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THE BELOVED EQUATION IN SHALLOW WATER

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ABSTRACT

A number of papers have been devoted to the differential equations describing the motion of waves in shallow water as modelled by Whitham, Broer and Kaup from some forty to thirty years ago. We examine these equations from the viewpoint of symmetry with the explicit intention to carry those considerations as far as possible to the end. As a subsequent exploration we look at the model equations when one or both of the critical parameters are zero and explore if there are any solutions of interest.

Keywords: Shallow-water waves; symmetry analysis; closed-form solution.

1 Prologue

The beloved equation is

$$y'' + 3yy' + y^3 = 0 \tag{1.1}$$

and appears in a bewildering array of contexts:

- theory of univalent functions [14]
- astrophysics [19, 16]
- fusion of pellets [11]
- mechanics [17, 6]
- motion on a geodesic in a space of constant curvature [8]
- a painful paradigm for some singularity analysts [3]
- not to mention today's instance

  and so forth.

The appellation is the anglican form of the name bestowed upon (1.1), *l'équation adoré*, by a slightly irreverent Mark Feix.

Given the number of papers which it has inspired, the name is not too far-fetched.

The beloved equation has a number of interesting features:

– eight Lie point symmetries and so linearisable by means of a point transformation
– equally linearisable to a third-order equation by means of a Riccati transformation
– in terms of singularity analysis principal branches which lead to a Right Painlevé Series on the one hand and to a Left Painlevé Series on the other hand.

2 Introduction

Whitham, Broer and Kaup [22, 7, 15] presented two equations for the motion of waves in shallow water as

$$u_t = uu_x + v_x + \beta u_{xx} \tag{2.1}$$

$$v_t = u_x v + uv_x + \beta v_{xx} + \alpha u_{xxx}, \tag{2.2}$$

where $u(t, x)$ is the horizontal velocity, $v(t, x)$ is the vertical displacement of the fluid from its equilibrium position and α and β are parameters related to the degree of diffusion.

The system, (2.1,2.2), has been studied by a variety of methods [15, 1, 21, 24, 13, 23, 20, 25].

Zhang *et al* [25] treat system (2.1,2.2) in terms of the optimal system of subgroups of the Lie point symmetries of the system:

– they present a closed-form solution in only one case

– they do pay considerable attention to the case that α and β are both zero, *ie* the problem of one-dimensional shallow-water equations over an horizontal base.

This talk

We:

- supplement the results of Zhang *et al* with a deeper analysis of the systems of ordinary differential equations which result from the reduction of system (2.1,2.2) using the several optimal subgroups for nonzero α and β
- this leads to a number of equations which are well known in the general literature on ordinary differential equations
- we conclude with an approach to the case of zero α and β different than that presented of Zhang *et al.*

3 The case of nonzero α and β

We rescale the system as

$$u_t = uu_x + kv_x + u_{xx} \tag{3.1}$$

$$v_t = u_x v + uv_x + v_{xx} + u_{xxx}, \tag{3.2}$$

where $k = \alpha/\beta^2$. Courtesy of the Mathematica add-on Sym [9, 10, 5] the Lie point symmetries are

$$\begin{aligned}
\Gamma_1 &= \partial_t \\
\Gamma_2 &= \partial_x \\
\Gamma_3 &= t\partial_x - \partial_u \\
\Gamma_4 &= 2t\partial_t + x\partial_x - u\partial_u - 2v\partial_v
\end{aligned}$$

for which the optimal system of one-dimensional subalgebras is composed of these four symmetries plus

$$\Gamma_5 = \Gamma_1 - \Gamma_3 = \partial_t - t\partial_x + \partial_u.$$

We consider the different possibilities for reduction in turn.

3.1 Γ_1 and Γ_2

Reduction under $\Gamma_2 = \partial_x$ leads to the trivial result that both u and v are constants.

