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ORBIT FUNCTIONS AND PROBLEMS OF MATHEMATICAL PHYSICS

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Abstract

Some valuable properties of recently introduced families of special functions are pointed out. In particular we focus at the following:

- Each function is an eigenfunction of the Laplace operator appropriate for G and their eigenvalues are known explicitly. Therefore orbit functions can be applied to the solution of the corresponding Neumann and Dirichlet boundary-value problems on the fundamental domains of Weyl groups.
- There is correspondence between multivariate exponential special functions defined using S_n and orbit functions defined by $W(A_{n-1})$. The definition starting from $W(A_{n-1})$ uses non-orthogonal basis of the simple roots of A_{n-1} , while the definition based on S_n leads naturally to variables given relative to an orthonormal basis.

ORBIT FUNCTIONS AND THEIR PROPERTIES

$$C_\lambda(x) = \sum_{\mu \in W_\lambda} e^{2\pi i \langle \mu, x \rangle}, \quad S_\lambda(x) = \sum_{\mu \in W_\lambda} (-1)^{l(\mu)} e^{2\pi i \langle \mu, x \rangle}, \quad E_\lambda(x) = \sum_{\mu \in W_e \lambda} e^{2\pi i \langle \mu, x \rangle}.$$

W_λ – Weyl group orbit of $\lambda \in \mathbb{R}^n$,

$l(\mu)$ – number of reflections from λ to μ ,

W_e – even subgroup of the Weyl group W .

- Since C -, S - and E -functions are the finite sums of exponential functions they are **continuous** and have **continuous derivatives** of all orders in \mathbb{R}^n .
- C -functions are invariant and S -functions are skew-invariant with respect to W and W^{aff} and that E -functions are invariant with respect to W_e and W_e^{aff} .

Therefore it is **enough to consider** orbit functions **only on the fundamental domain**.

Moreover the S -functions are **antisymmetric** with respect to $n - 1$ -dimensional boundary of F , hence they are zero on the boundary of F . The C -functions are **symmetric** with respect to $n - 1$ -dimensional boundary of F and their normal derivative at the boundary is equal to zero.

ORTHOGONAL BASES AND LAPLACE OPERATOR

Suppose the continuous variable x to be given relative to the orthogonal basis. In the case of Lie algebra A_n we use orthogonal coordinates x_1, x_2, \dots, x_{n+1} in the hyperplane $x_1 + x_2 + \dots + x_{n+1} = 0$ and coordinates x_1, x_2, \dots, x_n for B_n, C_n and D_n .

The Laplace operator in orthogonal coordinates has the form

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_k^2}, \quad \text{where } k = n \text{ (or } k = n + 1 \text{ for } A_n).$$

For the algebras A_n, B_n, C_n and D_n , the Laplace operator gives the same eigenvalues on every exponential function summand of an orbit function with eigenvalue $-4\pi\langle\lambda, \lambda\rangle$.

Hence, the functions $C_\lambda(x), E_\lambda(x)$ and $S_\lambda(x)$ are eigenfunctions of the Laplace operator:

$$\Delta \begin{pmatrix} C_\lambda(x) \\ E_\lambda(x) \\ S_\lambda(x) \end{pmatrix} = -4\pi^2\langle\lambda, \lambda\rangle \begin{pmatrix} C_\lambda(x) \\ E_\lambda(x) \\ S_\lambda(x) \end{pmatrix}.$$

The same relation holds in the case of direct sums of the algebras $A_{n_1}, B_{n_2}, C_{n_3}$ and D_{n_4} , but at the moment we do not know if the analogue of this statement can be formulated for exceptional Lie algebras.

