Vortices and Polynomials

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- Polynomials associated with rational solutions of the **second** and **fourth Painlevé equations**

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha \qquad \mathrm{P_{II}}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2-\alpha)w + \frac{\beta}{w} \qquad \mathrm{P_{IV}}$$

- Polynomials associated with rational solutions of some soliton equations including the **Korteweg-de Vries, nonlinear Schrödinger** and **Boussinesq equations**

$$u_t + 6uu_x + u_{xxx} = 0$$
$$\mathrm{i}u_t = u_{xx} - 2|u|^2u$$
$$u_{tt} + (u^2)_{xx} + \tfrac{1}{3}u_{xxxx} = 0$$

- The equations of motion for n point vortices with circulations Γ_j at positions z_j, in a background flow $w(z)$ are

$$\frac{\mathrm{d}z_j^*}{\mathrm{d}t} = \frac{1}{2\pi\mathrm{i}}\sum_{k=1}^{n}{}' \frac{\Gamma_k}{z_j - z_k} + \frac{w^*(z_j)}{2\pi\mathrm{i}}, \qquad j = 1, 2, \ldots, n$$

- Polynomial solutions of

$$\frac{\mathrm{d}^2P}{\mathrm{d}z^2}Q - 2\frac{\mathrm{d}P}{\mathrm{d}z}\frac{\mathrm{d}Q}{\mathrm{d}z} + P\frac{\mathrm{d}^2Q}{\mathrm{d}z^2} + 2\mu z\left(\frac{\mathrm{d}P}{\mathrm{d}z}Q - P\frac{\mathrm{d}Q}{\mathrm{d}z}\right) = 2\kappa PQ$$

Rational Solutions of the Second Painlevé Equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha \qquad \mathrm{P_{II}}$$

Theorem (**Yablonskii & Vorob'ev [1965]**)

$\mathrm{P_{II}}$ has rational solutions if and only if $\alpha = n$ with $n \in \mathbb{Z}$.

Theorem (**Kajiwara & Ohta [1996]**)

Define the polynomial $\varphi_k(z)$ by

$$\sum_{j=0}^{\infty} \varphi_j(z)\lambda^j = \exp\left(z\lambda - \tfrac{4}{3}\lambda^3\right)$$

then the **Yablonskii–Vorob'ev polynomials** *are given by*

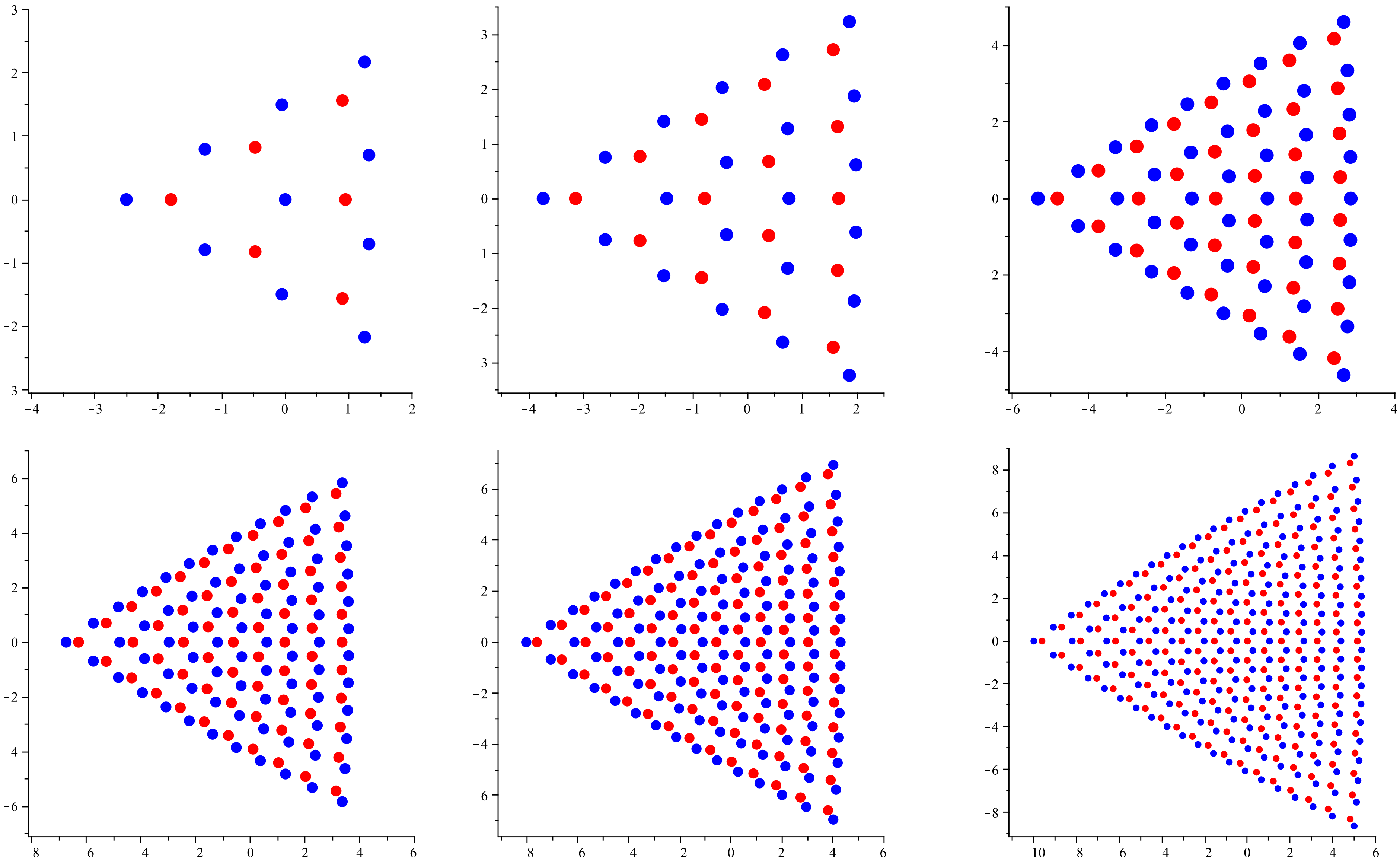
$$Q_n(z) = c_n \mathcal{W}(\varphi_1, \varphi_3, \ldots, \varphi_{2n-1})$$

