatible with the velocity of light;
- Non-relativistic models in principle are more simple and convenient than the relativistic ones;
- A correct definition of non-relativistic limit is by no means a simple problem, in general, and in the case of massless fields in particular;
- The very existence of a good non-relativistic approximation can serve as a selection rule for consistent relativistic theories.

To create group-theoretical grounds — to describe representations of the Poincaré and Galilei groups.

Relativistic theories in principle are more complicated then non-relativistic ones. On the other hand, the structure of subgroups of the Galilei group and of its representations are in many respects more complex than those of the Poincaré group and therefore it is perhaps not so surprising that the representations of the Poincaré group were described by Wigner in 1939, almost 15 years earlier than the representations of the Galilei group (Bargman, 1954) in spite of the fact that the relativity principle of classical physics was formulated by Galilei about three centuries prior to that of relativistic physics formulated by Einstein.

It appears that, as opposed to the Poincaré group, the Galilei group has the ordinary as well as the projective representations. Moreover, finite-dimensional indecomposable representations of the homogeneous Galilei group $HG(1, 3)$ are not classifiable. And they are the representations which play a key role in formulation of physical models satisfying the Galilei relativity principle!

1 Galilei group

The group of transformations in $R_3 \oplus R_1$:

$$\begin{aligned} &t \to t' = t + a, \\ &\mathbf{x} \to \mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{v}t + \mathbf{b}, \end{aligned} \tag{1}$$

where $a, \mathbf{b}$ and $\mathbf{v}$ real, $\mathbf{R}$ rotation matrix.

$\mathbf{HG(1,3)}$ a subgroup of $G(1,3)$ leaving invariant $\mathbf{x} = (0,0,0)$ at $t = 0$ and formed by:

$$\begin{aligned} &t \to t' = t, \\ &\mathbf{x} \to \mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{v}t \end{aligned} \tag{2}$$

Lie algebra $\mathbf{hg(1,3)}$ includes six basis elements: 3 rotation generators $S_a, a = 1,2,3$ and three generators of Galilean boosts η_a, with the commutation relations

$$[S_a, S_b] = \mathrm{i}\varepsilon_{abc} S_c, \tag{3}$$

$$[\eta_a, S_b] = \mathrm{i}\varepsilon_{abc} \eta_c, \ \ [\eta_a, \eta_b] = 0 \tag{4}$$

that is, they form a basis of the Lie algebra $hg(1,3)$ of the homogeneous Galilei group.

2 Vector representations.

All indecomposable representations of $HG(1,3)$ which, when restricted to the rotation subgroup, are decomposed to direct sums of vector and scalar representations, were found in [3]. These indecomposable representations (denoted as $D(m,n,\lambda)$) are labeled by triplets of numbers: n, m and λ. These numbers take the values

$$-1 \leq (n-m) \leq 2, \quad n \leq 3,$$

$$\lambda = \begin{cases} 0 \texttt{ if } m = 0, \\ 1 \texttt{ if } m = 2 \texttt{ or } n - m = 2, \\ 0, 1 \texttt{ if } m = 1, n \neq 3. \end{cases} \tag{5}$$

In accordance with (5) there exist ten non-equivalent indecomposable representations $D(m,n,\lambda)$. Their carrier spaces can include three types of rotational scalars

$$A, B, C$$

and five types of vectors

$$\mathbf{R}, \mathbf{U}, \mathbf{W}, \mathbf{K}, \mathbf{N}$$

whose transformation laws with respect to the Galilei boost are:

$$
\begin{aligned}
&A \to A' = A, \\
&B \to B' = B + \mathbf{v}\cdot\mathbf{R}, \\
&C \to C' = C + \mathbf{v}\cdot\mathbf{U} + \tfrac{1}{2}\mathbf{v}^2 A, \\
&\mathbf{R} \to \mathbf{R}' = \mathbf{R}, \\
&\mathbf{U} \to \mathbf{U}' = \mathbf{U} + \mathbf{v}A, \\
&\mathbf{W} \to \mathbf{W}' = \mathbf{W} + \mathbf{v}\times\mathbf{R}, \\
&\mathbf{K} \to \mathbf{K}' = \mathbf{K} + \mathbf{v}\times\mathbf{R} + \mathbf{v}A, \\
&\mathbf{N} \to \mathbf{N}' = \mathbf{N} + \mathbf{v}\times\mathbf{W} + \mathbf{v}B + \mathbf{v}(\mathbf{v}\cdot\mathbf{R}) - \tfrac{1}{2}\mathbf{v}^2\mathbf{R},
\end{aligned} \tag{6}
$$

where $\mathbf{v}$ is a vector whose components are parameters of the considered Galilei boosts, $\mathbf{v}\cdot\mathbf{R}$ and $\mathbf{v}\times\mathbf{R}$ are scalar and vector products of vectors $\mathbf{v}$ and $\mathbf{R}$ respectively.