ORBIT FUNCTIONS ARE EIGENFUNCTIONS OF OTHER OPERATORS

T_y – shift operator: $T_y e^{2\pi i \langle \lambda, x \rangle} := e^{2\pi i \langle \lambda, x+y \rangle}$; $w \in W$ acts as follows: $w e^{2\pi i \langle \lambda, x \rangle} := e^{2\pi i \langle \lambda, wx \rangle}$.

$$D_y := \sum_{w \in W} (-1)^{l(w)} w T_y, \quad \text{for } C \text{ – and } S \text{ – functions;}$$

$$D_y := \sum_{w \in W_e} (-1)^{l(w)} w T_y, \quad \text{for } E \text{ – functions;}$$

$$\widetilde{D}_y := \sum_{w \in W} w T_y.$$

C -, S - and E -functions are eigenfunctions of the above operators, namely

$$\begin{aligned} D_y C_\lambda(x) &= S_\lambda(y) C_\lambda(x), & D_y S_\lambda(x) &= C_\lambda(y) S_\lambda(x), \\ D_y E_\lambda(x) &= E_\lambda(y) E_\lambda(x), & \widetilde{D}_y S_\lambda(x) &= S_\lambda(y) S_\lambda(x). \end{aligned}$$

Orbit functions are also eigenfunctions of a modified Laplace operator

$$\widetilde{\Delta} := \sum_{w \in W} w \frac{\partial^2}{\partial x_i^2}, \quad \text{for } C \text{ – and } S \text{ – functions;}$$

$$\widetilde{\Delta} := \sum_{w \in W_e} w \frac{\partial^2}{\partial x_i^2}, \quad \text{for } E \text{ – functions.}$$

BOUNDARY VALUE PROBLEMS AND OTHER BASES

C -function is a solution of the Neumann boundary value problem on n -dimensional simplex F

$$\Delta f(x) = \Lambda f(x), \quad \frac{\partial f(x)}{\partial \nu} = 0 \quad \text{for } x \in \partial F.$$

S -function is a solution of the Dirichlet boundary value problem on n -dimensional simplex F

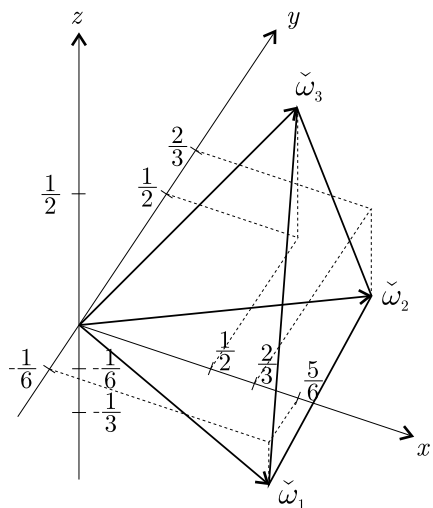
$$\Delta f(x) = \Lambda f(x), \quad f(x) = 0 \quad \text{for } x \in \partial F.$$

Let the continuous variable x is given relative to the ω -basis and Δ denotes the Laplace operator, where the differentiation ∂_{x_i} is made with respect to the direction given by ω_i .

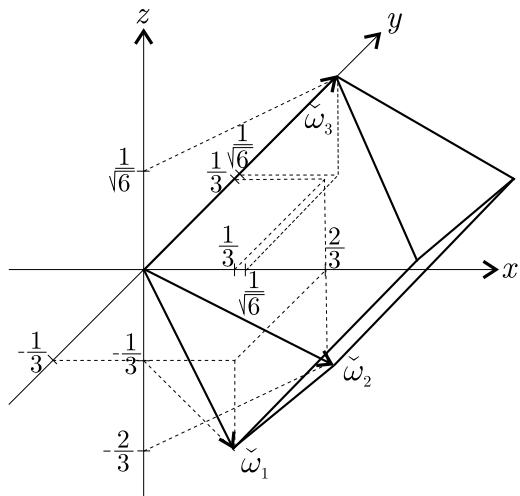
$$\Delta = \sum_{i,j=1}^n \frac{C_{ij}}{\langle \alpha_i, \alpha_i \rangle} \partial_{x_i} \partial_{x_j}, \quad \text{where } C \text{ is the Cartan matrix.}$$

It is known in Lie theory that the matrix of scalar products of the simple roots is positive definite, moreover our definition makes matrix $\frac{C_{ij}}{\langle \alpha_i, \alpha_i \rangle}$ symmetric, hence it can be diagonalized and the Laplace operator could be transformed to the sum of second derivatives by an appropriate change of variables.