where $\mathcal{W}(\varphi_1, \varphi_3, \ldots, \varphi_{2n-1})$ is the Wronskian and c_n a constant, and

$$w_n(z) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{Q_{n-1}(z)}{Q_n(z)} = \frac{Q'_{n-1}(z)}{Q_{n-1}(z)} - \frac{Q'_n(z)}{Q_n(z)}$$

satisfies $\mathrm{P_{II}}$ with $\alpha = n$.

Roots of some Yablonskii–Vorob’ev polynomials



Rational Solutions of the Second Painlevé Hierarchy
(PAC & Mansfield [2003])

The first three equations in the **second Painlevé hierarchy** are

$$w'' = 2w^3 + zw + \alpha_1 \qquad \mathrm{P}_{\mathrm{II}}^{[1]}$$

$$w'''' = 10w^2w'' + 10w\left(w'\right)^2 - 6w^5 + zw + \alpha_2 \qquad \mathrm{P}_{\mathrm{II}}^{[2]}$$

$$\begin{aligned} w'''''' &= 14w^2w'''' + 56ww'w''' + 42w\left(w''\right)^2 - 70\left[w^4 - \left(w'\right)^2\right]w'' \\ &\quad - 140w^3\left(w'\right)^2 + 20w^7 + zw + \alpha_3 \end{aligned} \qquad \mathrm{P}_{\mathrm{II}}^{[3]}$$

with α_1, α_2 and α_3 arbitrary constants. Rational solutions of $\mathrm{P}_{\mathrm{II}}^{[n]}$ have the form

$$w_n^{[n]}(z) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{Q_{n-1}^{[n]}(z)}{Q_n^{[n]}(z)}$$

for $n \geq 1$, where $Q_n^{[n]}(z)$ are monic polynomials of degree $\frac{1}{2}n(n+1)$ which can be expressed in the form of determinants (Wronskians) whose coefficients involve hypergeometric functions of the form

$$_1F_{2n}(a; b_1, b_2, \ldots, b_{2n}; \zeta), \qquad \zeta = \frac{z^{2n+1}}{(4n+2)^{2n}}$$

