

4th Workshop in
Group Analysis of Differential Equations and Integrable Systems
Protaras Oct. 2008

Quadratic algebras for superintegrable systems with quadratic integrals

C. Daskaloyannis, Y. Tanoudis and K. Ypsilantis
Aristotle University of Thessaloniki

Classical Mechanics on complex manifolds

$$\text{Manifold} \rightsquigarrow ds^2 = g_{ij} dx^i dx^j$$

Equation of Geodesics: Geodesic=Trajectory $\gamma(t)$ such that $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$

$$\frac{d^2 x^i}{dt^2} + \Gamma^i{}_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0, \quad \Gamma^i{}_{jk} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

Lagrangian formalism: $\mathcal{L} \equiv g_{ij} \dot{x}^i \dot{x}^j$

Equation of Geodesics \Leftrightarrow Equations Euler - Lagrange

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = \frac{\partial \mathcal{L}}{\partial x^i}$$

Hamiltonian Formalism: $p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}^i}$, $H = p_i \dot{x}^i - \mathcal{L} = H(p, x)$

$$\left. \begin{array}{l} \dot{p}_i = \frac{\partial H}{\partial x^i} \\ \dot{q}^i = -\frac{\partial H}{\partial p_i} \end{array} \right\} \rightsquigarrow \dot{A} = \{A, H\}_P \equiv \frac{\partial A}{\partial x^i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial x^i} \frac{\partial A}{\partial p_i}$$

Real Formulation

\rightsquigarrow

Complex Formulation

Geometry

$$\mathcal{L} = g_{ij} \dot{x}^i \dot{x}^j$$

\rightsquigarrow

Mechanics

$$\mathcal{L} = g_{ij} \dot{x}^i \dot{x}^j + V$$

Superintegrability in Classical Mechanics

Integrable system

A Hamiltonian system $H = H(p, x) = g^{ij}(x)p_i p_j + V(x)$ with $N = \dim M$
integrals (constants) of motion:

$$A_\nu = A_\nu(p, x), \quad \nu = 1, 2, \dots, N-1 \quad \text{and} \quad \{H, A_\nu\}_P = 0, \quad \{A_\mu, A_\nu\}_P = 0$$

Superintegrable system

An integrable system with $N - 1$ additional integrals of motion

$$B_\nu = B_\nu(p, x), \quad \nu = 1, 2, \dots, N-1 \quad \text{and} \quad \{H, B_\nu\}_P = 0,$$

$$\{B_\mu, B_\nu\}_P \neq 0, \quad \{A_\mu, B_\nu\}_P \neq 0$$

Superintegrability = absence of chaos = perfect order

**The integrals of motion can be considered as
generators of a Poisson Algebra**

Superintegrability in Quantum Mechanics

Quantum Integrable system

A Hamiltonian system $H = -\frac{\hbar^2}{2}\Delta + V(x)$ (Δ =Laplace operator) with
 $N = \dim M$ integrals (constants) of motion:

$$A_\nu, \quad \nu = 1, 2, \dots, N-1 \quad \text{and} \quad [H, A_\nu] = 0, \quad [A_\mu, A_\nu] = 0$$

Quantum Superintegrable system

A quantum integrable system with $N-1$ additional integrals of motion

$$B_\nu \quad \nu = 1, 2, \dots, N-1 \quad \text{and} \quad [H, B_\nu] = 0,$$

$$[B_\mu, B_\nu] \neq 0, \quad [A_\mu, B_\nu] \neq 0$$

The integrals of motion can be considered as generators of an associative Algebra

Quadratic algebra as a 'hidden' symmetry of the Hartmann potential

Ya I Granovskii, A S Zhedanov and I M Lutzenko

Physics Department, Donetsk State University, Donetsk 340055, USSR

Received 24 September 1990

Abstract. It is shown that operators, commuting with the Hamiltonian of the Hartmann potential form the quadratic Hahn algebra QH(3). The structure of this algebra and its *finite-dimensional representations* are described. An analysis of these representations is applied to obtain all the relevant physical results: energy spectrum, degree of degeneration and overlap functions.

$$\begin{aligned} [K_0, K_1] &= K_2 \\ [K_0, K_2] &= 2\{K_0, K_1\} \\ [K_2, K_1] &= 2K_1^2 + 4K_0 - 2\left(m^2 + 2\beta - 1 - \frac{\alpha^2}{2H}\right) \end{aligned}$$

This algebra **cannot** be reduced to a finite dimensional Lie algebra

Lie algebras and ternary relations

Let the Universal Enveloping Algebra $U(\mathfrak{g})$ of the Lie algebra with basis x_1, x_2, \dots, x_n satisfying the relations

$$[x_i, x_j] = \sum_m c_{ij}^m x_m$$

The generators satisfy the obvious "ternary" relations (which are trilinear)

$$T(x_i, x_j, x_k) \stackrel{\text{def}}{=} [x_i, [x_j, x_k]] = \sum_n d_{i;jk}^n x_n, \quad \text{where } d_{i;jk}^n = \sum_m c_{im}^n c_{jk}^m$$

Definition

Ternary algebra is an associative algebra satisfying whose the generators satisfy ternary trilinear relations

$$T(x_i, x_j, x_k) = \sum_n d_{i;jk}^n x_n$$

Parastatistics as examples of Ternary Algebras

H.S. Green, "A generalized method of field quantization", Phys. Rev., **90**, (1953),

270

Parafermionic algebra

$$\begin{aligned} \left[f_k, \left[f_\ell^\dagger, f_m \right] \right] &= 2\delta_{k\ell} f_m \\ \left[f_k, \left[f_\ell^\dagger, f_m^\dagger \right] \right] &= 2\delta_{k\ell} f_m^\dagger - 2\delta_{km} f_\ell^\dagger \\ \left[f_k, \left[f_\ell, f_m \right] \right] &= 0 \end{aligned}$$

Representation theory: N.I. Stoilova and J. Van der Jeugt, *The parafermion Fock space and explicit so(2n+1) representations*. J. Phys. A: Math. Theor. 41 (2008), 075202 .

Parabosonic algebra

$$\begin{aligned} \left[b_k, \left\{ b_\ell^\dagger, b_m \right\} \right] &= 2\delta_{k\ell} b_m \\ \left[b_k, \left\{ b_\ell^\dagger, b_m^\dagger \right\} \right] &= 2\delta_{k\ell} b_m^\dagger + 2\delta_{km} b_\ell^\dagger \\ \left[b_k, \left\{ b_\ell, b_m \right\} \right] &= 0 \end{aligned}$$

Representation Theory: T. Palev 80's and

S. Lievens, N. I. Stoilova, Van der Jeugt, "The paraboson Fock space and unitary irreducible representations of the Lie superalgebra osp(1/2n)", arXiv:0706.4196v1 [hep-th].

Parafermionic-like algebra

Parafermionic-like algebra \leftrightarrow Ternary algebra with

$$T(x_i, x_j, x_k) = [x_i, [x_j, x_k]] = \sum_m c_{i,jk} x_m$$

Any parafermionic like algebra corresponds to the universal enveloping algebra of a Lie algebra with $\frac{n(n+3)}{2}$ generators x_1, x_2, \dots and $[x_i, x_j]$.