Reduction under $\Gamma_1 = \partial_t$ implies a steady state. We write $u = U(x)$ and $v = V(x)$. The reduced system is

$$0 = UU' + kV' + U'' \tag{3.3}$$

$$0 = U'V + U'V + V'''. \tag{3.4}$$

Analysis:

- We integrate (3.3) to obtain

  $$V = \left(A - U' - \tfrac{1}{2}U^2\right)/k,$$

  where A is a constant of integration.

- We substitute for V into (3.4) to obtain

$$U''' + \frac{A}{k}U - \frac{1}{k}\left(UU'' + U'^2 + \tfrac{3}{2}U^2U'\right) = 0$$

which can be integrated once to give

$$U'' - \frac{1}{k}UU' - \frac{1}{2k}U^3 + \frac{A}{k} - B = 0, \tag{3.5}$$

where B is a further constant of integration.

- We recognise (3.5) as a generalisation of the well-known Painlevé-Ince equation. In the case that $k = -2/9$ (3.5) is linearisable to

$$w''' - \frac{9A}{2}w' - \frac{3B}{2}w = 0$$

by means of the Riccati transformation $U = 2w'/w$ and is trivially integrable.

- For the same value of k the equation has eight Lie point symmetries independently of the values of the constants of integration, A and B.

- For what happens for general values of k one recalls the discussion in the paper of Lemmer and Leach [18].

A more profitable line of investigation is to take the combination $\Gamma_1 + c\Gamma_2 = \partial_t + c\partial_x$ to obtain a travelling-wave solution. In terms of the independent variable, $y = x - ct$, and the dependent variables $W(y) = U(y) + c$ and $V(y)$ system (3.1,3.2) becomes

$$kV' + WW' + W'' = 0 \tag{3.6}$$
$$W'V + WV' + V'' + W''' = 0, \tag{3.7}$$

where now the prime denotes differentiation with respect to y. System (3.6,3.7) is integrated with constants of integration A and B/k respectively. V is eliminated from the second using the first and we obtain a second-order equation for W, namely

$$W''(k-1) - 2WW' - \tfrac{1}{2}W^3 + AW - B = 0. \tag{3.8}$$

One possibility is that $k = 1$. Then (3.8) can immediately be written as the quadrature

$$\int \frac{4W\,\mathrm{d}W}{W^3 - 2AW + 2B} + y = y_0.$$

The division into partial fractions gives

$$\frac{4\lambda_1}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_1)}\frac{1}{W+\lambda_1} + \frac{4\lambda_2}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)}\frac{1}{W+\lambda_2} + \frac{4\lambda_3}{(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)}\frac{1}{W+\lambda_3},$$

where the denominator factors as $(W+\lambda_1)(W+\lambda_2)(W+\lambda_3)$, and it is highly unlikely that the integral could be inverted to give W as an explicit function of $y - y_0$.

For other values of k (3.8) is again a generalised Painlevé-Ince equation. When $k = 1/9$, the equation can be linearised by means of the Riccati transformation $W = 4w'/3w$ to

$$w''' - \frac{9A}{8}w' + \frac{9B}{8}w = 0.$$

For given values of the constants of integration, A and B, the roots of the characteristic equation are readily found and the solution of the equation, hence $u(x-ct)$ and $v(x-ct)$, follows trivially.

3.2 $\Gamma_1 + \Gamma_3 = \partial_t + t\partial_x - \partial_u$

In terms of the independent variable, $y = x + \frac{1}{2}t^2$, and dependent variables, $U = u - t$ and $V = v$, system (3.1,3.2) becomes

$$\begin{aligned} &UU' + kV' + U'' - 1 = 0 \\ &(UV)' + V'' + U''' = 0. \end{aligned}$$

Both equations can be integrated and V eliminated using the integral of the first equation. We obtain

$$(k-1)U'' - 2UU' - \tfrac{1}{2}U^3 + (y - A)U + B + 1 = 0, \tag{3.9}$$

where A and B/k are the respective constants of integration.

