Group analysis of differential equations and integrable systems
Protaras (Cyprus) October 26–30, 2008.

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Simple Lie algebras, Involutive distrubutions of operator valued evolutionary vector fields, and 2D Toda chains.

arXiv:math-ph/0703082⟶*SIGMA*

Liouville equation

$$\boxed{u_{xy} = \exp(2u)}$$

$$\frac{\delta}{\delta u}\Big(\Big[-\tfrac{1}{2}u_x u_y - \tfrac{1}{2}\mathrm{e}^{2u}\Big]\,\mathrm{d}x\mathrm{d}y\Big) = 0$$

Conservation laws:

$$w := u_x^2 - u_{xx} \qquad \bar{w} := u_y^2 - u_{yy}$$

$$D_y(w) \doteq 2u_x\mathrm{e}^{2u} - \big(\mathrm{e}^{2u}\big)_x = 0 \qquad D_x(\bar{w}) \doteq 0$$

$$\forall\ f(x,[w,w_x,w_{xx},\ldots]) \in \ker \bar{D}_y \qquad \bar{f}(y,[\bar{w}])$$

Symmetries:

$$\square := u_x + \tfrac{1}{2}D_x$$

$$u_t = \varphi = \square(\phi(x,[w])) \in \operatorname{sym}\mathcal{E}$$

$$u_t = \varphi_{\mathcal{L}} = \square\Big(\frac{\delta\mathcal{H}(x,[w])}{\delta w}\Big) \in \operatorname{sym}\boldsymbol{\mathcal{L}}$$

Liouville, Shabat'79–95, Sakovich'94.

Commutation $[\operatorname{sym}\mathcal{E}, \operatorname{sym}\mathcal{E}] \subseteq \operatorname{sym}\mathcal{E}$

$$[\operatorname{im}\square, \operatorname{im}\square] \subseteq \operatorname{im}\square,$$

$$[\operatorname{im}\square, \operatorname{im}\bar{\square}] \doteq 0$$

$$[\operatorname{im}\bar{\square}, \operatorname{im}\bar{\square}] \subseteq \operatorname{im}\bar{\square}$$

Leibnitz: Let $p, q(x, [w])$,

$$[\square(p), \square(q)] = \square(\mathrm{L}_{\square(p)}(q) - \text{v.v}) + \mathrm{L}_{\square(p)}(\square)(q) - \text{v.v.}$$

Closure:

$$[\square(p), \square(q)] = \square\Big(\mathrm{L}_{\square(p)}(q) - \text{v.v}) + \{\underline{p_x \cdot q - p \cdot q_x}\}\Big)$$

mKdV: $u_t = -\frac{1}{2}u_{xxx} + u_x^3 = \square(w) \in \operatorname{sym}\mathcal{E}.$

Miura: $w = u_x^2 - u_{xx}.$

KdV: $w_t = -\frac{1}{2}w_{xxx} + 3ww_x.$

Magri: $\mathcal{P} = -\frac{1}{2}D_x^3 + 2uD_x + u_x.$

$$\mathcal{P} = \square^* \circ D_x \circ \square$$

$$[\mathcal{P}(p), \mathcal{P}(q)] = \mathcal{P}\big(\mathrm{L}_{\mathcal{P}(p)}(q) - \text{v.v.} + \{\underline{p_x \cdot q - p \cdot q_x}\}\big)$$

Systematic generalization ?

Miura; Sokolov'2001.

Frobenius operators

- A = matrix operator in total derivatives;
- $w[u]$ = Miura substitution.

Definition.
§1. Prohibit any changes: ($w \sim \operatorname{dom} A$), ($u \sim \operatorname{im} A$).

$$[\operatorname{im} A, \operatorname{im} A] \subseteq \operatorname{im} A.$$

Koszul bracket:

$$\big[A(p), A(q)\big] = A\big([p, q]_A\big).$$

Bi-differential **Sokolov bracket:**

$$[p, q]_A = \mathrm{L}_{A(p)}(q) - \text{v.v.} + \{\{p, q\}\}_A.$$

Remark. Operator $A \Longrightarrow$ involutive distribution $\big\langle \mathrm{L}_{A(\cdot)} \big\rangle$ on ∞ jets.

Koszul, §1: Sokolov'2001, §2: A.K.+J.vdL. Olver

Well-defined Frobenius operators

§2. Allow changes: $(\tilde{w}[w], \operatorname{dom} A)$, $\quad (\tilde{u}[u], \operatorname{im} A)$.

"Rec": $\operatorname{dom} A \sim \operatorname{im} A \sim$ **"vectors"** $\mathrm{L}_{(\cdot)}$:

$$A \mapsto \tilde{A} = \ell_{\tilde{u}}^{(u)} \circ A \circ \ell_{w}^{(\tilde{w})} \Big|_{\substack{w=w[u] \\ u=u[\tilde{u}]}}.$$

"Ham":

- $\operatorname{dom} A \sim$ "**covectors**" $\psi = \delta(\cdot)/\delta w$;
- $\operatorname{im} A \sim$ "**vectors**" $\mathrm{L}_{A(\psi)}$:

$$A \mapsto \tilde{A} = \ell_{\tilde{u}}^{(u)} \circ A \circ \left(\ell_{\tilde{w}}^{(w)}\right)^* \Big|_{\substack{w=w[u] \\ u=u[\tilde{u}]}}.$$

Example.

$$A \colon \frac{\delta \mathcal{H}}{\delta w} \longmapsto \varphi_{\mathcal{L}} \in \operatorname{sym} \mathcal{L}_{\mathrm{Toda}}$$

Now Frobenius operators are **well defined**.