Carrier spaces of these indecomposable representations of the group $HG(1,3)$ include such sets of scalars A, B, C and vectors $\mathbf{R}, \mathbf{U}, \mathbf{W}, \mathbf{K}, \mathbf{N}$ which transform among themselves w.r.t. transformations (6) but cannot be split to a direct sum of invariant subspaces. There exist exactly ten such sets:

$$\begin{array}{rcl}
\{A\} & \Longleftrightarrow & D(0,1,0), \\
\{\mathbf{R}\} & \Longleftrightarrow & D(1,0,0), \\
\{B,\mathbf{R}\} & \Longleftrightarrow & D(1,1,0), \\
\{A,\mathbf{U}\} & \Longleftrightarrow & D(1,1,1), \\
\{A,\mathbf{U},C\} & \Longleftrightarrow & D(1,2,1), \\
\{\mathbf{W},\mathbf{R}\} & \Longleftrightarrow & D(2,0,0), \\
\{\mathbf{R},\mathbf{W},B\} & \Longleftrightarrow & D(2,1,0), \\
\{A,\mathbf{K},\mathbf{R}\} & \Longleftrightarrow & D(2,1,1), \\
\{A,B,\mathbf{K},\mathbf{R}\} & \Longleftrightarrow & D(2,2,1), \\
\{B,\mathbf{N},\mathbf{W},\mathbf{R}\} & \Longleftrightarrow & D(3,1,1).
\end{array} \tag{7}$$

Thus, in contrary to the relativistic case, where are only three Lorentz covariant quantities which transform as vectors or scalars under rotations (i.e., as a relativistic four-vector, antisymmetric tensor of the second order and a scalar), there are ten indecomposable sets of the Galilei vectors and scalars which we have enumerated in equation (7). The corresponding vectors of carrier spaces can be one-, three-, four-, five-, six-, eight- and ten-dimensional.

3 Contractions of representations of the Lorentz algebra

It is well known that the Galilei algebra can be obtained from the Poincaré one by a limiting procedure called "the Inönü-Wigner contraction" [9]. In the simplest case a *contraction* is a limit procedure which transforms an N-dimensional Lie algebra $\mathcal{L}$ into an non-isomorphic Lie algebra $\mathcal{L}'$, also with N dimensions. The commutation relations of a *contracted Lie algebra* $\mathcal{L}'$ are given by:

$$[x, y]' \equiv \lim_{\varepsilon \to \varepsilon_0} W_\varepsilon^{-1}([W_\varepsilon(x), W_\varepsilon(y)], \qquad (8)$$

where $W_\varepsilon \in \mathrm{GL}(N, C)$ is a non-singular linear transformation of $\mathcal{L}$, with ε_0 being a singularity point of its inverse W_ε^{-1}.

Representations of these algebras can also be connected by this kind of contraction. However, this connection is more complicated for two reasons:

- First, contraction of a non-trivial representation of the Lorentz algebra yields to the representation of the homogeneous Galilei algebra in which generators of the Galilei boosts are represented trivially, so that to obtain a non-trivial representation it is necessary to apply in addition a similarity transformation which depends on a contraction parameter in a tricky way.

- Second, to obtain indecomposable representations of $hg(1,3)$, it is necessary, in general, to start with completely reducible representations of the Lie algebra of the Lorentz group.

In papers by Niederle & N (2006) and by de Montigny, Niederle & N (2006) representations of the Lorentz group which can be contracted to representations $D(m,n,\lambda)$ of the Galilei group were found and the related contractions specified. Here we present only two examples of such contractions.