In (3.9) we recognise a generalisation of (3.8). The possibility that $k = 1$ gives an Abel's equation of the second kind which is not variables separable. For $k = 1/9$ the same Riccati transformation gives the linear third-order equation

$$w''' + \frac{9}{8}(y - A)w' + \frac{9}{8}(B + 1)w = 0.$$

For general values of the parameters the solution of this equation can be expressed in terms of hypergeometric functions as

$$w = C_0 F\left[\left\{\frac{m}{3}\right\},\left\{\frac{1}{3},\frac{2}{3}\right\},-\frac{x^3}{9}\right] + C_1 x F\left[\left\{\frac{1}{3}+\frac{m}{3}\right\},\left\{\frac{2}{3},\frac{4}{3}\right\},-\frac{x^3}{9}\right]$$
$$+C_2 x^2 F\left[\left\{\frac{2}{3}+\frac{m}{3}\right\},\left\{\frac{4}{3},\frac{5}{3}\right\},-\frac{x^3}{9}\right]$$

and in the specific case that $B = -1/2$ (A remains a free parameter) the solution can be written in terms of products of Airy functions as

$$w = C_1 Ai\left[\frac{(-1)^{1/3}x}{2^{2/3}}\right]^2 + C_2 Ai\left[\frac{(-1)^{1/3}x}{2^{2/3}}\right] Bi\left[\frac{(-1)^{1/3}x}{2^{2/3}}\right] + C_3 Bi\left[\frac{(-1)^{1/3}x}{2^{2/3}}\right]^2,$$

where the C_i, $i = 0$, 3, are constants of integration and $x = y - A$.

The difference in the expressions for the solution can be attributed to the fact that for general values of the parameters the equation has only four Lie point symmetries whereas, when the parameter B takes that specific value, the equation possesses the full seven. Then it can be integrated using the integrating factor w to give an Ermakov-Pinney equation, the solutions of which are given by Pinney's formula from the solutions of the equivalent linear equation, *ie* Airy's equation.

3.3 $\Gamma_3 = t\partial_x - \partial_u$

This is a somewhat trivial case. The invariants of Γ_3 are t, $u + x/t$ and v. The solutions are easily found to be

$$u = \frac{1}{t}(C_1 - x) \quad \text{and} \quad v = \frac{C_2}{t}, \tag{3.10}$$

where C_1 and C_2 are the constants of integration.

3.4 $\Gamma_4 = 2t\partial_t + x\partial_x - u\partial_u - 2v\partial_v$

The invariants of Γ_4 are $y = x^2/t$, $U = xu$ and $V = tv$. System (3.1,3.2) becomes

$$0 = 4y^2U'' + 2yUU' + y(y-2)U' + 2U - U^2 + 2ky^2V' \tag{3.11}$$

$$0 = 8y^3U''' - 6yU' - 6U + 4y^3V'' + (2U + y + 2)y^2V' + (2yU' - U + y)yV \tag{3.12}$$

(3.11) has the integrating factor y^{-2} and so

$$V = (my + 2U - yU - U^2 - 4yU')/(2ky). \tag{3.13}$$

Then (3.12) becomes an equation $U(y)$, being

$$\begin{aligned} &16(k-1)y^3U''' - 16y^2UU'' - 8y^3U'' - 16y^2U'^2 - 6yU^2U' + 6y(4-y)UU' \\ &-12\left[y^3 - 2(m-2)y^2 + 12(k+1)y\right]U' + 3U^3 + (y-12)U^2 \\ &-\left(y^2 + my + 12k\right)U + my^2. \end{aligned} \tag{3.14}$$

Although some parts of (3.14) are reminiscent of structures found in elements of the Riccati sequence [12, 4], there does not appear to be sufficient similarity in the different parts of the equation for a linearising transformation such as one had for the special cases above for which $k = 1/9$. It is evident that, when $k = 1$, the equation does reduce to a second-order equation.