Frobenius Th.

PDE with ∞ symmetry algebras.

§2: A.K.+J.vdL. Thanks: B.A.Dubrovin

Commutation relations via $\{\{\,,\,\}\}_A$

Let $\mathcal{P} = \|\sum_\tau A_\tau^{\alpha\beta} \cdot D_\tau\|$ be **Hamiltonian operator**.

Th. Poisson bracket $\Longrightarrow$ Sokolov bracket:

$$\{\{p, q\}\}_A^i = \sum_{\sigma,\alpha} (-1)^\sigma \Big(D_\sigma \circ \Big[\sum_{\tau,\beta} D_\tau(p^\beta) \cdot \frac{\partial A_\tau^{\alpha\beta}}{\partial u_\sigma^i} \Big] \Big) (q^\alpha).$$

$i = 1, \ldots, m$ for $\vec{u} = (u^1, \ldots, u^m)$.

$$[\mathcal{P}(p), \mathcal{P}(q)] = \mathcal{P}\Big(\mathrm{L}_{\mathcal{P}(p)}(q) - \text{v.v.} + \{\{p, q\}\}_{\mathcal{P}} \Big).$$

Our claim: **Def.** + **Th.** $\Longrightarrow$ sym $\mathcal{E}_{\text{Toda}}$.

2D Toda systems

Exactly solvable exp-nonlinear $\mathcal{E}_{\text{Toda}}$:

$$u^i_{xy} = \exp\Bigl(\sum\nolimits_{j=1}^{\boldsymbol{m}} K^i_j u^j\Bigr).$$

Linear PDE ord = 1:

$$D_y(w) = 0 \quad \text{on } \mathcal{E}_{\text{Toda}}.$$

First integrals:

$$w^1, \ldots, w^{\boldsymbol{m}} \qquad \Longleftrightarrow \qquad K \text{ semi-simple}$$

Example A_2:

$$w^1 = u_{xx} + v_{xx} - u_x^2 + u_x v_x - v_x^2,$$
$$w^2 = u_{xxx} - 2u_x u_{xx} + u_x v_{xx} + u_x^2 v_x - u_x v_x^2.$$

Lagrangian:

$$\mathcal{L} = -\frac{\langle \alpha_i, \alpha_j \rangle}{|\alpha_i|^2 \cdot |\alpha_j|^2} \cdot u^i_x u^j_y + \mathrm{H}(u).$$

Momenta:

$$\mathfrak{m}^j := \frac{\partial \mathcal{L}}{\partial u^j_y} \qquad \Longrightarrow \qquad w = w[\mathfrak{m}]$$

Example A_2: $\quad \mathfrak{m}^1 := 2u_x - v_x$, $\mathfrak{m}^2 := 2v_x - u_x$.

$$w^1 = 3\mathfrak{m}^1_x + 3\mathfrak{m}^2_x - (\mathfrak{m}^1)^2 - \mathfrak{m}^1\mathfrak{m}^2 - (\mathfrak{m}^2)^2,$$
$$w^2 = 2\mathfrak{m}^1_{xx} + \mathfrak{m}^2_{xx} - 2\mathfrak{m}^1\mathfrak{m}^1_x - \mathfrak{m}^2\mathfrak{m}^1_x + \tfrac{2}{9}(\mathfrak{m}^1)^3$$
$$+ \tfrac{1}{3}(\mathfrak{m}^1)^2\mathfrak{m}^2 - \tfrac{1}{3}\mathfrak{m}^1(\mathfrak{m}^2)^2 - \tfrac{2}{9}(\mathfrak{m}^2)^3.$$

Minimal integrals w.

Liouville, Darboux, Toda, Leznov+Saveliev, Shabat.

Generators $\operatorname{sym}\mathcal{E}_{\mathsf{Toda}}$.

Operator

$$\square := \left(\ell_w^{(\mathfrak{m})}\right)^*.$$

Theorem.

(i) **Noether** symmetries:

$$\varphi_{\mathcal{L}} = \square\left(\frac{\delta\mathcal{H}}{\delta w}\right) \in \operatorname{sym}\mathcal{L} \qquad \forall\, \mathcal{H}(x,[w]).$$

(ii) **All** symmetries:

$$\varphi = \square\bigl(\phi\bigr) \in \operatorname{sym}\mathcal{E} \qquad \forall\, \phi(x,[w]) = (\phi^1,\ldots,\phi^r).$$

(iii) Commutation **closure**:

$$[\operatorname{im}\square, \operatorname{im}\square] \subseteq \operatorname{im}\square.$$

N.B.: Integrals w are **fixed**.

Example A_2:

$$\square = \begin{pmatrix} u_x + D_x & 2D_x^2 + 3u_xD_x + u_x^2 + 2u_xv_x - 2v_x^2 - u_{xx} + 2v_{xx} \\ v_x + D_x & D_x^2 - 2u_{xx} + v_{xx} + 2u_x^2 - 2u_xv_x - v_x^2 \end{pmatrix}.$$

Noether, Sakovich, Sokolov, Startsev, A.K.