To obtain the five-dimensional *indecomposable* representation $D(1,2,1)$ we have to start with a *direct sum* of the representations $D(\frac{1}{2},\frac{1}{2})$ and $D(0,0)$ of the Lorentz group. The corresponding generators of the algebra $so(1,3)$ have the form

$$\hat{S}_{\mu\nu} = \begin{pmatrix} S_{\mu\nu} & \cdot \\ \cdot & 0 \end{pmatrix} \tag{9}$$

where $\hat{S}_{\mu\nu}$ are skew symmetric matrices whose non-zero elements lies at μ-th row and ν-th column (and ν-th row and μ-th column).

The Inönü-Wigner contraction consists of the transformation to a new basis

$$S_{ab} \rightarrow S_{ab},\ S_{0a} \rightarrow \varepsilon S_{0a}$$

followed by a similarity transformation of all basis elements $S_{\mu\nu} \rightarrow S'_{\mu\nu} = VS_{\mu\nu}V^{-1}$ with a matrix V depending on a contraction parameter ε. Moreover, V depends on ε in such a way that all transformed generators S'_{ab} and $\varepsilon S'_{0a}$ are kept non-trivial and non-singular when $\varepsilon \rightarrow 0$ [9].

The matrix of the corresponding similarity transformation can be written as:

$$V_1 = \begin{pmatrix} I_{3\times 3} & 0_{3\times 1} & 0_{3\times 1} \\ 0_{1\times 3} & \frac{1}{2}\varepsilon & \frac{1}{2}\varepsilon \\ 0_{1\times 3} & -\varepsilon^{-1} & \varepsilon^{-1} \end{pmatrix}. \qquad (10)$$

As a result we obtain the following basis elements of representation $D(1,2,1)$ of the algebra $hg(1,3)$:

$$S_a = \begin{pmatrix} s_a & \mathbf{0}_{1\times 3} & \mathbf{0}_{1\times 3} \\ \mathbf{0}_{3\times 1} & 0 & 0 \\ \mathbf{0}_{3\times 1} & 0 & 0 \end{pmatrix}, \eta_a = \begin{pmatrix} \mathbf{0}_{3\times 3} & k_a^{\dagger} & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 0 & 0 \\ k_a & 0 & 0 \end{pmatrix} \qquad (11)$$

were s_a are matrices of spin one with the elements $(s_a)_{bc} = i\varepsilon_{abc}$, ε_{abc} is a skew symmetric unit tensor and k_a are 1×3 matrices of the form

$$k_1 = (\mathrm{i}, 0, 0)\,, \qquad k_2 = (0, \mathrm{i}, 0)\,, \qquad k_3 = (0, 0, \mathrm{i})\,.$$

$$(12)$$

Thus starting with a *direct sum* of representations of $so(1,4)$ we contract it to the *indecomposable* representation of $hg(1,3)$.

Considering the representation $D(1,0) \oplus D(0,1)$ of the Lorentz group whose generators are 6×6 matrices

$$S_{ab} = \varepsilon_{abc} \begin{pmatrix} s_c & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & s_c \end{pmatrix}, \; S_{0a} = \begin{pmatrix} \mathbf{0}_{3\times 3} & -s_a \\ s_a & \mathbf{0}_{3\times 3} \end{pmatrix}, \tag{13}$$

we can choose the corresponding contraction matrix in one of the following forms:

$$\begin{array}{l} V_2 = \begin{pmatrix} \varepsilon I_{3\times 3} & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & I_{3\times 3} \end{pmatrix} \\ \quad \texttt{or} \\ V_3 = \begin{pmatrix} I_{3\times 3} & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \varepsilon I_{3\times 3} \end{pmatrix}. \end{array} \tag{14}$$

As a result of such contractions we obtain the representation $D(2,0,0)$ of $hg(1,3)$.

4 Galilean massless fields

For constructions of Galilei invariant equations for massless fields it is possible to use at least three approaches:

- To start with equations invariant w.r.t. group $P(1,4)$, i.e., the Poincaré group in (1+4)-dimensional space. This group includes the Galilei group $HG(1,3)$ as a subgroup and so making reduction $P(1,4) \rightarrow HG(1,3)$ we obtain Galilei-invariant equations.
- To start with equations invariant w.r.t. the Poincar'e group $P(1,3)$ and to make a contraction to the Galilei group.
- To use our knowledge of indecomposable representations of $HG(1,3)$ and deduce Galilei-invariant equations using tools of Lie analysis, i.e., calculating absolute and relative differential invariants of an appropriate order.