Definition

Parafermionic-like Poisson Algebra The Parafermionic Poisson algebra is a Poisson algebra satisfying the ternary relations:

$$\{x_i, \{x_j, x_k\}_P\}_P = \sum_m c_{i,jk}^m x_m$$

Definition

Quadratic Parafermionic-like Poisson Algebra The Parafermionic Poisson algebra is a Poisson algebra satisfying the ternary relations:

$$\{x_i, \{x_j, x_k\}_P\}_P = \sum_{m,n} d_{i,jk}^{mn} x_m x_n + \sum_m c_{i,jk}^m x_m$$

Surfaces with geodesics possessing two quadratic integrals

G. Darboux, *Leçons sur la Théorie Générale des Surfaces* (1898), Vol. III, p.30 and 31

- Liouville surfaces: $ds^2 = (F(x+y) + G(x-y))dx dy$

$$H = (F(x+y) + G(x-y))\dot{x}\dot{y}$$

$$I = (F(x+y) + G(x-y))^2 \left(\dot{x}^2 + \dot{y}^2 - 2 \frac{F(x+y) - G(x-y)}{F(x+y) + G(x-y)} \dot{x}\dot{y} \right)$$

- Lie surfaces: $ds^2 = (F(y)x + G(y))dx dy$

$$H = (F(y)x + G(y))\dot{x}\dot{y}$$

$$I = (F(y)x + G(y))^2 \left(\dot{x}^2 - 2 \frac{\int F(y) dy}{F(y)x + G(y)} \dot{x}\dot{y} \right)$$

Integrable Systems in Liouville coordinates

Liouville Integrable Systems

$$H = \frac{p_\xi p_\eta}{F(\xi + \eta) + G(\xi - \eta)} + \underbrace{\frac{f(\xi + \eta) + g(\xi - \eta)}{F(\xi + \eta) + G(\xi - \eta)}}_{V(\xi, \eta)}$$

$$A = \underbrace{p_\xi^2 + p_\eta^2}_{\downarrow 2} - 2 \frac{F(\xi + \eta) - G(\xi - \eta)}{F(\xi + \eta) + G(\xi - \eta)} p_\xi p_\eta + 4 \frac{f(\xi + \eta) G(\xi - \eta) - g(\xi - \eta) F(\xi + \eta)}{F(\xi + \eta) + G(\xi - \eta)}$$

Lie Integrable Systems

$$H = \frac{p_\xi p_\eta}{F(\eta) \xi + G(\eta)} + \underbrace{\frac{f(\eta) \xi + g(\eta)}{F(\eta) \xi + G(\eta)}}_{V(\xi, \eta)}$$

$$A = \underbrace{p_\xi^2}_{\downarrow 2} - 2 \frac{\int F(\eta) d\eta}{F(\eta) \xi + G(\eta)} p_\xi p_\eta - 2 \frac{(f(\eta) \xi + g(\eta)) \int F(\eta) d\eta}{F(\eta) \xi + G(\eta)} + 2 \int f(\eta) d\eta$$

Surfaces with geodesics possessing three quadratic integrals

- G. Darboux, *Leçons sur la Théorie Générale des Surfaces* (1898) ► Vol. IV p.386, Note by Koenigs: The surfaces, whose the geodesics possess **three** (or more) integrals quadratic in velocity

$$ds^2 = g(x, y) dx dy \rightsquigarrow H = g(x, y) \dot{x} \dot{y}$$

$$A = g^2(x, y) (\dot{x}^2 + \dot{y}^2 + \dots)$$

$$B = g^2(x, y) (r(x)\dot{x}^2 + s(y)\dot{y}^2 + \dots)$$

- There are **5** classes of these metrics (tableau VII)

TABLEAU VII.

Réciproques des ds^2 de courbure constante non nulle.

$$1. \quad \left\{ \begin{aligned} ds^2 = & \Lambda_0 [p(x+y) - p(x-y)] dx dy \\ & + \Lambda_1 [p(x+y + \omega_1) - p(x-y + \omega_1)] dx dy \\ & + \Lambda_2 [p(x+y + \omega_2) - p(x-y + \omega_2)] dx dy \\ & + \Lambda_3 [p(x+y + \omega_3) - p(x-y + \omega_3)] dx dy. \end{aligned} \right.$$

Coefficients de transformation $[p(x), p(y)]$.

$$2. \quad \left\{ \begin{aligned} ds^2 = & \Lambda_0 \left[\frac{1}{\sin^2(x+y)} - \frac{1}{\sin^2(x-y)} \right] dx dy \\ & \Lambda_1 \left[\frac{1}{\cos^2(x+y)} - \frac{1}{\cos^2(x-y)} \right] dx dy \\ & + \Lambda_2 [\cos 2(x+y) - \cos 2(x-y)] dx dy \\ & + \Lambda_3 [\cos 4(x+y) - \cos 4(x-y)] dx dy. \end{aligned} \right.$$

Coefficients de transformation $\left(\frac{1}{\sin^2 2x}, \frac{1}{\sin^2 2y} \right)$.

$$3. \quad \left\{ \begin{aligned} ds^2 = & \Lambda_0 [\sin 4(x+y) - \sin 4(x-y)] dx dy \\ & + \Lambda_1 [\cos 4(x+y) - \cos 4(x-y)] dx dy \\ & + \Lambda_2 [\sin 2(x+y) - \sin 2(x-y)] dx dy \\ & + \Lambda_3 [\cos 2(x+y) - \cos 2(x-y)] dx dy. \end{aligned} \right.$$

Coefficients de transformation $\left(0, \frac{1}{\sin^2 2y} \right)$.

$$4. \quad \left\{ \begin{aligned} ds^2 = & \Lambda_0 \left[\frac{1}{(x+y)^2} - \frac{1}{(x-y)^2} \right] dx dy \\ & + \Lambda_1 [(x+y)^2 - (x-y)^2] dx dy \\ & + \Lambda_2 [(x+y)^4 - (x-y)^4] dx dy \\ & + \Lambda_3 [(x+y)^6 - (x-y)^6] dx dy. \end{aligned} \right.$$

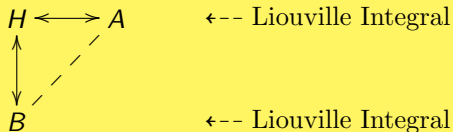
Coefficients de transformation $\left(\frac{1}{x^2}, \frac{1}{y^2} \right)$.

$$5. \quad \left\{ \begin{aligned} ds^2 = & \Lambda_0 [(x+y)^4 - (x-y)^4] dx dy \\ & + \Lambda_1 [(x+y)^2 - (x-y)^2] dx dy \\ & + \Lambda_2 [(x+y)^2 - (x-y)^2] dx dy \\ & + \Lambda_3 [(x+y) - (x-y)] dx dy. \end{aligned} \right.$$

Coefficients de transformation $\left(0, \frac{1}{y^2} \right)$.