Structural relations $\operatorname{sym}\mathcal{E}_{\mathsf{Toda}}$.

(iv) Under $\tilde{w}=\tilde{w}[w]$, "**covectors**"

$$\phi\mapsto\tilde{\phi}=\left[\left(\ell_{\tilde{w}}^{(w)}\right)^{*}\right]^{-1}(\phi).$$

Operator $\square$ becomes **well-defined** Frobenius.

(v) **Hamiltonian** operator:

$$\mathcal{P}=\square^{*}\circ\left(\ell_{\mathfrak{m}}^{(u)}\right)^{*}\circ\square.$$

(vi) Bi-differential Sokolov brackets **coincide**:

$$\{\{\,,\,\}\}_{\square}=\{\{\,,\,\}\}_{\mathcal{P}}.$$

All coefficients of $\mathcal{P}$ and $\{\{\,,\,\}\}_{\mathcal{P}}$ **depend on** $[w]$**.**

Example A_2: $\mathcal{P}=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}$,

$$\begin{aligned}
A_{11}&=2D_x^3+2w^1D_x+w_x^1,\\
A_{21}&=D_x^4+w^1D_x^2+3w^2D_x+w_x^2\\
A_{12}&=-D_x^4-w^1D_x^2+(3w^2-2w_x^1)\cdot D_x+(2w_x^2-w_{xx}^1),\\
A_{22}&=-\tfrac{2}{3}D_x^5-\tfrac{4}{3}w^1D_x^3-2w_x^1D_x^2\\
&\quad+(2w_x^2-2w_{xx}^1-\tfrac{2}{3}(w^1)^2)\cdot D_x+\tfrac{1}{3}(3w_{xx}^2-2w_{xxx}^1-2w^1w_x^1).
\end{aligned}$$

The bracket:

$$\begin{aligned}
\{\{\vec{p},\vec{q}\}\}_{\hat{A}_k}&=\left[p_x^1q^1-p^1q_x^1\right]+\left[p_{xx}^1q^2-p^2q_{xx}^1\right]\\
&\quad+\tfrac{2}{3}(p^2q_{xxx}^2-p_{xxx}^2q^2)+\tfrac{2}{3}w^1(p^2q_x^2-p_x^2q^2),\\
\{\{\vec{p},\vec{q}\}\}_{\hat{A}_k}&=\left[p_x^2q^1-p^1q_x^2+2(p_x^1q^2-p^2q_x^1)\right]+p^2q_{xx}^2-p_{xx}^2q^2.
\end{aligned}$$

A.K.+J.vdL. Thanks: Dubrovin.

Root system B_2.

$$u_{xy} = \exp(2u - 2v), \qquad v_{xy} = \exp(-u + 2v).$$

The integrals;

$$w^1 = u_{xx} + 2v_{xx} - 2(v_x)^2 + 2v_x u_x - u_x^2,$$

$$\begin{aligned} w^2 = v_{4x} &+ v_x(u_{xxx} - 2v_{xxx}) + u_{xx}v_x(v_x - 2u_x) \\ &+ v_{xx}(4v_x u_x - 2(v_x)^2 - (u_x)^2) + v_{xx}(u_{xx} - v_{xx}) \\ &+ (v_x)^4 + (v_x)^2(u_x)^2 - 2(v_x)^3 u_x. \end{aligned}$$

Frobenius operator:

$$\square = \left(\square^1, \square^2\right) = \begin{pmatrix} \square_{11} & \square_{12} \\ \square_{21} & \square_{22} \end{pmatrix}, \qquad \text{where } \square^1 = \begin{pmatrix} u_x + 2D_x \\ v_x + \frac{3}{2}D_x \end{pmatrix},$$

and

$$\begin{aligned} \square_{12} &= D_x^3 + \left(u_{xx} - u_x^2\right) \cdot D_x + \left(2v_x^2 u_x - 4v_{xx}v_x + 2u_{xx}v_x - 2v_x u_x^2 + 2v_{xxx}\right); \\ \square_{22} &= D_x^3 + v_x D_x^2 + \left(2v_x u_x - v_x^2 - u_x^2 + v_{xx} + u_{xx}\right) \cdot D_x \\ &\quad + \left(4v_x^2 u_x - 2v_{xx}v_x + 2u_{xx}v_x - 2v_x u_x^2 + 2v_{xxx} - 2v_x^3\right). \end{aligned}$$

Hamiltonian operator $\mathcal{P} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ has the components

$$\begin{aligned} A_{11} &= 10D_x^3 + 4w^1 D_x + 2w_x^1, \\ A_{21} &= 3D_x^5 + 3w^1 D_x^3 + 3w_x^1 D_x^2 + 8w^2 D_x + 2w_x^2, \\ A_{12} &= 3D_x^5 + 3w^1 D_x^3 + 6w_x^1 D_x^2 + (3w_{xx}^1 + 8w^2) \cdot D_x + 6w_x^2, \\ A_{22} &= D_x^7 + 2w^1 D_x^5 + 5w_x^1 D_x^4 + (6w_{xx}^1 + 6w^2 + (w^1)^2) \cdot D_x^3 \\ &\quad + (4w_{xxx}^1 + 3w^1 w_x^1 + 9w_x^2) \cdot D_x^2 \\ &\quad + \left(w_{4x}^1 + 7w_{xx}^2 + (w_x^1)^2 + 4w^1 w^2 + w^1 w_{xx}^1\right) \cdot D_x \\ &\quad + 2 \cdot (w_x^1 w^2 - w_{xxx}^2 - w_x^2 w^1). \end{aligned}$$