The first and second approaches were used by many authors. However, in this way it is possible to obtain particular results only. The third approach is the most powerful but it can be used provided we know representations of $HG(1,3)$. Since now we know indecomposable representations for vector fields, it is possible to describe all Galilei-invariant equations for such fields.

J. Niederle and me had found a completed lists of relative functional and relative first order differential invariants. We also classify possible Galilei-invariant equations which can be obtained from known relativistic models using the contraction procedure. I will not torture you by demonstration of the corresponding classification tables but restrict myself to showing some examples which are seem to be interesting.

4.1 Galilei limits of the Maxwell equations

According to the Lévy-Leblond and Le Bellac analysis from 1967 there are two Galilean limits of the Maxwell equations.

In the so-called "magnetic" Galilean limit we receive pre-Maxwellian electromagnetism. The corresponding equations for magnetic field **H** and electric field **E** read

$$\begin{array}{ll} \nabla \times \mathbf{E}_m - \frac{\partial \mathbf{H}_m}{\partial t} = 0, & \nabla \cdot \mathbf{E}_m = e j^0_m, \\ \nabla \times \mathbf{H}_m = e \mathbf{j}_m, & \nabla \cdot \mathbf{H}_m = 0, \end{array} \qquad (15)$$

where $j = (j^0_m, \mathbf{j}_m)$ is an electric current and e denotes an electric charge.

Equations (15) are invariant with respect to the Galilei transformations (1) provided vectors $\mathbf{H}_m$, $\mathbf{E}_m$ and electric current j cotransform as

$$\begin{array}{l} \mathbf{H}_m \to \mathbf{H}_m, \ \mathbf{E}_m \to \mathbf{E}_m - \mathbf{v} \times \mathbf{H}_m, \\ \mathbf{j}_m \to \mathbf{j}_m, \quad j_m^0 \to j_m^0 + \mathbf{v} \cdot \mathbf{j}_m. \end{array} \tag{16}$$

Introducing a Galilean vector-potential $A_m = (A^0, \mathbf{A})$ such that

$$\mathbf{H}_m = \nabla \times \mathbf{A}, \quad \mathbf{E}_m = -\frac{\partial \mathbf{A}}{\partial t} - \nabla A^0 \tag{17}$$

we conclude that transformation laws for A have the following form:

$$A^0 \to A^0 + \mathbf{v} \cdot \mathbf{A}, \ \mathbf{A} \to \mathbf{A}. \tag{18}$$

The other Galilean limit of the Maxwell equations, i.e., the "electric" one looks as

$$\begin{array}{ll} \nabla \times \mathbf{H}_e + \frac{\partial \mathbf{E}_e}{\partial t} = e\mathbf{j}_e, & \nabla \cdot \mathbf{E}_e = e j_e^4, \\ \nabla \times \mathbf{E}_e = 0, & \nabla \cdot \mathbf{H}_e = 0, \end{array} \tag{19}$$

with the Galilean transformation lows of the following form

$$\begin{array}{ll} \mathbf{H}_e \to \mathbf{H}_e + \mathbf{v} \times \mathbf{E}_e, & \mathbf{E}_e \to \mathbf{E}_e, \\ \mathbf{j}_e \to \mathbf{j}_e + \mathbf{v} j_e^4, & j_e^4 \to j_e^4. \end{array} \tag{20}$$

Vectors $\mathbf{H}_e$ and $\mathbf{E}_e$ can be expressed via vector- potentials as

$$\mathbf{H}_e = \nabla \times \mathbf{A}, \quad \mathbf{E}_e = -\nabla A^4 \tag{21}$$

with the corresponding Galilei transformations for the vector-potential:

$$A^4 \to A^4, \;\; \mathbf{A} \to \mathbf{A} + \mathbf{v}A^4. \qquad (22)$$

The Galilean limits of the Maxwell equations admit clear interpretation in the representation theory. There are exactly two non-equivalent representations of the homogeneous Galilei group the carrier spaces of which are four-vectors – the representations $D(1,1,0)$ and $D(1,1,1)$. The first of them corresponds to the magnetic limit of Maxwell equations, the second one - to the electric limit.