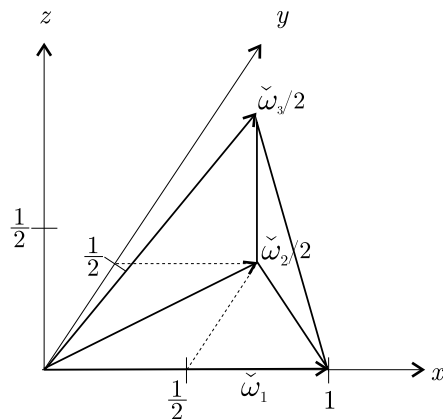
We can write the explicit form of the Laplace operator given in the ω -basis for any semisimple Lie algebra.

A₃

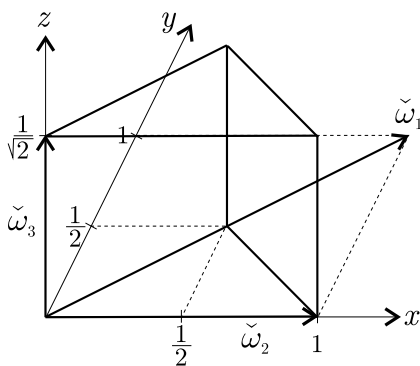
$$\Delta = \partial_{x_1}^2 - \partial_{x_1} \partial_{x_2} + \partial_{x_2}^2 - \partial_{x_2} \partial_{x_3} + \partial_{x_3}^2$$

A₂ × A₁

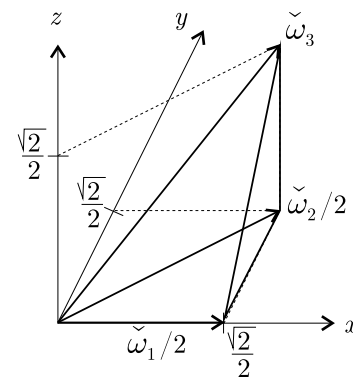
$$\Delta = \partial_{x_1}^2 - \partial_{x_1} \partial_{x_2} + \partial_{x_2}^2 + \partial_{x_3}^2$$

B₃

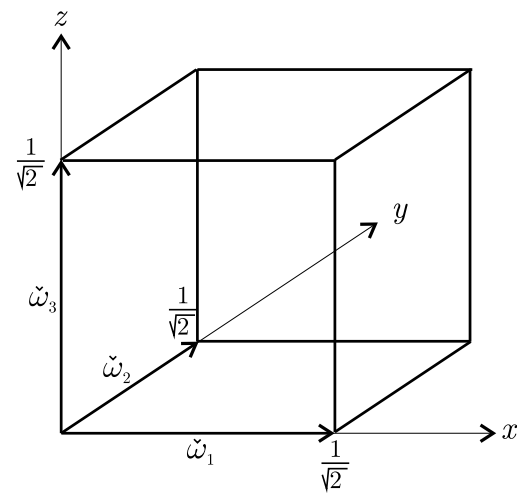
$$\Delta = \partial_{x_1}^2 - \partial_{x_1} \partial_{x_2} + \partial_{x_2}^2 - 2\partial_{x_2} \partial_{x_3} + 2\partial_{x_3}^2$$

C₂ × A₁

$$\Delta = 2\partial_{x_1}^2 - 2\partial_{x_1} \partial_{x_2} + \partial_{x_2}^2 + \partial_{x_3}^2$$

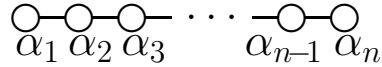
C₃

$$\Delta = 2\partial_{x_1}^2 - 2\partial_{x_1} \partial_{x_2} + 2\partial_{x_2}^2 - 2\partial_{x_2} \partial_{x_3} + \partial_{x_3}^2$$