Sokolov bracket:

$$\begin{aligned} \{\{\vec{p}, \vec{q}\}\}_\square^1 &= 2(p^1 q_x^1 - p_x^1 q^1) + 3(p_{xx}^1 q_x^2 - p_x^2 q_{xx}^1) + (p_{4x}^2 q_x^2 - p_x^2 q_{4x}^2) \\ &\quad + w^1(p_{xx}^2 q_x^2 - p_x^2 q_{xx}^2) + 2w^2(p^2 q_x^2 - p_x^2 q^2); \\ \{\{\vec{p}, \vec{q}\}\}_\square^2 &= 6(p^2 q_x^1 - p_x^1 q^2) + 2(p^1 q_x^2 - p_x^2 q^1) + 2(p^2 q_{xxx}^2 - p_{xxx}^2 q^2) \\ &\quad + (p_{xx}^2 q_x^2 - p_x^2 q_{xx}^2) + 2w^1(p^2 q_x^2 - p_x^2 q^2). \end{aligned}$$

Root system G_2.

The Toda system for $K = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ is

$$u_{xy} = \exp(2u - v), \qquad v_{xy} = \exp(-3u + 2v).$$

The differential orders of the integrals w.r.t. the momenta equal 1 and 5, respectively:

$$w^1 = u_{xx} + \tfrac{1}{3}v_{xx} - u_x^2 + u_x v_x - \tfrac{1}{3}v_x^2,$$

$$\begin{aligned} w^2 = {} & u_{6x} - 2u_x u_{5x} + u_x v_{5x} + 10u_{4x}u_x v_x - 8u_{4x}u_x^2 - \tfrac{7}{3}u_{4x}v_x^2 + \tfrac{7}{3}u_{4x}v_{xx} \\ & - \tfrac{5}{3}v_{4x}u_x v_x + \tfrac{2}{3}v_{4x}u_x^2 - \tfrac{1}{9}v_{4x}v_x^2 + \tfrac{1}{9}v_{4x}v_{xx} + \tfrac{10}{3}v_{4x}u_{xx} + \tfrac{46}{3}u_{xxx}v_{xx}u_x + 20u_{xxx}u_{xx}v_x \\ & - 40u_{xxx}u_{xx}u_x + \tfrac{10}{3}u_{xxx}v_{xxx} - \tfrac{19}{3}u_{xxx}v_{xx}v_x + 16u_{xxx}u_x^3 - 18u_{xxx}u_x^2 v_x - \tfrac{1}{3}u_{xxx}v_x^3 \\ & + \tfrac{14}{3}u_{xxx}u_x v_x^2 - \tfrac{19}{3}v_{xxx}u_{xx}v_x + \tfrac{40}{3}v_{xxx}u_{xx}u_x - 8v_{xxx}u_x^3 - \tfrac{4}{9}v_{xxx}v_{xx}v_x - \tfrac{13}{3}v_{xxx}v_{xx}u_x \\ & + \tfrac{26}{3}v_{xxx}u_x^2 v_x - 2v_{xxx}u_x v_x^2 + \tfrac{2}{9}v_x^3 v_{xxx} + \tfrac{1}{18}v_{xxx}^2 - \tfrac{17}{3}u_{xx}v_{xx}^2 + 40u_{xx}^2 u_x^2 - 28u_{xx}^2 u_x v_x \\ & - 2u_{xx}v_{xx}v_x^2 + \tfrac{25}{6}u_{xx}^2 v_x^2 - 16u_{xx}^3 + \tfrac{1}{3}u_{xx}v_x^4 - 5u_{xx}u_x v_x^3 + 15u_{xx}u_x^2 v_x^2 - 12u_{xx}u_x^3 v_x \\ & + \tfrac{58}{3}u_{xx}v_{xx}u_x v_x - 34u_{xx}v_{xx}u_x^2 + \tfrac{49}{3}u_{xx}^2 v_{xx} + \tfrac{10}{3}v_{xx}v_x^3 u_x - \tfrac{8}{27}v_{xx}^3 + 12v_{xx}v_x u_x^3 \\ & - \tfrac{34}{3}v_{xx}v_x^2 u_x^2 - \tfrac{2}{9}v_{xx}v_x^4 + \tfrac{25}{3}v_{xx}^2 u_x^2 - 4v_{xx}^2 u_x v_x + \tfrac{2}{3}v_{xx}^2 v_x^2 - 3v_{xx}u_x^4 - \tfrac{2}{3}u_x v_x^5 + \tfrac{2}{27}v_x^6 \\ & + \tfrac{3}{2}u_x^4 v_x^2 + \tfrac{13}{6}u_x^2 v_x^4 - 3u_x^3 v_x^3. \end{aligned}$$

Root system G_2.