4 The case $\alpha = 0$ and $\beta = 0$

In this case the system (2.1,2.2) becomes

$$\begin{aligned} u_t &= uu_x + v_x && (4.1)\\ v_t &= u_x v + uv_x. && (4.2)\end{aligned}$$

If we define $u = w_x$, (4.1) may be integrated to give

$$v = w_t - \tfrac{1}{2}w_x^2. \tag{4.3}$$

We substitute for u and v into (4.2) to obtain a single equation for w. It is

$$w_{tt} - 2w_x w_{tx} + w_{xx}\left(\tfrac{3}{2}w_x^2 - w_t\right) = 0 \tag{4.4}$$

which has the Lie point symmetries

$$\begin{array}{ll} \Gamma_1 = \partial_x & \Gamma_4 = x\partial_x - 2w\partial_w \\ \Gamma_2 = \partial_w & \Gamma_5 = t\partial_x - x\partial_w \\ \Gamma_3 = \partial_t & \Gamma_6 = t\partial_t - w\partial_w. \end{array}$$

Γ_1 and Γ_3 lead to $A + Bt$ and $A + Bx$ for (4.4) respectively.
A travelling-wave solution of system (4.1,4.2), equally (4.4), gives both u and v are constant.

4.1 $\Gamma_4 = x\partial_x - 2w\partial_w$

The invariants for Γ_4 are t and w/x^2. We write $w = x^2 f(t)$ and obtain

$$\ddot{f} - 10f\dot{f} + 12f^3 = 0. \tag{4.5}$$

Equation (4.5) is an instance of the generalised Painlevé-Ince equation with the generic two symmetries of invariance under translation in t and rescaling. It one investigates (4.5) using the techniques of singularity analysis, one finds that the leading-order term is a simple pole with coefficient either $-\frac{1}{2}$ or $-\frac{1}{3}$. For the former both resonances are at -1 and the analysis fails. For the latter the second resonance is at $\frac{2}{3}$ and the expansion about the movable singularity is in the powers of $(t - t_0)^{2/3}$, where t_0 is the location of the movable singularity. Since the equation has the two Lie point symmetries, it can be reduced to a variables separable first-order equation which can be integrated. In terms of the variables $x = \log f$ and $y = \dot{f}/f^2$ the solution can be written as

$$\frac{(y-3)^3}{(y-2)^2} = K\exp[-2x].$$

4.2 $\Gamma_5 = t\partial_x - x\partial_w$

The invariants are t and $w + x^2/(2t)$. We write $w = f(t) - x^2/(2t)$ and obtain

$$t\ddot{f} - \dot{f} = 0$$

which has the easily obtained solution

$$f = A + Bt^2$$

so that w follows immediately.

4.3 $\Gamma_6 = t\partial_t - w\partial_w$

The invariants are x and tw. In terms of $w = f(x)/t$ the reduced ordinary differential equation is

$$3f'^2f'' + 2ff'' + 4f'^2 + 4f = 0. \tag{4.6}$$

This also possesses the two symmetries of invariance under translation in the independent variable and rescaling. It is not in a satisfactory form for singularity analysis since the exponent of the leading-order term is 2. This is rectified by means of the replacement $f \longrightarrow 1/q$ so that the exponent is now -2. The coefficient of the leading-order term is either -2 or -3. The former gives a double -1 resonance whereas the latter gives -1 and 2 and thereby constitutes a viable principal branch. In terms of the variables $x = \log f$ and $y = f'/f$ the separable first-order equation has the solution

$$\frac{3y+4}{(y+2)^2} = K\mathrm{e}^x.$$

4.4 Potential representation of $v_t = u_x v + u v_x$, (4.2)

We set $v = w_x$. Then (4.2) can be solved to give $u = w_t/w_x$ and (4.1) becomes