A₁ × A₁ × A₁

$$\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$$

LIE ALGEBRA A_n



$$C = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 2 \end{pmatrix},$$

$$C^{-1} = \frac{1}{n+1} \begin{pmatrix} 1 \cdot n & 1 \cdot (n-1) & 1 \cdot (n-2) & 1 \cdot (n-3) & \dots & 1 \cdot 3 & 1 \cdot 2 & 1 \cdot 1 \\ 1 \cdot (n-1) & 2 \cdot (n-1) & 2 \cdot (n-2) & 2 \cdot (n-3) & \dots & 2 \cdot 3 & 2 \cdot 2 & 2 \cdot 1 \\ 1 \cdot (n-2) & 2 \cdot (n-2) & 3 \cdot (n-2) & 1 \cdot (n-3) & \dots & 3 \cdot 3 & 3 \cdot 2 & 3 \cdot 1 \\ 1 \cdot (n-3) & 2 \cdot (n-3) & 3 \cdot (n-3) & 1 \cdot (n-3) & \dots & 4 \cdot 3 & 4 \cdot 2 & 4 \cdot 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 \cdot 3 & 2 \cdot 3 & 3 \cdot 3 & 4 \cdot 3 & \dots & (n-2) \cdot 3 & (n-2) \cdot 2 & (n-2) \cdot 1 \\ 1 \cdot 2 & 2 \cdot 2 & 3 \cdot 2 & 4 \cdot 2 & \dots & (n-2) \cdot 2 & (n-1) \cdot 2 & (n-1) \cdot 1 \\ 1 \cdot 1 & 2 \cdot 1 & 3 \cdot 1 & 4 \cdot 1 & \dots & (n-2) \cdot 1 & (n-1) \cdot 1 & n \cdot 1 \end{pmatrix}.$$

Consider an orthonormal basis

$$e_i \in \mathbb{R}^{n+1}, \quad \langle e_i, e_j \rangle = \delta_{ij}, \quad 1 \leq i, j \leq n+1$$

A possible choice of the simple roots in the e -basis is:

$$\alpha_i = e_i - e_{i+1}, \quad i = 1, \dots, n$$

α_i are realized in \mathbb{R}^{n+1} , but they span the hyperplane $\mathcal{H}: x_1 + x_2 + \dots + x_n + x_{n+1} = 0$ (n -dimensional subspace of \mathbb{R}^{n+1}).

WEYL GROUP OF A_n AND DUAL BASIS

$W(A_n)$ acts in \mathbb{R}^{n+1} as permutations of the adjacent i -th and $(i+1)$ -th coordinates.

Indeed, let r_i , $1 \leq i \leq n$ be generating elements of $W(A_n)$, i.e, reflections through the hyperplanes perpendicular to α_i , then

$$\begin{aligned} r_i x &= x - \frac{2\langle x, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i = (x_1, x_2, \dots, x_{n+1}) - (x_i - x_{i+1})(e_i - e_{i+1}) \\ &= (x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_{n+1}). \end{aligned}$$

Such transpositions generate the full permutation group S_{n+1} . Thus $W(A_n)$ is isomorphic to S_{n+1} .

Introduce the ω -basis as the \mathbb{Z} -dual basis to simple roots α_i (which is also the basis in $\mathcal{H} \subset \mathbb{R}^{n+1}$ and a basis of \mathbb{R}^n):

$$\langle \alpha_i, \omega_j \rangle = \delta_{ij}, \quad 1 \leq i \leq n.$$

The bases α and ω are related by the Cartan matrix:

$$\alpha = C\omega, \quad \omega = C^{-1}\alpha$$

COORDINATES IN ORTHOGONAL AND ω -BASES

Consider a point $\lambda \in \mathcal{H}$ with coordinates l_j and λ_i in e - and ω -bases respectively.

$$\lambda = \sum_{i=1}^n \lambda_i \omega_i = \sum_{j=1}^{n+1} l_j e_j, \quad \text{with } l_1 + l_2 + \cdots + l_{n+1} = 0$$