Roots of Polynomials Associated with the P_{II} Hierarchy

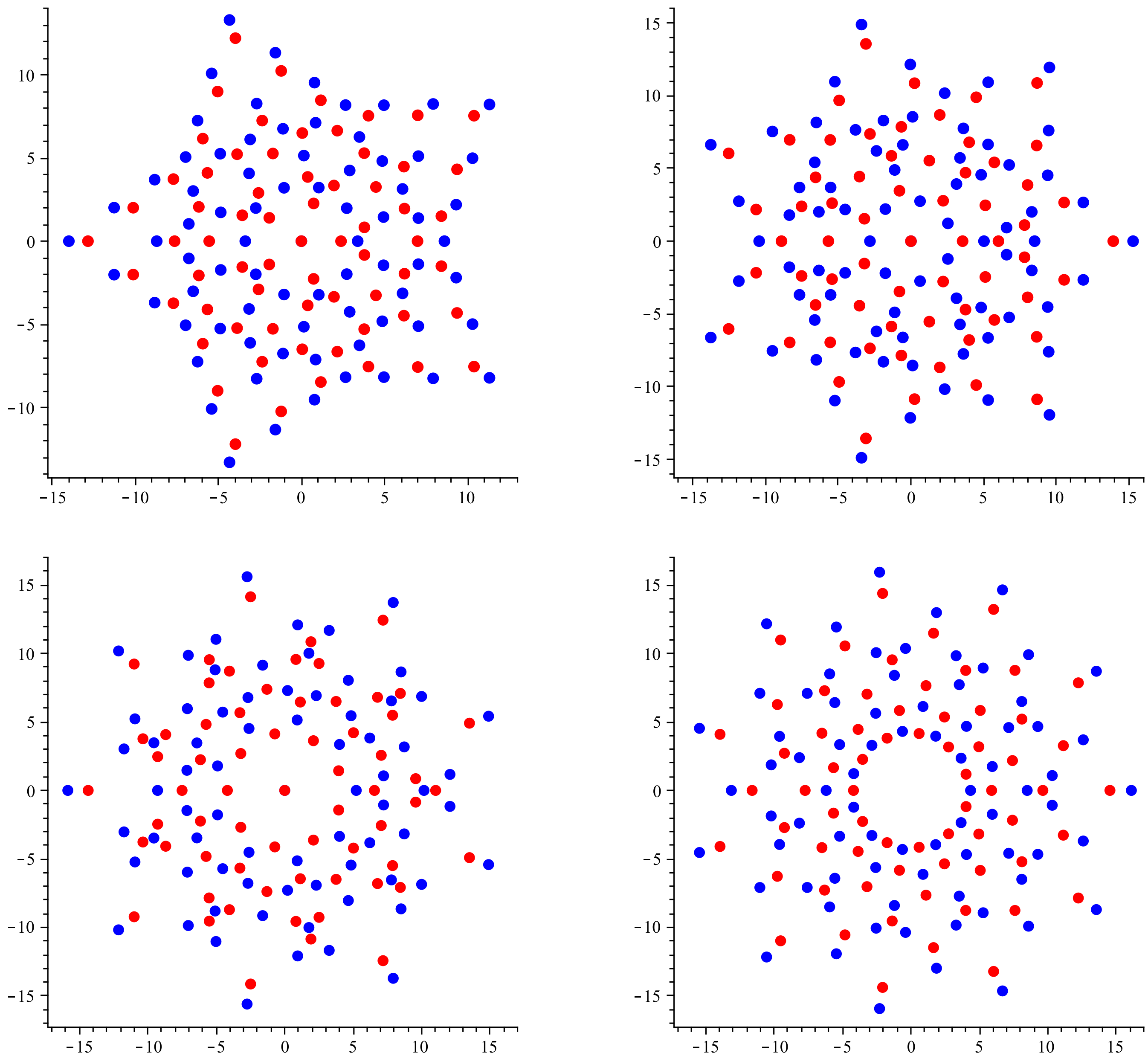
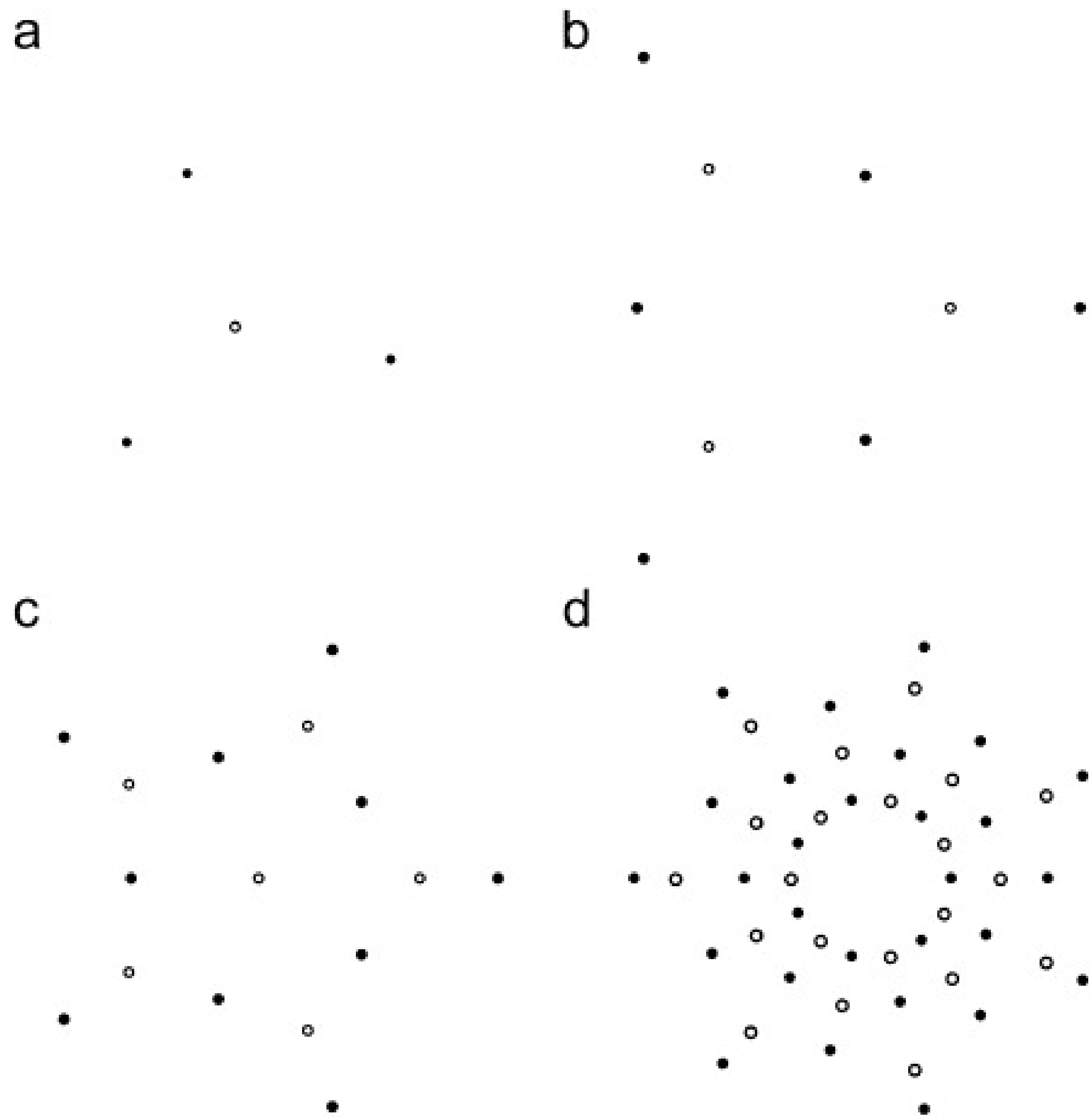