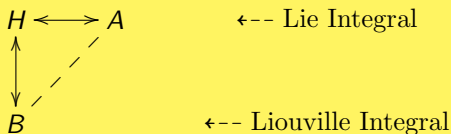
Classical Superintegrable Systems

- Class I:



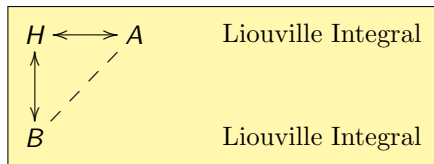
Separable in **two** coordinate systems

- Class II:



Separable to **one** coordinate system

Class I Superintegrable Systems in Liouville coordinates

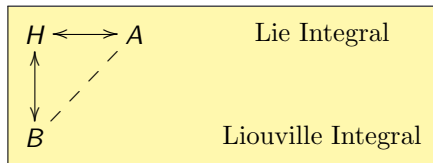


$$H = \frac{p_\xi p_\eta}{F(\xi + \eta) + G(\xi - \eta)} + \frac{f(\xi + \eta) + g(\xi - \eta)}{F(\xi + \eta) + G(\xi - \eta)}$$

$$A = \downarrow p_\xi^2 + \downarrow p_\eta^2 - 2 \frac{F(\xi + \eta) - G(\xi - \eta)}{F(\xi + \eta) + G(\xi - \eta)} p_\xi p_\eta + 4 \frac{f(\xi + \eta) G(\xi - \eta) - g(\xi - \eta) F(\xi + \eta)}{F(\xi + \eta) + G(\xi - \eta)}$$

$$B = \downarrow A(\xi) p_\xi^2 + \downarrow B(\eta) p_\eta^2 - 2 p_\xi p_\eta \frac{\beta(\xi, \eta)}{F(\xi + \eta) + G(\xi - \eta)} + Q(\xi, \eta)$$

Class II: Superintegrable Systems in Lie coordinates



$$H = \frac{p_\xi p_\eta}{F(\eta)\xi + G(\eta)} + \underbrace{\frac{f(\eta)\xi + g(\eta)}{F(\eta)\xi + G(\eta)}}_{V(x,y)}$$

$$A = p_\xi^2 - 2 \frac{\int F(\eta) d\eta}{F(\eta)\xi + G(\eta)} p_\xi p_\eta - 2 \frac{(f(\eta)\xi + g(\eta)) \int F(\eta) d\eta}{F(\eta)\xi + G(\eta)} + 2 \int f(\eta) d\eta$$

$$B = A(\xi) p_\xi^2 + B(\eta) p_\eta^2 - 2 p_\xi p_\eta \frac{\beta(\xi, \eta)}{F(\xi + \eta) + G(\xi - \eta)} + Q(\xi, \eta)$$

Poisson Quadratic Algebra of Integrals

$$\begin{aligned}
 H &= \frac{p_\xi p_\eta}{g(\xi, \eta)} + V(\xi, \eta) \\
 A &= p_\xi^2 + k p_\eta^2 - 2c(\xi, \eta) \frac{p_\xi p_\eta}{g(\xi, \eta)} + q(\xi, \eta), \quad k = 0 \text{ or } 1 \\
 B &= a^2(\xi) p_\xi^2 + b^2(\eta) p_\eta^2 - 2\beta(\xi, \eta) \frac{p_\xi p_\eta}{g(\xi, \eta)} + Q(\xi, \eta)
 \end{aligned}$$

M integral of motion $\{M, H\}_P = 0$, polynomial function of the momenta of **even order**

$$M = \sum_{k+l=\text{even}}^{2n} \alpha_{k,l}(\xi, \eta) p_\xi^k p_\eta^l$$

$$p_\xi^2 = \frac{\begin{vmatrix} A + 2cH - q - 2cV & k \\ B + 2\beta H - Q - 2\beta V & b^2(\xi, \eta) \end{vmatrix}}{\begin{vmatrix} 1 & k \\ a^2(\xi, \eta) & b^2(\xi, \eta) \end{vmatrix}}, \quad p_\eta^2 = \frac{\begin{vmatrix} 1 & A + 2cH - q - 2cV \\ a^2(\xi, \eta) & B + 2\beta H - Q - 2\beta V \end{vmatrix}}{\begin{vmatrix} 1 & k \\ a^2(\xi, \eta) & b^2(\xi, \eta) \end{vmatrix}}$$

$$p_\xi p_\eta = gH - V$$

$$M = \sum_{0 \leq i+j+k \leq n} c_{ijk}(\xi, \eta) A^i B^j H^k \rightsquigarrow c_{ijk}(\xi, \eta) = \text{const}$$

$\rightsquigarrow \{A, \{A, B\}_P\}_P$ and $\{B, \{A, B\}_P\}_P$ are even polynomials of order 4 in momenta

$\{A, \{A, B\}_P\}_P$ and $\{B, \{A, B\}_P\}_P$ are quadratic polynomials of A, B, H .

Superintegrable systems with two quadratic integrals of motion \rightsquigarrow

Parafermionic-like Poisson Quadratic Algebra

$$\{H, A\}_P = 0, \quad \{H, B\}_P = 0, \quad C = \{A, B\}_P \neq 0$$

Parafermionic-like Poisson Quadratic Algebra

$$\begin{aligned}\{A, \{A, B\}_P\}_P &= \alpha A^2 + 2\gamma AB + \delta A + \epsilon B + \zeta \\ \{B, \{A, B\}_P\}_P &= aA^2 - \gamma B^2 - 2\alpha AB + dA - \delta B + z\end{aligned}$$

$$\begin{aligned}\exists \text{ Casimir } K &= \{A, B\}_P^2 - 2\alpha A^2 B - 2\gamma AB^2 - 2\delta AB - \\ &\quad - \epsilon B^2 - 2\zeta B + \frac{2}{3}aA^3 + dA^2 + 2zA = \\ &= k_0 + k_1 H + k_2 H^2 + k_3 H^3\end{aligned}$$

$$\{K, A\}_P = \{K, B\}_P = \{K, C\}_P = 0$$

$$\{A, B\}_P^2 = 2F(A, H, B), \Rightarrow \begin{aligned}\{A, \{A, B\}_P\}_P &= \partial F / \partial B \\ \{B, \{A, B\}_P\}_P &= -\partial F / \partial A\end{aligned}$$

Classification Poisson Algebras \rightsquigarrow

Classification Classical Superintegrable systems

Classification of superintegrable systems

Any two dimensional superintegrable systems corresponds to some (parafermionic-like) Quadratic Poisson Algebra with **two** generators

Theorem

Classification of Poisson Algebras

⇒ ∃ **six** distinct classes of superintegrable systems

⇒ *All the known two dimensional systems are classified in these six classes*

Classification Table of quadratic superintegrable systems

		$A(\xi)$	$B(\eta)$
I_1	$\gamma = 0, \alpha = 0, a = -6$	ξ	η
I_2	$\gamma = 0, \alpha \neq 0, a = 8$	ξ^2	η^2
I_3	$\gamma = 2, \alpha = -32, a = 0$	$(e^\xi + e^{-\xi})^2$	$(e^\eta + e^{-\eta})^2$
II_1	$\gamma = 0, \alpha = 0, a = -6$	1	1
II_2	$\gamma = 0, \alpha \neq 0, a = 8$	ξ	η
II_3	$\gamma = 2, \alpha = -32, a = 0$	ξ^2	η^2