The Frobenius operator $\square$ is

$$\square = (\square^1, \square^2) = \begin{pmatrix} \square_{11} & \square_{12} \\ \square_{21} & \square_{22} \end{pmatrix}, \qquad \text{where } \square^1 = \begin{pmatrix} u_x + 3D_x \\ v_x + 5D_x \end{pmatrix}$$

and

$$\begin{aligned}
\square_{12} = {}& 2D_x^5 + u_x D_x^4 + \Big(15u_xv_x - 14u_x^2 - 5v_x^2 + 14u_{xx} + 5v_{xx}\Big)\cdot D_x^3 \\
&+ \Big(8v_{xxx} - 16v_{xx}v_x + 10u_x^2v_x + \tfrac{82}{3}v_{xx}u_x - 44u_{xx}u_x - \tfrac{10}{3}u_xv_x^2 + 24u_{xx}v_x - 8u_x^3 + 26u_{xxx}\Big)\cdot D_x^2 \\
&+ \Big(12u_x^3v_x - 16v_{xxx}v_x - 38u_{xxx}u_x - 4u_x^2v_x^2 + 4u_{xx}v_x^2 + 24u_{xxx}v_x + 44u_{xx}v_{xx} \\
&\quad + 2v_{xx}u_x^2 + \tfrac{70}{3}v_{xxx}u_x + 10u_{xx}u_x^2 + 21u_{4x} + 8v_{4x} - 16v_{xx}^2 - 14u_{xx}u_xv_x \\
&\quad + \tfrac{4}{3}v_{xx}u_xv_x - 51u_{xx}^2 - 9u_x^4\Big)\cdot D_x \\
&+ \Big(15u_x^3v_x^2 - 23u_{xx}u_xv_x^2 - \tfrac{44}{3}v_{xxx}u_xv_x + \tfrac{4}{3}u_xv_x^4 + 18u_{xx}u_x^2v_x - 12v_{xx}u_x^2v_x \\
&\quad + \tfrac{46}{3}v_{xx}u_xv_x^2 + 28u_{xx}v_{xx}u_x - 4u_{xx}v_{xx}v_x + 10u_{xxx}u_xv_x + 12u_{5x} + 3v_{5x} \\
&\quad + u_{xxx}v_x^2 - 20u_{4x}u_x + 4v_{xx}^2v_x + \tfrac{40}{3}v_{4x}u_x - 6v_{4x}v_x + 9u_{4x}v_x - 72u_{xxx}u_{xx} \\
&\quad - 20v_{xxx}v_{xx} + 26v_{xxx}u_{xx} - 8u_x^2v_x^3 + 6u_{xx}v_x^3 - 8u_{xx}^2u_x - 4v_{xx}v_x^3 \\
&\quad + 26u_{xxx}v_{xx} + 2v_{xxx}v_x^2 - 8u_{xxx}u_x^2 - \tfrac{40}{3}v_{xx}^2u_x - 9u_x^4v_x - 3u_{xx}^2v_x + 14v_{xxx}u_x^2\Big),
\end{aligned}$$

$$\begin{aligned}
\square_{22} = {}& 3D_x^5 + \Big(20u_xv_x - 20u_x^2 + \tfrac{23}{3}v_{xx} - \tfrac{23}{3}v_x^2 + 20u_{xx}\Big)\cdot D_x^3 \\
&+ \Big(40u_{xxx} - 2u_xv_x^2 - 80u_{xx}u_x - \tfrac{73}{3}v_{xx}v_x + 2u_x^2v_x + 40v_{xx}u_x + \tfrac{37}{3}v_{xxx} + 38u_{xx}v_x - \tfrac{1}{3}v_x^3\Big)\cdot D_x^2 \\
&+ \Big(38u_{xxx}v_x - 25v_{xx}^2 - \tfrac{73}{3}v_{xxx}v_x - 68u_{xxx}u_x - 23u_x^2v_x^2 + 36u_x^3v_x + 9u_{xx}v_x^2 \\
&\quad + 66u_{xx}v_{xx} + 2v_{xx}u_x^2 + 34v_{xxx}u_x + 36u_{xx}u_x^2 - 86u_{xx}^2 - 18u_x^4 + 34u_{4x} \\
&\quad + \tfrac{37}{3}v_{4x} - 44u_{xx}u_xv_x + 2v_{xx}u_xv_x - \tfrac{1}{3}v_x^4 + 5u_xv_x^3\Big)\cdot D_x \\
&+ \Big(36u_{xx}u_x^2v_x - 22v_{xxx}u_xv_x - \tfrac{2}{3}v_x^5 + 36u_x^3v_x^2 + 7u_xv_x^4 - 46u_{xx}u_xv_x^2 - 18v_{xx}u_x^2v_x \\
&\quad + 23v_{xx}u_xv_x^2 + 40u_{xx}v_{xx}u_x - 6u_{xx}v_{xx}v_x + 4u_{xxx}u_xv_x + 20u_{5x} + \tfrac{14}{3}v_{5x} \\
&\quad + 3u_{xxx}v_x^2 - 40u_{4x}u_x + 6v_{xx}^2v_x + 20v_{4x}u_x - 9v_{4x}v_x + 18u_{4x}v_x - 120u_{xxx}u_{xx} \\
&\quad - \tfrac{95}{3}v_{xxx}v_{xx} + 40v_{xxx}u_{xx} - 25u_x^2v_x^3 + 12u_{xx}v_x^3 - 6v_{xx}v_x^3 + 40u_{xxx}v_{xx} + 3v_{xxx}v_x^2 \\
&\quad - 20v_{xx}^2u_x - 18u_x^4v_x - 14u_{xx}^2v_x + 20v_{xxx}u_x^2\Big).
\end{aligned}$$

Root system G_2.