Figure 5 in H. Aref [*Fluid Dyn. Res.*, 39 (2007) pp5–23]



Rational Solutions of the Korteweg-de Vries Equation

The **Korteweg-de Vries** (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0$$

which is the best known example of a soliton equation solvable by inverse scattering (**Gardner, Green, Kruskal & Miura [1967]**), has the scaling reduction

$$u(x,t) = W(z)/(3t)^{2/3}, \qquad z = x/(3t)^{1/3}$$

where $W(z)$ satisfies

$$\frac{\mathrm{d}^3W}{\mathrm{d}z^3} + 6W\frac{\mathrm{d}W}{\mathrm{d}z} = 2W + z\frac{\mathrm{d}W}{\mathrm{d}z} \tag{1}$$

whose solution is expressible in terms of the solution w of $\mathrm{P_{II}}$ (**Fokas & Ablowitz [1982]**)

$$W = -\frac{\mathrm{d}w}{\mathrm{d}z} - w^2, \qquad w = \frac{1}{2W - z}\left(\frac{\mathrm{d}W}{\mathrm{d}z} + \alpha\right)$$

It can be shown that rational solutions of (1) have the form

$$W_n(z) = 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln Q_n(z)$$

where $Q_n(z)$ are the Yablonskii–Vorob'ev polynomials and so

$$u(x,t) = \frac{2}{(3t)^{2/3}}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln Q_n(z), \qquad z = x/(3t)^{1/3}$$