C.Dask. K. Ypsilantis JMP (2006) and C. Dask., Y. Tanoudis JMP (2008)

Class I_1 of superintegrable systems

$$A(\xi) = \xi \quad B(\eta) = \eta$$

$$\begin{aligned} F(u) &= 4\lambda u^2 + \kappa u + \nu/2, & G(v) &= -\lambda v^2 + \mu/v^2 + \nu/2 \\ f(u) &= 4\ell u^2 + k u + n/2, & g(v) &= -\ell v^2 + m/v^2 + n/2 \end{aligned}$$

$$ds^2 = (F(\xi + \eta) + G(\xi - \eta)) d\xi d\eta,$$

$$H = \frac{p_\xi p_\eta + f(\xi + \eta) + g(\xi - \eta)}{F(\xi + \eta) + G(\xi - \eta)}$$

$$A = p_\xi^2 + p_\eta^2 - 2p_\xi p_\eta \frac{F(\xi + \eta) - G(\xi - \eta)}{F(\xi + \eta) + G(\xi - \eta)} + 4 \frac{f(\xi + \eta)G(\xi - \eta) - g(\xi - \eta)F(\xi + \eta)}{F(\xi + \eta) + G(\xi - \eta)}$$

$$\tilde{F}(u) = \frac{\lambda u^6}{256} + \frac{\kappa u^4}{128} + \frac{\nu u^2}{16} - \frac{\mu}{u^2}, \quad \tilde{G}(v) = -\frac{\lambda v^6}{256} - \frac{\kappa v^4}{128} - \frac{\nu v^2}{16} + \frac{\mu}{v^2}$$

$$\tilde{f}(u) = \frac{\ell u^6}{256} + \frac{k u^4}{128} + \frac{n u^2}{16} - \frac{m}{u^2}, \quad \tilde{g}(v) = -\frac{\ell v^6}{256} - \frac{k v^4}{128} - \frac{n v^2}{16} + \frac{m}{v^2}$$

$$B = p_X^2 + p_Y^2 - 2p_X p_Y \frac{\tilde{F}(X + Y) - \tilde{G}(X - Y)}{\tilde{F}(X + Y) + \tilde{G}(X - Y)} + 4 \frac{\tilde{f}(X + Y)\tilde{g}(X - Y) - \tilde{g}(X - Y)\tilde{f}(X + Y)}{\tilde{F}(X + Y) + \tilde{G}(X - Y)}$$

$$X = 2\sqrt{\xi}, \quad p_X = \sqrt{\xi}p_\xi, \quad Y = 2\sqrt{\eta}, \quad p_Y = \sqrt{\eta}p_\eta$$

$$\begin{aligned} \alpha = 0 \quad \beta = 0 \quad \gamma = 0 \quad \delta = 16(\kappa H - k)\epsilon = 256(\lambda H - \ell) \quad \zeta = -32(\kappa H - k)(\nu H - n) \\ a = -6 \quad d = 8(\nu H - n)z = 8(\nu H - n)^2 - 128(\lambda H - \ell)(\mu H - m) \end{aligned}$$

Class I_2 of superintegrable systems

$$A(\xi) = \xi^2 \quad B(\eta) = \eta^2$$

$$F(u) = \lambda u^2 + \frac{\kappa}{u^2} + \frac{\nu}{2}, \quad G(v) = -\lambda v^2 + \frac{\mu}{v^2} + \frac{\nu}{2}$$

$$f(u) = \ell u^2 + \frac{k}{u^2} + \frac{n}{2}, \quad g(v) = -\ell v^2 + \frac{m}{v^2} + \frac{n}{2}$$

$$\tilde{F}(u) = 4\lambda e^{2u} + \nu e^u, \quad \tilde{G}(v) = \frac{\kappa e^v}{(1+e^v)^2} + \frac{\mu e^v}{(-1+e^v)^2}$$
$$\tilde{f}(u) = 4\ell e^{2u} + n e^u, \quad \tilde{g}(v) = \frac{k e^v}{(1+e^v)^2} + \frac{m e^v}{(-1+e^v)^2}$$

$$X = \ln \xi, \quad p_X = \xi p_\xi, \quad Y = \ln \eta, \quad p_Y = \eta p_\eta$$

$$\alpha = 8 \quad \beta = 0 \quad \gamma = 0 \quad \delta = 0$$

$$\epsilon = 256(\lambda H - \ell) \quad \zeta = -32(\nu H - n)^2 + 256(\lambda H - \ell)((\mu - \kappa)H - (m - k))$$

$$a = 0 \quad d = 0 \quad z = 32((\kappa + \mu)H - (k + m))(\nu H - n)$$

Potentials on a manifold with curvature zero

		κ	λ	μ	ν	ref [C]	ref [D]	ref [E]
F_1	I_1	0	0	0	\cdot	E_2	(a) ($r \neq 0$)	V^α
F_2	I_2	0	0	0	\cdot	E_1	(b) ($r \neq 0$)	V^b
F_3		0	\cdot	0	0	E_{16}	(g) ($r \neq 0$)	V^c
F_4	II_1	\cdot	0	0	0	E_{20}	(c)	V^d
F_5		0	0	\cdot	0	E_{11}	(e) ($r = 0$)	
F_6	II_2	0	0	0	\cdot	E_9	(e) ($r \neq 0$)	
F_7		0	0	\cdot	0	E_{10}		
F_8	II_3	0	0	0	\cdot	E_8	(f)	
F_9		0	0	$-\nu$	\cdot	E_7	(i) ($r \neq 0$)	
F_{10}		0	\cdot	0	0	E_{17}	(d)	
F_{11}		$-\lambda$	\cdot	0	0	E_{19}	(h) ($r \neq 0$)	

[C] Kalnins E.G., Kress J.M., Pogosyan G.S. and Miller W., J. Phys. A **34** 4705 (2001)

[D] Rañada M. F., J. Math Phys. **38** 4165 (1997)

[E] Rañada M. F. and Santander M., J. Math. Phys. **40** 5026 (1999)

Quadratic Parafermionic algebra for two dim. superintegrable systems

$$\begin{aligned}[A, [A, B]] &= \alpha A^2 + \beta B^2 + \gamma \{A, B\} + \delta A + \epsilon B + \zeta \\ [B, [A, B]] &= a A^2 - \gamma B^2 - \alpha \{A, B\} + d A - \delta B + z\end{aligned}$$

$$\begin{aligned}\delta &= \delta_1 H + \delta_0, & \epsilon &= \epsilon_1 H + \epsilon_0, \\ \zeta &= \zeta_2 H^2 + \zeta_1 H + \zeta_0, & d &= d_1 H + d_0, & z &= z_2 H^2 + z_1 H + z_0,\end{aligned}$$