4.2 Extended Galilei electromagnetism

Let us begin with relativistic equations for vector-potential A^μ

$$\Box A^\nu = ej^\nu \qquad (23)$$

in the Lorentz gauge, i.e., fulfilling

$$\partial_\mu A^\mu = 0 \;\; \texttt{or} \;\; \partial_0 A^0 = \nabla \cdot \mathbf{A}. \qquad (24)$$

Consider in addition the inhomogeneous d'Alembert equation for a relativistic scalar field denoted as A^4:

$$\Box A^4 = ej^4. \qquad (25)$$

Introducing the related vectors of the field strengthes in the standard form:

$$\mathbf{H} = \nabla \times \mathbf{A}, \quad \mathbf{E} = \frac{\partial \mathbf{A}}{\partial x_0} - \nabla A^0, \\ \mathbf{F} = \nabla A^4, \; F^0 = \frac{\partial A^4}{\partial x_0} \tag{26}$$

we get the Maxwell equations for **E** and **H**:

$$\nabla \times \mathbf{E} - \frac{\partial \mathbf{H}}{\partial x_0} = 0, \quad \nabla \cdot \mathbf{H} = 0, \\ \nabla \times \mathbf{H} + \frac{\partial \mathbf{E}}{\partial x_0} = e\mathbf{j}, \qquad \nabla \cdot \mathbf{E} = ej^0 \tag{27}$$

and the following equations for **F** and F^0

$$\frac{\partial F^0}{\partial x_0} + \nabla \cdot \mathbf{F} = ej^4, \\ \nabla \times \mathbf{F} = 0, \quad \frac{\partial \mathbf{F}}{\partial x_0} = \nabla F^0. \tag{28}$$

Clearly the system of equations (27) and (28) is completely decoupled. Its Galilean counterpart obtained using the Inönü-Wigner contraction appears to be, rather surprisingly, coupled.

Let us start with the system of equations (23)-(25) which describes a decoupled system of relativistic equations for the five-component function

$$A = \texttt{column}(A^1, A^2, A^3, A^0, A^4) \\ = \texttt{column}(\mathbf{A}, A^0, A^4). \tag{29}$$

Moreover, the components (A^1, A^2, A^3, A^0) transform as a four-vector and A^4 transforms as a scalar. The related generators (9) of the Lorentz group realize a direct sum of representations of the algebra $so(1,3)$, namely $D(\frac{1}{2}, \frac{1}{2}) \oplus D(0,0)$.

Making contraction $D(\frac{1}{2}, \frac{1}{2}) \oplus D(0,0) \to D(1,2,1)$ which was presented in the equations (9)-(11) it is possible to reduce decoupled relativistic system (23) and (25) to a system of the coupled equations invariant with respect to the Galilei group:

$$\Box A'^k = ej'^k, \quad \frac{\partial A'^4}{\partial t} = \nabla \cdot \mathbf{A}'. \tag{30}$$

Under the Galilei transformations components of the contracted potential cotransform in accordance with the representation $D(1,2,1)$, i.e.,

$$A^0 \to A^0 + \mathbf{v} \cdot \mathbf{A} + \frac{\mathbf{v}^2}{2} A^4, \;\; \mathbf{A} \to \mathbf{A} + \mathbf{v} A^4, \;\; A^4 \to A^4, \tag{31}$$

and

$$j^4 \to j^4, \;\; \mathbf{j} \to \mathbf{j} + \mathbf{v} j^4, \;\; j^0 \to j^0 + \mathbf{v} \cdot \mathbf{j} + \frac{1}{2} v^2 j^4. \tag{32}$$

Transformations (1), (31) and (32) keep the system (30) invariant. The corresponding field strengthes

$$\begin{aligned} &\mathbf{W} = \nabla \times \mathbf{A}', \quad \mathbf{N} = \frac{\partial \mathbf{A}'}{\partial t} - \nabla A'^0, \\ &\mathbf{R} = \nabla A'^4, \quad B = \frac{\partial A'^4}{\partial t} \end{aligned} \tag{33}$$

satisfy the following equations

$$\begin{aligned} &\mathcal{C} \equiv \nabla \cdot \mathbf{N} - \frac{\partial}{\partial t} B - e j^0 = 0, \\ &\boldsymbol{\mathcal{U}} \equiv \nabla \times \mathbf{W} + \nabla B - e\mathbf{j} = 0, \\ &\mathcal{A} \equiv \nabla \cdot \mathbf{R} - e j^4 = 0, \\ &\boldsymbol{\mathcal{N}} \equiv \frac{\partial}{\partial t}\mathbf{W} + \nabla \times \mathbf{N} = 0, \\ &\boldsymbol{\mathcal{W}} \equiv \frac{\partial}{\partial t}\mathbf{R} - \nabla B = 0, \\ &\boldsymbol{\mathcal{R}} \equiv -\nabla \times \mathbf{R} = 0, \\ &\mathcal{B} \equiv \nabla \cdot \mathbf{W} = 0. \end{aligned} \tag{34}$$