Using the fact that the coefficients A_{ij} of the Hamiltonian operator $\mathcal{P} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ are differential functions of the integrals w, we deduce for G_2 that

$$A_{11} = \tfrac{14}{3}D_x^3 + 2w^1 D_x + w^1_x,$$

$$A_{21} = 3D_x^7 + \tfrac{68}{3}w^1 D_x^5 + \tfrac{95}{3}w^1_x D_x^4 + \big((w^1)^2 + \tfrac{95}{3}w^1_{xx}\big)\cdot D_x^3 + \big(9w^1_{xxx} + 34w^1 w^1_x\big)\cdot D_x^2 + 6w^2 D_x + w^2_x,$$

$$\begin{aligned} A_{12} = {} & 3D_x^7 + \tfrac{68}{3}w^1 D_x^5 + \tfrac{245}{3}w^1_x D_x^4 + \big((w^1)^2 + \tfrac{395}{3}w^1_{xx}\big)\cdot D_x^3 + \big(\tfrac{368}{3}w^1_{xxx} - 28w^1 w^1_x\big)\cdot D_x^2 \\ & + \big(6w^2 + \tfrac{191}{3}w^1_{4x} - 62w^1_{xx}w^1 - 62(w^1_x)^2\big)\cdot D_x + \big(5w^2_x + \tfrac{41}{3}w^1_{5x} - 32w^1 w^1_{xxx} - 96w^1_{xx}w^1_x\big), \end{aligned}$$

$$\begin{aligned} A_{22} = {} & 2D_x^{11} + 30w^1 D_x^9 + 135w^1_x D_x^8 + \big(414w^1_{xx} + \tfrac{338}{3}(w^1)^2\big)\cdot D_x^7 + \big(819w^1_{xxx} + \tfrac{2366}{3}w^1 w^1_x\big)\cdot D_x^6 \\ & + \big(1119w^1_{4x} + 4w^2 + \tfrac{4870}{3}w^1_{xx}w^1 + \tfrac{1846}{3}(w^1_x)^2 + 8(w^1)^3\big)\cdot D_x^5 \\ & + \big(1065w^1_{5x} + 10w^2_x + \tfrac{6260}{3}w^1 w^1_{xxx} + 1220w^1_{xx}w^1_x + 60w^1_x(w^1)^2\big)\cdot D_x^4 \\ & + \Big(699w^1_{6x} + 26w^2_{xx} + \tfrac{5402}{3}w^1_{4x}w^1 + 36w^1 w^2 + \tfrac{1096}{3}w^1_{xxx}w^1_x - \tfrac{652}{3}(w^1_{xx})^2 - 428w^1_{xx}(w^1)^2 \\ & \quad - 88(w^1_x)^2 w^1 + 54(w^1)^4\Big)\cdot D_x^3 \\ & + \Big(303w^1_{7x} + 29w^2_{xxx} + \tfrac{3026}{3}w^1_{5x}w^1 + 54w^2_x w^1 - \tfrac{518}{3}w^1_{4x}w^1_x + 54w^1_x w^2 - \tfrac{5722}{3}w^1_{xxx}w^1_{xx} \\ & \quad - 702w^1_{xxx}(w^1)^2 - 1908w^1_{xx}w^1_x w^1 - 252(w^1_x)^3 + 324w^1_x(w^1)^3\Big)\cdot D_x^2 \\ & + \Big(78w^1_{8x} + 15w^2_{4x} + 328w^1_{6x}w^1 + 58w^2_{xx}w^1 - \tfrac{560}{3}w^1_{5x}w^1_x + 36w^2_x w^1_x - \tfrac{4030}{3}w^1_{4x}w^1_{xx} \\ & \quad - 514w^1_{4x}(w^1)^2 + 54w^2 w^1_{xx} - 36w^2(w^1)^2 - 1086(w^1_{xxx})^2 - 2000w^1_{xxx}w^1_x w^1 \\ & \quad - 1652(w^1_{xx})^2 w^1 - 1120w^1_{xx}(w^1_x)^2 + 396w^1_{xx}(w^1)^3 + 540(w^1_x)^2(w^1)^2\Big)\cdot D_x \\ & + \Big(9w^1_{9x} + 3w^2_{5x} + \tfrac{140}{3}w^1_{7x}w^1 + 20w^2_{xxx}w^1 - \tfrac{148}{3}w^1_{6x}w^1_x + 20w^2_{xx}w^1_x - \tfrac{994}{3}w^1_{5x}w^1_{xx} \\ & \quad - 138w^1_{5x}(w^1)^2 + 18w^2_x w^1_{xx} - 18w^2_x(w^1)^2 - 678w^1_{4x}w^1_{xxx} - 708w^1_{4x}w^1 w^1_x + 18w^2 w^1_{xxx} \\ & \quad - 36w^2 w^1_x w^1 - w^1_{xxx}\cdot\big(1424w^1_{xx}w^1 + 524(w^1_x)^2 - 144(w^1)^3\big) - 680(w^1_{xx})^2 w^1_x \\ & \quad + 648w^1_{xx}w^1_x(w^1)^2 + 216(w^1_x)^3 w^1\Big). \end{aligned}$$

Root system G_2.