The Casimir $[K, A] = [K, B] = 0$:

$$\begin{aligned}K &= [A, B]^2 - \alpha \{A^2, B\} - \gamma \{A, B^2\} + \\ &+ \left(\alpha \gamma - \delta + \frac{a\beta}{3} \right) \{A, B\} - \frac{2\beta}{3} B^3 + \left(\gamma^2 - \epsilon - \frac{\alpha\beta}{3} \right) B^2 + \\ &+ \left(-\gamma\delta + 2\zeta - \frac{\beta d}{3} \right) B + \frac{2a}{3} A^3 + \left(d + \frac{a\gamma}{3} + \alpha^2 \right) A^2 + \\ &+ \left(\frac{a\epsilon}{3} + \alpha\delta + 2z \right) A = h_0 + h_1 H + h_2 H^2 + h_3 H^3\end{aligned}$$

quadratic algebra **representation theory** in JMP **42**, 1100 (2001)

the energy eigenvalues are determined
by the condition of existing finite dimensional representations

Deformed oscillator representation

deformed oscillator algebra

$$\mathcal{P} = \mathbb{C} \langle\langle b^\dagger, b, \mathcal{N} \rangle\rangle$$

$$[\mathcal{N}, b^\dagger] = b^\dagger, \quad [\mathcal{N}, b] = -b, \quad b^\dagger b = \Phi(\mathcal{N}), \quad bb^\dagger = \Phi(\mathcal{N} + 1)$$

$\Phi(x)$ is a "well behaved" real function :

$$\Phi(0) = 0, \quad \text{and} \quad \Phi(x) > 0 \quad \text{for} \quad x > 0$$

\exists a Fock representation with basis $|n\rangle$, $n = 0, 1, \dots$

$$\begin{aligned} \mathcal{N}|n\rangle &= n|n\rangle & b^\dagger|n\rangle &= \sqrt{\Phi(n+1)}|n+1\rangle, & n &= 0, 1, \dots \\ b|0\rangle &= 0 & b|n\rangle &= \sqrt{\Phi(n)}|n-1\rangle, & n &= 1, 2, \dots \end{aligned}$$

$$A = A(\mathcal{N})$$

$$B = b(\mathcal{N}) + b^\dagger \rho(\mathcal{N}) + \rho(\mathcal{N}) b$$

$$[A, B] = b^\dagger \Delta A(\mathcal{N}) \rho(\mathcal{N}) - \rho(\mathcal{N}) \Delta A(\mathcal{N}) b$$

$$\Delta A(\mathcal{N}) = A(\mathcal{N} + 1) - A(\mathcal{N})$$

where the $A(x)$, $\Phi(x)$ and $\rho(x)$ are functions, which will be determined.

Case 1: $\gamma \neq 0$

$$A(k) = \frac{\gamma}{2} \left((n+u)^2 - 1/4 - \frac{\epsilon}{\gamma^2} \right)$$

$$\rho(\mathcal{N}) = \frac{1}{3 \cdot 2^{12} \cdot \gamma^8 (\mathcal{N} + u)(1 + \mathcal{N} + u)(1 + 2(\mathcal{N} + u))^2}$$

$$\begin{aligned} \Phi(\mathcal{N}) = & -3072\gamma^6 K(-1 + 2(\mathcal{N} + u))^2 - \\ & -48\gamma^6(\alpha^2\epsilon - \alpha\delta\gamma + a\epsilon\gamma - d\gamma^2) \cdot \\ & \cdot (-3 + 2(\mathcal{N} + u))(-1 + 2(\mathcal{N} + u))^4(1 + 2(\mathcal{N} + u)) + \\ & + \gamma^8(3\alpha^2 + 4a\gamma) \cdot \\ & \cdot (-3 + 2(\mathcal{N} + u))^2(-1 + 2(\mathcal{N} + u))^4(1 + 2(\mathcal{N} + u))^2 + \\ & + 768(\alpha\epsilon^2 - 2\delta\epsilon\gamma + 4\gamma^2\zeta)^2 + \\ & + 32\gamma^4(-1 + 2(\mathcal{N} + u))^2(-1 - 12(\mathcal{N} + u) + 12(\mathcal{N} + u)^2) \cdot \\ & \cdot (3\alpha^2\epsilon^2 - 6\alpha\delta\epsilon\gamma + 2a\epsilon^2\gamma + 2\delta^2\gamma^2 - 4d\epsilon\gamma^2 + 8\gamma^3z + 4\alpha\gamma^2\zeta) - \\ & - 256\gamma^2(-1 + 2(\mathcal{N} + u))^2 \cdot \\ & \cdot (3\alpha^2\epsilon^3 - 9\alpha\delta\epsilon^2\gamma + a\epsilon^3\gamma + 6\delta^2\epsilon\gamma^2 - 3d\epsilon^2\gamma^2 + 2\delta^2\gamma^4 + \\ & + 2d\epsilon\gamma^4 + 12\epsilon\gamma^3z - 4\gamma^5z + 12\alpha\epsilon\gamma^2\zeta - 12\delta\gamma^3\zeta + 4\alpha\gamma^4\zeta) \end{aligned}$$

Case 2: $\gamma = 0$, $\epsilon \neq 0$

$$A(k) = \sqrt{\epsilon}(n + u).$$

$$\rho(\mathcal{N}) = 1$$

$$\begin{aligned} \Phi(\mathcal{N}) &= \\ &= \frac{1}{4} \left(-\frac{K}{\epsilon} - \frac{z}{\sqrt{\epsilon}} - \frac{\delta}{\sqrt{\epsilon}} \frac{\zeta}{\epsilon} + \frac{\zeta^2}{\epsilon^2} \right) - \\ &\quad - \frac{1}{12} \left(3d - a\sqrt{\epsilon} - 3\alpha \frac{\delta}{\sqrt{\epsilon}} + 3 \left(\frac{\delta}{\sqrt{\epsilon}} \right)^2 - 6 \frac{z}{\sqrt{\epsilon}} + 6\alpha \frac{\zeta}{\epsilon} - 6 \frac{\delta}{\sqrt{\epsilon}} \frac{\zeta}{\epsilon} \right) (\mathcal{N} + u) \\ &\quad + \frac{1}{4} \left(\alpha^2 + d - a\sqrt{\epsilon} - 3\alpha \frac{\delta}{\sqrt{\epsilon}} + \left(\frac{\delta}{\sqrt{\epsilon}} \right)^2 + 2\alpha \frac{\zeta}{\epsilon} \right) (\mathcal{N} + u)^2 - \\ &\quad - \frac{1}{6} \left(3\alpha^2 - a\sqrt{\epsilon} - 3\alpha \frac{\delta}{\sqrt{\epsilon}} \right) (\mathcal{N} + u)^3 + \frac{1}{4} \alpha^2 (\mathcal{N} + u)^4 \end{aligned}$$