These equations are covariant with respect to the Galilei group. In the central part of them we have relative differential invariants which transform between themselves like Galilean vectors and scalars.

In contrast to a decoupled relativistic system of equations (27) and (28) its Galilei counterpart (34) appears to be a coupled system of equations for vectors $\mathbf{R}, \mathbf{W},\ \mathbf{N}$ and scalar B.

The system of equations (34) is the most extended decoupled system of the first order equations for scalar and vector fields, invariant w.r.t. the Galilei transformations. It includes 10 dependent variables while the

Galilean versions of the Maxwell equations includes 6 dependent variables.

4.3 Reduced Galilean electromagnetism

The considered vector representations of the Galilei group are indecomposable but reducible. It means that we can reduce the number of dependent components in (34 without violating its Galilei invariance. For example, vector $\mathbf{R}$ and the fourth component j^4 of the current form invariant subspaces with respect to the Galilei transformations. Thus we can impose the Galilei-invariant conditions

$$\mathbf{R} = 0 \ \ \texttt{or} \ \ \nabla A^4 = 0, \quad j^4 = 0 \tag{35}$$

and reduce system (34) to the following one

$$\begin{array}{l} \frac{\partial}{\partial t}\tilde{\mathbf{H}} + \nabla \times \tilde{\mathbf{E}} = 0, \\ \nabla \times \tilde{\mathbf{H}} = e\mathbf{j}, \ \nabla \cdot \tilde{\mathbf{H}} = 0, \\ \nabla \cdot \tilde{\mathbf{E}} = \frac{\partial}{\partial t}S + ej^0, \\ \nabla S = 0, \end{array} \tag{36}$$

where we have used notation $\tilde{\mathbf{H}} = \mathbf{W}|_{\mathbf{R}\equiv 0}$, $\tilde{\mathbf{E}} = \mathbf{N}|_{\mathbf{R}\equiv 0}$ and $S = B|_{\mathbf{R}\equiv 0}$.

Vectors $\tilde{\mathbf{E}}$, $\tilde{\mathbf{H}}$ and scalar S belong to a carrier space of the representation $D(2,1,1)$. Their Galilei transformation laws are

$$\tilde{\mathbf{E}} \rightarrow \tilde{\mathbf{E}} + \mathbf{v} \times \tilde{\mathbf{H}} + \mathbf{v}S, \quad \tilde{\mathbf{H}} \rightarrow \tilde{\mathbf{H}}, \ S \rightarrow S. \quad (37)$$

In accordance with (37) S belongs to an invariant subspace of the Galilei transformations, so we can impose the following additional Galilei-invariant condition

$$S = 0 \ \ \texttt{or} \ \ \frac{\partial A^4}{\partial t} = 0. \quad (38)$$

As a result we come to equations (15), i.e., to the magnetic limit of Maxwell's equations.

In analogous way it is possible to find other 6 versions of the reduced equations for Galilean vector fields, among them we find the magnetostatics equations:

$$\nabla \times \hat{\mathbf{H}} = e\mathbf{j}, \quad \nabla \cdot \hat{\mathbf{H}} = 0. \quad (39)$$

We see that, in contrast to a relativistic theory, there exist a big variety of linear equations for massless vector fields invariant with respect to the Galilei group. We found a complete list of them.

5 Nonlinear equations for vector fields

Starting with indecomposable vector representations of the group $HG(1,3)$ it is possible to find out various classes of partial differential equations invariant w.r.t. the Galilei group. In the above we have restricted ourselves to linear Galilean equations for vector and scalar fields and now we shall present nonlinear equations.