$$w_x^2 w_{tt} - 2w_x w_t w_{tx} + w_{xx}\left(w_t^2 - w_x^3\right) = 0, \tag{4.7}$$

which apart from the ultimate term is a two-dimensional Bateman equation. The Lie point symmetries of (4.7) are

$$\begin{aligned} \Sigma_1 &= \partial_t & \Sigma_4 &= t\partial_x \\ \Sigma_2 &= \partial_x & \Sigma_5 &= t\partial_t - 2w\partial_w \\ \Sigma_3 &= \partial_w & \Sigma_6 &= x\partial_x + 3w\partial_w. \end{aligned}$$

Reduction using the first four symmetries leads to no result.

4.5 $\Sigma_5 = t\partial_t - 2w\partial_w$

We write (4.7) in terms of the reduction $w = f(x)/t^2$ as

$$f''\left(4f^2 - f'^3\right) - 2ff'^2 = 0$$

which has two Lie point symmetries and so can be reduced to a variables separable first-order equation. This can be integrated to give

$$f'^2(f' - 6) = C_1 \exp[-3f].$$

This can be integrated to give x in terms of f, but the expression is so complicated that it is pointless to list it here and one could not expect to be able to invert it.

4.6 $\Sigma_6 = x\partial_x + 3w\partial_w$

The invariants are t and w/x^3 so that we write $w = x^3 f(t)$. Equation (4.7) becomes

$$f\ddot{f} - \tfrac{4}{3}\dot{f}^2 - 18f^3 = 0$$

which takes the more transparent form

$$\ddot{h} + \frac{6}{h^2} = 0 \tag{4.8}$$

under $f \longrightarrow h^{-3}$. Equation (4.8) is an Emden-Fowler equation of index $(0, -2)$ and the implicit solution is

$$t - t_0 = \sqrt{Ih(12 + Ih)} + 12\,\mathrm{arcsinh}\left[\tfrac{1}{2}\sqrt{Ih/3}\right].$$

5 Discussion

We have considered in some detail the two cases, $\alpha \neq 0$ and $\beta \neq 0$, and $\alpha = 0$ and $\beta = 0$. For completeness we provide a brief summary of the results for the two cases in which one of the parameters is zero and the other is nonzero.

5.1 $\alpha \neq 0$ and $\beta = 0$

Without loss of generality the value of α can be taken as unity. System (2.1,2.2) is now

$$u_t = uu_x + v_x \tag{5.1}$$

$$v_t = u_x v + uv_x + u_{xxx}. \tag{5.2}$$

We write u in terms of a potential function as w_x. As in the previous section we may integrate (5.1) to obtain (4.3) and substitute for v in (5.2). The equation for w is

$$w_{tt} - 2w_x w_{xt} + \tfrac{3}{2} w_x^2 w_{xx} - w_t w_{xx} - w_{xxxx} = 0 \tag{5.3}$$

which has the Lie point symmetries

$$\begin{aligned} \Delta_1 &= \partial_t & \qquad \Delta_3 &= \partial_w \\ \Delta_2 &= \partial_x & \qquad \Delta_4 &= 2t\partial_t + x\partial_x \end{aligned}$$

so that there has been a noticeable decrease in symmetry by comparison with systems (3.1,3.2) and (4.1,4.2).

We note that the alternate reduction using the second equation as for (4.1,4.2) is not feasible for this system.

5.1.1 Δ_1 and Δ_2

In the case of Δ_1 we can write $w = f(x)$ so that (5.3) becomes

$$f'''' - \tfrac{3}{2} f'^2 f'' = 0$$

which can be integrated to give $f''(x)$ in terms of an elliptic function, the precise nature of which depends upon the values of the constants of integration. We note that the equation in $f'(x)$ passes the Painlevé Test.

Reduction by Δ_2 leads to the trivial solution $A + Bt$.