The energy eigenvalues with degeneracy $p + 1$ can be calculated by solving the system of algebraic equations:

$$\begin{aligned} \Phi(0) &= 0 \\ \Phi(p + 1) &= 0 \end{aligned}$$

$$H = \frac{1}{2} \left(p_x^2 + p_y^2 + \frac{k}{r} + \mu_1 \frac{\sqrt{r+x}}{r} + \mu_2 \frac{\sqrt{r-x}}{r} \right)$$

$$A = \frac{1}{2} \left(\{ (yp_x - xp_y), p_y \} + \frac{\mu_1(r-x)\sqrt{r+u}}{r} - \frac{\mu_2(r+x)\sqrt{r-u}}{r} - \frac{kx}{r} \right)$$

$$B = \frac{1}{2} \left(\{ xp_y - yp_x, p_x \} - \frac{\mu_1 x \sqrt{r-u}}{r} + \frac{\mu_2 x \sqrt{r+u}}{r} - \frac{ky}{r} \right)$$

$$\alpha = 0, \quad \gamma = 0, \quad \delta = 0, \quad \epsilon = -2\hbar^2 H, \quad \zeta = \hbar^2 \mu_1 \mu_2 / 2,$$

$$a = 0, \quad d = 2\hbar^2 H, \quad z = -\hbar^2 (\mu_1^2 - \mu_2^2) / 4$$

$$K = \hbar^2 k^2 H / 2 + \hbar^2 k (\mu_1^2 + \mu_2^2) / 4 + \hbar^4 H^2$$

$$\epsilon = \sqrt{-2E} / \hbar, \quad \lambda = k / \hbar^2,$$

$$\nu_1 = \mu_1 / \hbar^2, \quad \nu_2 = \mu_2 / \hbar^2, \quad \nu^2 = \nu_1^2 + \nu_2^2$$

$$\Phi(x) = \frac{\hbar^4}{16\epsilon^4} \left(\nu_1^2 - \lambda\epsilon^2 + 2(x+u - \frac{1}{2})\epsilon^3 \right) \cdot \left(\nu_2^2 - \lambda\epsilon^2 - 2(x+u - \frac{1}{2})\epsilon^3 \right)$$

$$\left. \begin{array}{l} \Phi(0) = 0, \\ \Phi(\rho+1) = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} u_1 \longrightarrow 2(\rho+1)\epsilon^3 - 2\lambda\epsilon^2 + \nu^2 = 0 \\ u_2 \longrightarrow 2(\rho+1)\epsilon^3 + 2\lambda\epsilon^2 - \nu^2 = 0 \end{array} \right.$$

Difficulties in three dimensions

Difficulties:

- All the 2 dim. manifolds are described by "conformal flat metrics"

$$ds^2 = g(x, y) (dx^2 + dy^2)$$

An analogous theorem there is not for the 3 dim. manifolds.

- In 3 dimensions there is not a general theory of metrics accepting integrals of motion. The mostly known cases are on the flat space.

Result:

The closest theory to the 2 dim. case is the theory of integrable and superintegrable systems on conformally flat manifolds.

$$ds^2 = g(x, y, z)(dx^2 + dy^2 + dz^2)$$

Studied and Classified by Kalnins-Kress-Miller using algebraic varieties

Known three dimensional systems on a flat space

- Evans list
NW Evans, *Superintegrability in Classical Mechanics*, Phys. Rev. A41, 5666 (1990)
5 potentials (2 potentials with 4 parameters (non degenerate) and 3 with 3 parameters (degenerate))
- Kalnins–Kress–Miller list
 - Nondegenerate 3D Complex Euclidean Superintegrable Systems and Algebraic Varieties. E G Kalnins, J M Kress and W Miller Jr. J. Math. Phys. 48 (2007) 113518. (arXiv:0708.3044)
(9 potentials)
 - Fine structure for 3D second order superintegrable systems: 3-parameter potentials. E G Kalnins, J M Kress and W Miller Jr. J. Phys. A: Math. Theor. 40 (2007) 5875-5892.
(5 potentials)

Main results of Kalnins-Kress-Miller

The classical systems on 3-dim conformal flat spaces are classified using techniques from algebraic varieties.

Theorem (5 \longrightarrow 6)

Let V be a nondegenerate potential (=depending on 4 parameters)corresponding to a conformally flat space in 3 dimensions

$$ds^2 = g(x, y, z)(dx^2 + dy^2 + dz^2)$$

that is superintegrable and there are 5 functionally independent constants of the motion $\mathcal{L} = \{S_\ell : \ell = 1, \dots, 5\}$ **There is always a 6th quadratic integral S_6 that is functionally dependent on \mathcal{L} , but linearly independent.**

Results of Kalnins-Kress-Miller 5 \longrightarrow 6 theorem

There are 6 independent equations of 6 linear independent variables

$$\{px^2, py^2, pz^2, p_x p_y, p_x p_z, p_y p_z\}$$

$$H = \frac{px^2 + py^2 + pz^2}{g(x, y, z)} + V(x, y, z)$$

$$S_i = a_i(x, y, z)px^2 + b_i(x, y, z)py^2 + c_i pz^2 + \\ + d_i(x, y, z)p_x p_y + e_i(x, y, z)p_x p_z + f_i(x, y, z)p_y p_x + Q_i(x, y, z)$$

where $i = 1, 2, 3, 4, 5$

$$px^2 = A_{xx}(x, y, z)H + B_{xx}(x, y, z)S_1 + C_{xx}S_2 + \\ + D_{xx}(x, y, z)S_3 + E_{xx}(x, y, z)S_4 + F_{xx}(x, y, z)S_5 + R_{xx}(x, y, z) \\ p_x p_y = A_{xy}(x, y, z)H + B_{xy}(x, y, z)S_1 + C_{xy}S_2 + \\ + D_{xy}(x, y, z)S_3 + E_{xy}(x, y, z)S_4 + F_{xy}(x, y, z)S_5 + R_{xy}(x, y, z) \\ \text{etc etc}$$

M integral of motion $\{M, H\}_P = 0$, polynomial function of the momenta of **even order**

$$M = \sum_{k+\ell+m=\text{even}}^{2n} \alpha_{k,\ell,m}(x, y, z) p_x^k p_y^\ell p_z^m$$

$$M = \sum_{0 \leq i_0 + \dots + i_5 = \text{even}}^{2n} c_{i_0 \dots i_5}(x, y, z) H^{i_0} S_1^{i_1} S_2^{i_2} S_3^{i_3} S_4^{i_4} S_5^{i_5} \rightsquigarrow c_{i_0 \dots i_5}(x, y, z) = \text{constants}$$

$\{S_i, \{S_j, S_k\}_P\}_P$ are even polynomials of order 4 in momenta

Quadratic Poisson Algebra

In the case of the nondegenerate or degenerate superintegrable systems on a conformally flat manifold the integrals of motion satisfy a **special form** of parafermionic-like quadratic Poisson Algebra (or quadratic associative algebra in quantum case)

Classical case
$$\{S_i, \{S_j, S_k\}_P\}_P = \sum_{mn} d_{i;jk}^{m,n} S_m S_n + \sum_m c_{i;jk}^m S_m + f_{i;jk}$$

Quantum case

$$[S_i, [S_j, S_k]]_P = -\hbar^4 \left(\frac{1}{2} \sum_{mn} D_{i;jk}^{m,n}(\hbar) \{S_m, S_n\} + \sum_m C_{i;jk}^m(\hbar) S_m + F_{i;jk}(\hbar) \right)$$

with

5 generators (KKM $V_i, V_{ii}, V_{iii}, V_{iv}, V_v, V_{vi}$), Evans V_1, V_5) or

4 generators (KKM V_{vii} , Evans V_3)

The potentials V_2 (Coulomb+2 Winternitz terms) and V_4 do not satisfy a parafermionic-like algebra as it was defined previously, but their algebra is characteristic "similar" to the "special" parafermionic one.