Finally, we calculate the Sokolov bracket on the domain of $\square$:

$$
\begin{aligned}
\{\{\vec{p},\vec{q}\}\}^1_\square = {} & \bigl[p^1_x q^1 - p^1 q^1_x\bigr] + \bigl[\tfrac{41}{3}(p^1_{5x}q^2 - p^2 q^1_{5x}) + \tfrac{14}{3}(p^1_{4x}q^2_x - p^2_x q^1_{4x}) + \tfrac{14}{3}(p^1_{xxx}q^2_{xx} - p^2_{xx}q^1_{xxx})\\
&+ 9(p^2_{xxx}q^1_{xx} - p^1_{xx}q^2_{xxx}) + 32w^1(p^2 q^1_{xxx} - p^1_{xxx}q^2) + 34w^1(p^2_x q^1_{xx} - p^1_{xx}q^2_x)\bigr]\\
&+ 9(p^2_{9x}q^2 - p^2 q^2_{9x}) + 3(p^2_{8x}q^2_x - p^2_x q^2_{8x})\\
&+ \tfrac{140}{3}w^1(p^2_{7x}q^2 - p^2 q^2_{7x}) + 3(p^2_{7x}q^2_{xx} - p^2_{xx}q^2_{7x}) + 6(p^2_{xxx}q^2_{6x} - p^2_{6x}q^2_{xxx})\\
&+ 376w^1_x(p^2_{6x}q^2 - p^2 q^2_{6x}) + \tfrac{4}{3}w^1(p^2_x q^2_{6x} - p^2_{6x}q^2_x) + \tfrac{62}{3}w^1(p^2_{5x}q^2_{xx} - p^2_{xx}q^2_{5x})\\
&+ \tfrac{304}{3}w^1_x(p^2_{5x}q^2_x - p^2_x q^2_{5x}) + \tfrac{2834}{3}w^1_{xx}(p^2_{5x}q^2 - p^2 q^2_{5x}) + 138(w^1)^2(p^2 q^2_{5x} - p^2_{5x}q^2)\\
&+ \tfrac{248}{3}w^1_x(p^2_{4x}q^2_{xx} - p^2_{xx}q^2_{4x}) + \tfrac{640}{3}w^1_{xx}(p^2_{4x}q^2_x - p^2_x q^2_{4x}) + \tfrac{4184}{3}w^1_{xxx}(p^2_{4x}q^2 - p^2 q^2_{4x})\\
&+ 44w^1(p^2_{xxx}q^2_{4x} - p^2_{4x}q^2_{xxx}) + 672w^1w^1_x(p^2 q^2_{4x} - p^2_{4x}q^2) + 176(w^1)^2(p^2_x q^2_{4x} - p^2_{4x}q^2_x)\\
&+ \tfrac{830}{3}w^1_{xx}(p^2_{xxx}q^2_{xx} - p^2_{xx}q^2_{xxx}) + \tfrac{1060}{3}w^1_{xxx}(p^2_{xxx}q^2_x - p^2_x q^2_{xxx}) + \tfrac{4022}{3}w^1_{4x}(p^2_{xxx}q^2 - p^2 q^2_{xxx})\\
&+ 452(w^1_x)^2(p^2 q^2_{xxx} - p^2_{xxx}q^2) + 144(w^1)^3(p^2_{xxx}q^2 - p^2 q^2_{xxx}) + 18w^2(p^2_{xxx}q^2 - p^2 q^2_{xxx})\\
&+ 26(w^1)^2(p^2_{xx}q^2_{xxx} - p^2_{xxx}q^2_{xx}) + 1352w^1_{xx}w^1(p^2 q^2_{xxx} - p^2_{xxx}q^2)\\
&+ 576w^1w^1_x(p^2_x q^2_{xxx} - p^2_{xxx}q^2_x) + \tfrac{368}{3}w^1_{4x}(p^2_{xx}q^2_x - p^2_x q^2_{xx}) + 1360w^1w^1_{xxx}(p^2 q^2_{xx} - p^2_{xx}q^2)\\
&+ 1592w^1_{xx}w^1_x(p^2 q^2_{xx} - p^2_{xx}q^2) + 648w^1_x(w^1)^2(p^2_{xx}q^2 - p^2 q^2_{xx}) + 68(w^1_x)^2(p^2_x q^2_{xx} - p^2_{xx}q^2_x)\\
&+ 772w^1_{5x}(p^2_{xx}q^2 - p^2 q^2_{xx}) + 36w^2_x(p^2_{xx}q^2 - p^2 q^2_{xx}) + 36(w^1)^3(p^2_{xx}q^2_x - p^2_x q^2_{xx})\\
&+ 584w^1_{xx}w^1(p^2_x q^2_{xx} - p^2_{xx}q^2_x) + \tfrac{772}{3}w^1_{6x}(p^2_x q^2 - p^2 q^2_x) + 1020(w^1_{xx})^2(p^2 q^2_x - p^2_x q^2)\\
&+ 38w^2_{xx}(p^2_x q^2 - p^2 q^2_x) + 1360w^1_{xxx}w^1_x(p^2 q^2_x - p^2_x q^2) + 680w^1_{4x}w^1(p^2 q^2_x - p^2_x q^2)\\
&+ 36w^1w^2(p^2 q^2_x - p^2_x q^2) + 648w^1_{xx}(w^1)^2(p^2_x q^2 - p^2 q^2_x) + 648(w^1_x)^2w^1(p^2_x q^2 - p^2 q^2_x);\\
\{\{\vec{p},\vec{q}\}\}^2_\square = {} & p^2_x q^1 - p^1 q^2_x + 5(p^1_x q^2 - p^2 q^1_x) + 3(p^2_{5x}q^2 - p^2 q^2_{5x}) + (p^2_{xx}q^2_{xxx} - p^2_{xxx}q^2_{xx})\\
&+ 20w^1(p^2_{xxx}q^2 - p^2 q^2_{xxx}) + 40w^1_x(p^2_{xx}q^2 - p^2 q^2_{xx}) + 2w^1(p^2_{xx}q^2_x - p^2_x q^2_{xx})\\
&+ 38w^1_{xx}(p^2_x q^2 - p^2 q^2_x) + 18(w^1)^2(p^2 q^2_x - p^2_x q^2).
\end{aligned}
$$