Using $\alpha = C\omega$, i.e. $\omega_i = \sum_{k=1}^n C_{ik}^{-1} \alpha_k$, one finds the relations between λ_i and l_j :

$$\begin{aligned} l_1 &= \sum_{k=1}^n \lambda_k C_{k1}^{-1} \\ l_j &= \lambda_1 (C_{1j}^{-1} - C_{1j-1}^{-1}) + \lambda_2 (C_{2j}^{-1} - C_{2j-1}^{-1}) + \cdots \\ &\quad + \lambda_{n-1} (C_{n-1j}^{-1} - C_{n-1j-1}^{-1}) + \lambda_n (C_{nj}^{-1} - C_{nj-1}^{-1}), \quad j = 2, \dots, n, \\ l_{n+1} &= - \sum_{k=1}^n \lambda_k C_{kn}^{-1}, \end{aligned}$$

Relations between the coordinates can be written explicitly

MULTIVARIATE EXPONENTIAL FUNCTIONS

For a fixed point $\lambda = (l_1, l_2, \dots, l_m)$ of \mathbb{R}^m symmetric multivariate exponential function E_λ^- of $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$ is defined as the function

$$E_\lambda^+(x) := \det^+ \begin{pmatrix} e^{2\pi i l_1 x_1} & e^{2\pi i l_1 x_2} & \dots & e^{2\pi i l_1 x_m} \\ e^{2\pi i l_2 x_1} & e^{2\pi i l_2 x_2} & \dots & e^{2\pi i l_2 x_m} \\ \vdots & \vdots & \ddots & \vdots \\ e^{2\pi i l_m x_1} & e^{2\pi i l_m x_2} & \dots & e^{2\pi i l_m x_m} \end{pmatrix},$$

where

$$\det^+(a_{ij})_{i,j=1}^m = \sum_{s \in S_m} a_{1,s(1)} a_{2,s(2)} \cdots a_{m,s(m)} = \sum_{s \in S_m} a_{s(1),1} a_{s(2),2} \cdots a_{s(m),m}.$$

Thereby

$$E_\lambda^+(x) = \sum_{s \in S_m} e^{2\pi i l_1 x_{s(1)}} \cdots e^{2\pi i l_m x_{s(m)}} = \sum_{s \in S_m} e^{2\pi i (\lambda, s(x))} = \sum_{s \in S_m} e^{2\pi i (s(\lambda), x)}.$$

It is enough to consider the function $E_\lambda^+(x)$ on the hyperplane $x \in \mathcal{H}$ for the $\lambda = (l_1, l_2, \dots, l_m)$ such that $l_1 \geq l_2 \geq \dots \geq l_m$.

Therefore **C-functions** $C_\lambda(x)$ defined for $\lambda \in D^{++} \subset \mathbb{R}^n$ and $x \in R^n$ are equivalent to symmetric multivariate exponential functions $\mathbf{E}_\lambda^+(\mathbf{x})$ defined on R^{n+1} for $\lambda \in D^{++} \subset \mathbb{R}^{n+1}$ and $x \in \mathcal{H} \subset \mathbb{R}^{n+1}$.

CONCLUSIONS

- The same equivalence relations can be established between S -orbit functions and E^- -antisymmetric multivariate exponential functions as well as between E -multivariate exponential functions based on alternating group and E -orbit functions based on subgroup of W with even number of reflections.
- Notions of multivariate trigonometric functions lead us to the idea of new yet to be defined classes of W -orbit functions based on trigonometric sine and cosine functions.
- Analogously, we can introduce multivariate exponential functions built on discrete transformations group (where basic group action is simply negation of the i -th coordinate). Such functions are to be connected in similar way to the orbit functions related to the Lie group B_n .
- The established connection allows one to work in orthonormal basis instead ω -basis.
- The fact that orbit functions are eigenfunctions of the wide range of differential operators gives us an infinite sets of solutions of corresponding equations.
- Completeness of the families of orbit functions and orthogonality of the functions in these families allows us to expand functions defined on fundamental regions.