Winternitz- Smorodinsky Oscillator = V_i Kalnins-Kress-Miller = V_1 Evans

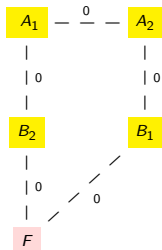
This potential get six, known, linearly independent integrals H, A_1, A_2, B_1, B_2, F .

$$H = p_x^2 + p_y^2 + p_z^2 + \delta(x^2 + y^2 + z^2) + \frac{\alpha_1}{x^2} + \frac{\alpha_2}{y^2} + \frac{\alpha_3}{z^2},$$

$$A_1 = p_x^2 + \delta x^2 + \frac{\alpha_1}{x^2}, \quad A_2 = p_z^2 + \delta z^2 + \frac{\alpha_3}{z^2},$$

$$B_1 = J_z^2 + kx^2 + \frac{\alpha_2 x^2}{y^2} + \frac{\alpha_1 y^2}{x^2}, \quad B_2 = J_x^2 + \frac{\alpha_3 y^2}{z^2} + \frac{\alpha_2 z^2}{y^2},$$

$$F = J_x^2 + J_y^2 + J_z^2 + \alpha_1 \frac{(y^2 + z^2)}{x^2} + \alpha_2 \frac{(x^2 + z^2)}{y^2} + \alpha_3 \frac{(x^2 + y^2)}{z^2}$$



Line= zero Poisson bracket

$$C_1^2 = \{A_1, B_1\}^2 = 2F_1(A_1, A_2, H, B_1)$$

$$F_1 = -8(A_1^2(\alpha_2 + B_1) + A_1 B_1(A_2 - H) + \delta B_1^2 + \alpha_1((A_1 + A_2 - H)^2 - 4\alpha_2 \delta))$$

$$C_2^2 = \{A_2, B_2\}^2 = 2F_2(A_1, A_2, H, B_2)$$

$$F_2 = -8((A_1 + A_2 - H)(\alpha_3 A_1 + A_2(\alpha_3 + B_2) - \alpha_3 H) + \delta B_2^2 + \alpha_2(A_2^2 - 4\alpha_3 \delta))$$

All six integrals form an Parafermionic-like Quadratic Algebra.

KKM potential V_V

This potential get six linearly independent integrals H, A_1, A_2, B_1, B_2, F .

$$H = p_x^2 + p_y^2 + p_z^2 + \alpha(4x^2 + y^2 + z^2) + \beta x + \frac{\gamma}{(y + iz)^2} + \frac{\delta(y - iz)}{(y + iz)^3}$$

$$A_1 = p_x^2 + 4\alpha x^2 + \beta x, \quad A_2 = J_x^2 + \frac{2y(\gamma y + iz(\gamma - 2\delta))}{(y + iz)^2}$$

$$B_1 = J_y p_z - J_z p_y + \frac{\beta}{4}(y^2 + z^2) + x(\alpha(y^2 + z^2) - \frac{2\delta y}{(y + iz)^3} + \frac{\delta - \gamma}{(y + iz)^2})$$

$$B_2 = (p_z - ip_y)^2 + \frac{\delta - \alpha(y + iz)^4}{(y + iz)^2}, \quad F = (p_z - ip_y)(J_y + iJ_z) - \frac{1}{4}(y + iz)^2(4\alpha x + \beta) - \frac{\delta x}{(y + iz)^2}$$

$$\{A_1, A_2\} = \{A_1, B_2\} = \{A_2, B_1\} = \{A_2, F\} = \{B_2, F\} = 0$$

$$C_1^2 = \{A_1, B_1\}^2 = 2F_1, \quad C_2^2 = \{A_2, B_2\}^2 = 2F_2$$

$$F_1 = \frac{1}{2}(4A_1^3 - 8A_1^2 H - 16\alpha B_1^2 + 4\beta H B_1 - \beta^2(A_2 - \gamma + \delta) + 4A_1(H^2 - \beta B_1 - 4\alpha(A_2 - \gamma + \delta)))$$

$$F_2 = 8(\gamma(B_2(A_1 + B_2 - H) - \alpha\gamma) + \delta(-B_2^2 + (A_1 - H^2) + 4\alpha\gamma) - 4\alpha\delta^2 - A_2(B_2^2 + 4\alpha\delta))$$

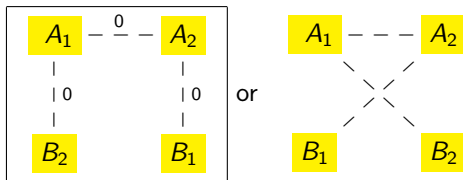
Degenerate Evans potentials

There are four integrals of motion A_1, A_2, B_1, B_2

V_1 and V_5 are non degenerate potentials

V_3 is a Parafermionic-like Poisson Quadratic Algebra,

V_2 and V_4 do not close a Poisson Quadratic Algebra but all of them have the "special" structure:



$$\{A_1, A_2\} = \{A_1, B_2\} = \{A_2, B_1\} = 0$$

$$\{A_1, B_1\}^2 = 2F_1(A_1, A_2, H, B_1) = \text{cubic function}$$

$$\{A_2, B_2\}^2 = 2F_2(A_1, A_2, H, B_2) = \text{cubic function}$$

Poisson Algebra

$$\begin{aligned} \{A_1, A_2\}_P &= \{A_1, B_2\}_P = \{A_2, B_1\}_P = 0 \\ \{A_1, B_1\}_P^2 &= 2F_1(A_1, A_2, H, B_1) = \text{cubic function} \\ \{A_2, B_2\}_P^2 &= 2F_2(A_1, A_2, H, B_2) = \text{cubic function} \end{aligned}$$

$$\{A_1, B_1\}_P = C_1, \quad \{A_2, B_2\}_P = C_2, \quad \{B_1, B_2\}_P = D$$

$$\{A_i, C_i\}_P = \frac{\partial F_i}{\partial B_i}, \quad \{B_i, C_i\}_P = -\frac{\partial F_i}{\partial A_i}$$

$$\{C_1, B_2\}_P = \{A_1, D\}_P, \quad \{C_2, B_1\}_P = -\{A_2, D\}_P$$

$$\{C_1, B_2\}_P C_1 - \frac{\partial F_1}{\partial A_2} C_2 - \frac{\partial F_1}{\partial B_1} D = 0$$