The equalities

$$\bigl[\mathcal{P}(\vec{p}),\mathcal{P}(\vec{q})\bigr] = \mathcal{P}\Bigl(\mathrm{L}_{\mathcal{P}(\vec{p})}(\vec{q}) - \mathrm{L}_{\mathcal{P}(\vec{q})}(\vec{p}) + \{\{\vec{p},\vec{q}\}\}_{\mathcal{P}}\Bigr)$$

and

$$\bigl[\square(\vec{p}),\square(\vec{q})\bigr] = \square\Bigl(\mathrm{L}_{\square(\vec{p})}(\vec{q}) - \mathrm{L}_{\square(\vec{q})}(\vec{p}) + \{\{\vec{p},\vec{q}\}\}_{\square}\Bigr)$$

hold, where $\{\{\vec{p},\vec{q}\}\}_\square = \{\{\vec{p},\vec{q}\}\}_{\mathcal{P}}$ for any $\vec{p}$, $\vec{q}(x,[w])$. This yields all the commutation relations between symmetries $\varphi = \square(\cdot)$ of the 2D Toda chain associated with the root system G_2.

Algebra of Frobenius operators.

Span $\mathcal{A} = \langle A_1, \ldots, A_N \rangle$:

$$\Big[\sum\nolimits_i \operatorname{im} A_i, \sum\nolimits_j \operatorname{im} A_j\Big] \subseteq \sum\nolimits_k \operatorname{im} A_k.$$

Bracket of operators:

$$[A_i, A_j](p,q) := \big[A_i(p), A_j(q)\big].$$

Bi-differential structural constants:

$$[A_i, A_j] = \sum\nolimits_{k=1}^{r} A_k \circ \mathbf{c}_{ij}^k.$$

Remark. 2D Toda $\Longrightarrow$ **generator** of such algebras.

Affine connection.

Christoffel symbols:

$$[A_i(p), A_j(q)] = A_j\bigl(\mathrm{L}_{A_i(p)}(q)\bigr) - A_i\bigl(\mathrm{L}_{A_j(q)}(p)\bigr) + \sum_{k=1}^{N} A_k\bigl(\Gamma^k_{ij}(p,q)\bigr).$$

Properties.

$$\Gamma^\ell_{\ell\ell} = \{\{\,,\,\}\}_{A_\ell}, \qquad \Gamma^k_{ij}(p,q) = -\Gamma^k_{ji}(q,p).$$

Connection form $\Gamma \mapsto \tilde{\Gamma} = g\Gamma g^{-1} + \mathrm{d}g\, g^{-1}$:

$$\Gamma^k_{ij}(p,q) \mapsto \Gamma^{\tilde{k}}_{\tilde{i}\tilde{j}}\bigl(\tilde{p},\tilde{q}\bigr) = \bigl(g \circ \Gamma^k_{ij}\bigr)\bigl(g^{-1}\tilde{p}, g^{-1}\tilde{q}\bigr) + \delta^{\tilde{k}}_i \cdot \mathrm{L}_{\tilde{A}_j(\tilde{q})}(g)\bigl(g^{-1}\tilde{p}\bigr) - \delta^{\tilde{k}}_j \cdot \mathrm{L}_{\tilde{A}_i(\tilde{p})}(g)\bigl(g^{-1}\tilde{q}\bigr).$$

Connection:

$$\nabla^A\colon\ \mathrm{Der}_{\mathrm{Int}}\Bigl(\bigl(\mathrm{dom}\,A, [\,,\,]_A\bigr),\ \bigl(\mathrm{im}\,A, [\,,\,]\bigr)\Bigr) \to \mathrm{Der}\Bigl(\bigl(\mathrm{im}\,A, [\,,\,]\bigr),\ \bigl(\mathrm{im}\,A\text{-modules}\bigr)\Bigr)$$

Lift derivations:

$$\nabla^A_{[\psi,\cdot]_A} = \bigl[A(\psi), \cdot\bigr].$$

- **Curvature** = 0.
- Commutative hierarchies = **geodesics**.