Travelling-wave solution using $\Delta_1 + c\Delta_2 = \partial_t + c\partial_x$

We write $w = f(x - ct)$. Then (5.3) is

$$f'''' = \left(\tfrac{3}{2}f'^2 + 3cf' + c^2\right) f''. \tag{5.4}$$

Despite the fact that (5.4) has only two Lie point symmetries its solution can be reduced to the integral of an elliptic function containing three constants of integration for general values of those parameters. Simpler solutions are available for specific values of the parameters. For example, if the two constants of integration from the double integration of (5.4) written as an equation in $g = f'$ are set to zero, the solution of (5.4) is

$$f(x) = \sqrt{\frac{2}{c}}\,\arcsin\left[\sqrt{K_3}\exp[c(x - x_0)]\right] + K_4,$$

where K_3 and K_4 are the constants of integration from the third and fourth quadratures respectively. Equation (5.4) possesses just three Lie point symmetries and so its reducibility to a quadrature in general indicates the existence of nonlocal symmetries of the useful variety.

5.1.2 $\Delta_4 = 2t\partial_t + x\partial_x$

This is the worst symmetry of the four to use to perform a reduction of (5.3) as both Δ_1 and Δ_2 are lost as point symmetries of the reduced equation. Although it is possible for a partial differential equation to gain point symmetries on reduction due to parallel equations leading to the same reduced equation [2], this is not one of them and the reduced equation is

$$16p^4f'''' + 48p^3f''' + 12p^2f'' - 24p^4f'^2f'' - 12p^4f'f'' - p^4f'' - 12p^3f'^3 - 10p^3f'^2 - 2p^3f' = 0,$$

where $p = x^2/t$ and $f(p) = w(t,x)$. There seems to be no further obvious route to reduction apart from the one consequent upon the existence of the symmetry ∂_f.

5.2 $\alpha = 0$ and $\beta \neq 0$

Now system (2.1,2.2) is

$$u_t = uu_x + v_x + u_{xx} \tag{5.5}$$

$$v_t = u_x v + uv_x + v_{xx}, \tag{5.6}$$

where β may be set at unity without loss of generality.

The Lie point symmetries of system (5.5,5.6) are

$$\begin{array}{ll} \Sigma_1 = \partial_t & \Sigma_3 = t\partial_x - \partial_u \\ \Sigma_2 = \partial_x & \Sigma_4 = 2t\partial_t + x\partial_x - u\partial_u - 2v\partial_v. \end{array}$$

5.2.1 $\Sigma_1 + c\Sigma_2 = \partial_t + c\partial_x$

One seeks a travelling-wave solution by setting $y = x - ct$, $u(t, x) = U(y)$ and $v(t, x) = V(y)$. The resulting pair of ordinary differential equations may each be integrated once to give the first-order system

$$U' + V + \tfrac{1}{2}U^2 + cU + A = 0 \tag{5.7}$$

$$V' + UV + cV + B = 0. \tag{5.8}$$

So far no useful further reduction of this system has been found.

5.2.2 $\Sigma_3 = t\partial_x - \partial_u$

System (5.5,5.6) can be reduced to a pair of elementary quadratures in terms of the variables $u = U(t) - x/t$ and $v = V(t)$. The solutions

$$u = \frac{A + x}{t} \quad \text{and} \quad v = \frac{B}{t}$$

are perhaps not the most exciting.

5.2.3 $\Sigma_4 = 2t\partial_t + x\partial_x - u\partial_u - 2v\partial_v$

In terms of the variables $u = x^{-1}U(y)$ and $v = t^{-1}V(y)$, where $y = x^2/t$, system (5.5,5.6) reduces to

$$\begin{aligned} 4y^2U'' + y(y-1)U' + \tfrac{1}{2}yUU' - U^2 + 2U + 2y^2V' = 0 \\ 4y^2V'' + y(y+2)V' + 2y(UV)' - UV + yV = 0. \end{aligned}$$

So far no further reduction of this system has been obtained.

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