$$\{C_2, B_1\}_P C_2 - \frac{\partial F_2}{\partial A_1} C_1 + \frac{\partial F_2}{\partial B_2} D = 0$$

$$\frac{\begin{vmatrix} \{A_1, D\}_P & -\frac{\partial F_1}{\partial A_2} \\ \frac{\partial F_2}{A_1} & \{A_1, D\}_P \end{vmatrix}}{D} = \frac{\begin{vmatrix} -\frac{\partial F_1}{\partial B_1} & \{A_1, D\}_P \\ -\frac{\partial F_2}{\partial B_2} & \frac{\partial F_2}{A_1} \end{vmatrix}}{C_2} = \frac{\begin{vmatrix} -\frac{\partial F_1}{\partial A_2} & -\frac{\partial F_1}{\partial B_1} \\ \{A_1, D\}_P & -\frac{\partial F_2}{\partial B_2} \end{vmatrix}}{C_1} =$$

$$= \{C_1, C_2\}_P = L_1(A_1, A_2, H, B_1, B_2)C_1 + L_2(A_1, A_2, H, B_1, B_2)C_2 + L_3(A_1, A_2, H, B_1, B_2)D$$

$L_i(A_1, A_2, H, B_1, B_2)$ linear functions

Poisson Algebras for the degenerate (3 parameters potentials)

The degenerate (3 parameter) Evans potentials with on "special" structures have the property

$$M_1(A_1, A_2, H, B_1, B_2) \{A_1, B_1\} + M_2(A_1, A_2, H, B_1, B_2) \{A_2, B_2\} + M_3(A_1, A_2, H, B_1, B_2) \{B_1, B_2\} = 0$$

where $M_i(A_1, A_2, H, B_1, B_2)$ linear functions. That means that there relations are of the form

$$\{A_i, \{A_i, B_i\}_P\}_P = \partial F_i / \partial B_i$$

$$\{B_i, \{A_i, B_i\}_P\}_P = -\partial F_i / \partial A_i$$

$$M_3 \{A_1, \{B_1, B_2\}_P\}_P = G_1(A_1, A_2, B_1, B_2, \{A_1, B_1\}_P, \{A_2, B_2\}_P, \{B_1, B_2\}_P)$$

$$M_3 \{A_2, \{B_1, B_2\}_P\}_P = G_2(A_1, A_2, B_1, B_2, \{A_1, B_1\}_P, \{A_2, B_2\}_P, \{B_1, B_2\}_P)$$

$$M_3^2 \{B_1, \{B_1, B_2\}_P\}_P = G_3(A_1, A_2, B_1, B_2, \{A_1, B_1\}_P, \{A_2, B_2\}_P, \{B_1, B_2\}_P)$$

$$M_3^2 \{B_2, \{B_1, B_2\}_P\}_P = G_4(A_1, A_2, B_1, B_2, \{A_1, B_1\}_P, \{A_2, B_2\}_P, \{B_1, B_2\}_P)$$

RESUMÉ: The three dimensional superintegrable problems seem to satisfy a non linear ternary algebra

$$\text{Evans } V_2 : H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - \frac{k}{\sqrt{x^2 + y^2 + z^2}} + \frac{k_1}{x^2} + \frac{k_2}{y^2}$$

$$2B_2 \{A_1, B_1\}_P + B_1 \{A_2, B_2\}_P + 2(A_1 - A_2) \{B_1, B_2\}_P = 0$$

$$\text{Evans } V_3 : H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + k_1 \frac{x}{y^2 \sqrt{x^2 + y^2}} + \frac{k_2}{y^2} + \frac{k_3}{z^2}$$

Parafermionic-like algebra with 4 generators

$$2H \{A_1, B_1\}_P + B_1 \{A_2, B_2\}_P + 2(H - A_2) \{B_1, B_2\}_P = 0$$

$$\text{Evans } V_4 : H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \frac{k_1 x}{y^2 \sqrt{x^2 + y^2}} + \frac{k_2}{y^2} + k_3 z$$

$$B_2 \{A_1, B_1\}_P - k_3 \{A_2, B_2\}_P + 2(A_1 - H) \{B_1, B_2\}_P = 0$$

Remarks

- In all the studied three dim. superintegrable systems the integrals of motion satisfy a "special" form of Quadratic Poisson algebra with one of two "special" structures = Quadratic subalgebras
- The "special" structures are traces of two dim. superintegrable systems in the three dim. systems. Their classification, could lead to a classification of the three dim. superintegrable systems.
- The cases where there are two "special" structures, in the quantum case could lead to a representation theory with coupled deformed oscillators

Open problems

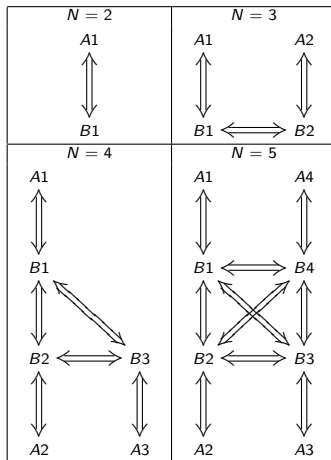
- Classification and Representation Theory of Parafermionic-like linear and quadratic Ternary Algebras
- Representation Theory of superintegrable Polynomial Algebras.

"Special" structure in N-Dimensions

$$\{A_i, A_j\} = \{A_i, B_j\} = \{A_j, B_i\} = 0$$

$$\{A_i, B_i\}^2 = 2F_i(A_1, A_2, \dots, A_N, H, B_i) = \text{cubic function}$$

⇔ no zero Poisson bracket



$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \frac{-k}{\sqrt{x^2 + y^2 + z^2}} + \frac{k_1}{x^2} + \frac{k_2}{y^2} + \frac{k_3}{z^2}$$

$$A_1 = \frac{1}{2}(J_1^2 + J_2^2 + J_3^2) + \frac{k_1(x^2 + y^2 + z^2)}{x^2} + \frac{k_2(x^2 + y^2 + z^2)}{y^2} + \frac{k_3(x^2 + y^2 + z^2)}{z^2}$$

$$A_2 = \frac{1}{2}J_3^2 + \frac{k_1(x^2 + y^2)}{x^2} + \frac{k_2(x^2 + y^2)}{y^2}$$

$$B_1 = \left(J_1 p_y - J_2 p_x - 2z \left(\frac{-k}{2\sqrt{x^2 + y^2 + z^2}} + \frac{k_1}{x^2} + \frac{k_2}{y^2} + \frac{k_3}{z^2} \right) \right)^2 + \frac{2k_3}{z^2} (x p_x + y p_y + z p_z)^2$$

$$B_2 = \frac{1}{2}J_2^2 + \frac{k_1 z^2}{x^2} + \frac{k_3 x^2}{z^2}$$

$$F = \left(-J_1 p_z + J_3 p_x - 2y \left(\frac{-k}{2\sqrt{x^2 + y^2 + z^2}} + \frac{k_1}{x^2} + \frac{k_2}{y^2} + \frac{k_3}{z^2} \right) \right)^2 + \frac{2k_2}{y^2} (x p_x + y p_y + z p_z)^2$$