5.1 Galilei electromagnetic field in medium

Let us consider the Maxwell equations for electromagnetic field in a medium

$$\frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H}, \quad \nabla \cdot \mathbf{D} = 0,$$
$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \quad \nabla \cdot \mathbf{B} = 0. \tag{40}$$

Here $\mathbf{E}$ and $\mathbf{H}$ are vectors of electric and magnetic field strengths and $\mathbf{D}$ and $\mathbf{B}$ denote the corresponding vectors of electric and magnetic inductions. The system (40) is underdetermined and has to be completed by constitutive equations which represent the medium properties. The simplest constitutive equations correspond to a case where $\mathbf{B}$ and $\mathbf{D}$ are proportional to **H** and **E** respectively, i.e.,

$$\mathbf{B} = \mu\mathbf{H} \ \texttt{and} \quad \mathbf{D} = \kappa\mathbf{E}, \tag{41}$$

Here μ and ε are constants.

In general μ and ε can be functions of **E** and **H** so that the related theories are essentially nonlinear. There are even more complex constitutive equations, e.g.,

$$\mathbf{B} = \mu\mathbf{H} + \nu\mathbf{E}, \quad \mathbf{D} = \kappa\mathbf{E} + \lambda\mathbf{H}, \tag{42}$$

where μ, ν, κ and λ are some functions of **H** and **E**.

Let us note that system (40) by itself, i.e., without constitutive equations, is invariant with respect to a very extended group $IGL(4, R)$ which includes both the Poincaré and the Galilei groups as subgroups. And just constitutive equations, e.g., (41) or (42), reduce this group to the Poincaré group.

Since we are studying Galilean aspects of electrodynamics, it is naturally to pose a problem, weather there exist such constitutive equations which reduce the symmetry of system (40) to the Galilei group.

For this purpose we shall search for Galilei-invariant constitutive equations in the form (42). Such equations are Galilei-invariant provided μ, ν, κ and λ are invariants of Galilei transformations and, in addition,

$$\sigma\kappa = \nu, \quad \mu = \sigma\lambda, \tag{43}$$

where σ is an invariant of the Galilei group.

The Galilei transformations of vectors $\mathbf{H}$, $\mathbf{E}$, $\mathbf{D}$ and $\mathbf{B}$, which keep equations (40) invariant, have the form:

$$\mathbf{E} \to \mathbf{E}+\mathbf{v}\times\mathbf{B},\ \mathbf{H} \to \mathbf{H}-\mathbf{v}\times\mathbf{D},\ \mathbf{D} \to \mathbf{D},\ \mathbf{B} \to \mathbf{B}. \tag{44}$$

A list of independent bilinear invariants of these transformations reads:

$$\mathbf{E}\cdot\mathbf{B},\ \mathbf{H}\cdot\mathbf{D},\ \mathbf{D}^2,\ \mathbf{B}^2,\ \mathbf{E}\cdot\mathbf{D}-\mathbf{H}\cdot\mathbf{B}. \tag{45}$$

Notice that all the other invariants are their functions.

5.2 Galilean Born-Infeld equations

The relativistic Born-Infeld equations include system

$$\begin{aligned} &\frac{\partial \mathbf{D}}{\partial t} = \nabla\times\mathbf{H}, \quad \nabla\cdot\mathbf{D} = 0, \\ &\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}, \quad \nabla\cdot\mathbf{B} = 0 \end{aligned} \tag{46}$$

and the constitutive equations

$$\mathbf{D} = \frac{1}{L}(\mathbf{E}+(\mathbf{B}\cdot\mathbf{E})\mathbf{B}),\ \ \mathbf{H} = \frac{1}{L}(\mathbf{B}-(\mathbf{B}\cdot\mathbf{E})\mathbf{E}), \tag{47}$$

where $L = (1 + \mathbf{B}^2 - \mathbf{E}^2 - \mathbf{B}\cdot\mathbf{E})^{1/2}$. Equations (46), (47)are Lorentz-invariant. Making the Inönü-Wigner contraction of the related representation of the Lorentz group we can reduce this system to the following form:

$$\begin{aligned} &\frac{\partial \mathbf{D}'}{\partial t} = \nabla \times \mathbf{H}', \quad \nabla \cdot \mathbf{D}' = 0, \\ &\nabla \times \mathbf{E}' = 0, \quad \nabla \cdot \mathbf{B}' = 0 \end{aligned} \tag{48}$$

with the constitutive equations

$$\mathbf{D}' = \frac{\mathbf{E}'}{\sqrt{1-\mathbf{E}'^2}}, \quad \mathbf{H}' = \frac{\mathbf{B}'}{\sqrt{1-\mathbf{E}'^2}} - \frac{(\mathbf{B}'\cdot\mathbf{E}')\mathbf{E}'}{\sqrt{1-\mathbf{E}'^2}}. \tag{49}$$

Equations (48), (49) are Galilei-invariant. Moreover, under Galilei boosts vectors $\mathbf{D}', \mathbf{H}', \mathbf{B}'$ and $\mathbf{E}'$ cotransform as

$$\begin{aligned} &\mathbf{D}' \to \mathbf{D}', \quad \mathbf{H}' \to \mathbf{H}' + \mathbf{v} \times \mathbf{D}', \\ &\mathbf{B}' \to \mathbf{B}' + \mathbf{v} \times \mathbf{E}', \quad \mathbf{E}' \to \mathbf{E}'. \end{aligned} \tag{50}$$

One more contracted version of the Born-Infeld equations looks as the system:

$$\begin{aligned} &\nabla \times \mathbf{H} = 0, \quad \nabla \cdot \mathbf{D} = 0, \\ &\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \quad \nabla \cdot \mathbf{B} = 0, \end{aligned} \tag{51}$$

which is supplemented with the Galilei-invariant constitutive equations

$$\mathbf{D}' = \frac{\mathbf{E}'}{\sqrt{1+\mathbf{B}^2}} + \frac{(\mathbf{B}'\cdot\mathbf{E}')\mathbf{B}'}{\sqrt{1+\mathbf{B}^2}}, \quad \mathbf{H}' = \frac{\mathbf{B}'}{\sqrt{1+\mathbf{B}^2}}. \tag{52}$$

The corresponding transformation laws read

$$\begin{array}{l} \mathbf{D}' \to \mathbf{D}' - \mathbf{v}\times\mathbf{H}', \quad \mathbf{H}' \to \mathbf{H}', \\ \mathbf{B}' \to \mathbf{B}', \quad \mathbf{E}' \to \mathbf{E}' - \mathbf{v}\times\mathbf{B}'. \end{array} \tag{53}$$

Thus there exist two Galilei limits for the Maxwell equations in media which we present in the above.

5.3 Quasilinear wave equations

Finally, let me present a quasilinear equation for Galilean 10-vector:

$$\begin{array}{l} \frac{\partial}{\partial t}B - \nabla\cdot\mathbf{N} + \nu\mathbf{W}\cdot\mathbf{N} + \lambda\mathbf{R}\cdot\mathbf{W} \\ +\sigma(B^2 - \mathbf{R}\cdot\mathbf{N}) + \omega\mathbf{R}^2 + \mu B = ej^0, \\ \frac{\partial\mathbf{R}}{\partial t} + \nabla\times\mathbf{W} + \nu(B\mathbf{W} + \mathbf{R}\times\mathbf{N}) \\ +\sigma(\mathbf{R}\times\mathbf{W} + B\mathbf{R}) + \mu\mathbf{R} = e\mathbf{j}, \\ \nabla\cdot\mathbf{R} + \nu\mathbf{R}\cdot\mathbf{W} + \sigma\mathbf{R}^2 = ej^4, \\ \frac{\partial}{\partial t}\mathbf{W} + \nabla\times\mathbf{N} + \rho\mathbf{N} = 0, \\ \frac{\partial}{\partial t}\mathbf{R} - \nabla B + \rho\mathbf{W} = 0, \\ -\nabla\times\mathbf{R} + \rho\mathbf{R} = 0, \\ \nabla\cdot\mathbf{W} + \rho B = 0. \end{array} \tag{54}$$

Formula (54) presents the most general Galilei-invariant quasilinear system which can be obtained from "the most extended" linear system (34) by adding linear terms and quadratic non-linearities. It is rather interesting since includes a number of important systems corresponding to special value of arbitrary parameters which are denoted by Greek letters. In particular it includes a Galilean version of the Carrol-Field-Jackiw model.

6 Discussion

The main of the presented results are:

1. Completed description of indecomposable vector representations of the homogeneous Galilei group. 2. Completed list of relative functional and first order differential invariants. 3. Linear and quasilinear Galilei-invariant equations for vector fields.

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