Generalized Rational Solutions of the KdV Equation

Theorem

Define the polynomials $\varphi_n(x;\boldsymbol{\kappa}_n)$ *by*

$$\sum_{n=0}^{\infty} \varphi_n(x;\boldsymbol{\kappa}_n)\lambda^n = \exp\left(x\lambda - \sum_{j=2}^{\infty} \frac{\kappa_j \lambda^{2j-1}}{2j-1}\right)$$

where $\boldsymbol{\kappa}_n = (\kappa_2, \kappa_3, \ldots, \kappa_n)$, *with* $\kappa_2, \kappa_3, \ldots, \kappa_n$ *arbitrary constants, and then define*

$$\Theta_n(x;\boldsymbol{\kappa}_n) = c_n \mathcal{W}_x(\varphi_1, \varphi_3, \ldots, \varphi_{2n-1})$$

where $\mathcal{W}_x(\varphi_1, \varphi_3, \ldots, \varphi_{2n-1})$ *is the Wronskian with respect to* x *and* c_n *is a constant, which is a polynomial in* x *of degree* $\frac{1}{2}n(n+1)$. *Then the KdV equation has rational solutions in the form*

$$u(x,t) = 2\frac{\partial^2}{\partial x^2} \ln \Theta_n(x; 12t, \kappa_3, \kappa_4, \ldots, \kappa_n)$$

The polynomials $\Theta_n(x;\boldsymbol{\kappa}_n)$ are known as the **Adler–Moser polynomials**, or **Burchnall–Chaundy polynomials**, which are generalizations of the **Yablonskii–Vorob'ev polynomials** and, as we shall see later, arise in the description of stationary vortex patterns.

Rational Solutions of the Fourth Painlevé Equation

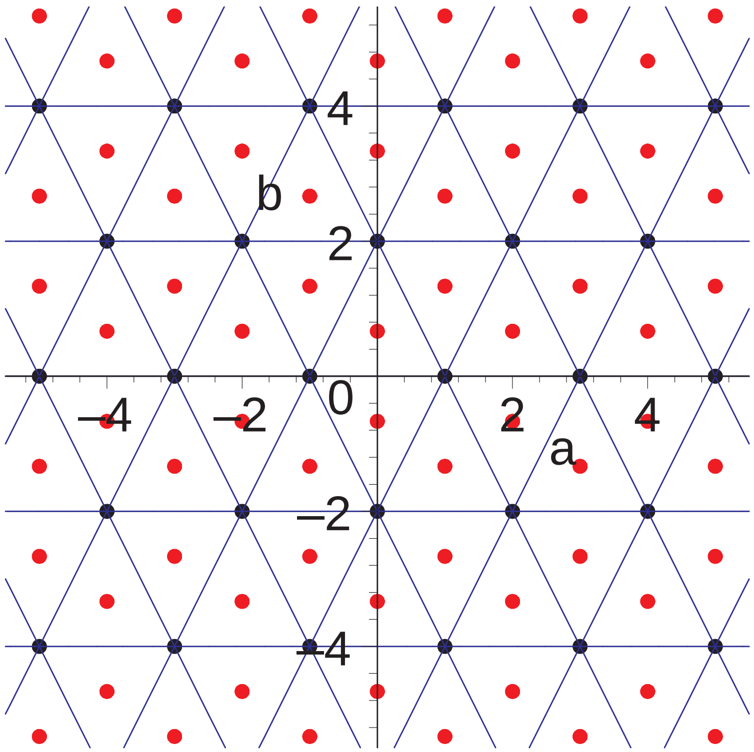
$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2-\alpha)w + \frac{\beta}{w} \qquad \mathrm{P_{IV}}$$

Theorem

$\mathrm{P_{IV}}$ *has rational solutions if and only if*

$$(\boldsymbol{\alpha},\boldsymbol{\beta}) = \left(\boldsymbol{m}, -2(2\boldsymbol{n}-\boldsymbol{m}+1)^2\right) \quad \textit{or} \quad (\boldsymbol{\alpha},\boldsymbol{\beta}) = \left(\boldsymbol{m}, -2(2\boldsymbol{n}-\boldsymbol{m}+\tfrac{1}{3})^2\right)$$

with $m,n\in\mathbb{Z}$*. Further the rational solutions for these parameter values are unique.*



$$a=\alpha, \qquad b=\sqrt{-2\beta^2}$$

$\mathrm{P_{IV}}$ — Generalized Hermite Polynomials

Theorem (**Kajiwara & Ohta [1998], Noumi & Yamada [1998]**)

Define the generalized Hermite polynomial $H_{m,n}(z)$, which has degree mn, by

$$H_{m,n}(z) = a_{m,n}\,\mathcal{W}\left(H_m(z), H_{m+1}(z), \ldots, H_{m+n-1}(z)\right), \qquad m, n \geq 1$$

where $\mathcal{W}(\varphi_1, \varphi_2, \ldots, \varphi_n)$ is the Wronskian, $H_n(z)$ is the n^{th} Hermite polynomial and $a_{m,n}$ is a constant. Then

$$w_{m,n}^{(\mathrm{i})} = w(z; \alpha_{m,n}^{(\mathrm{i})}, \beta_{m,n}^{(\mathrm{i})}) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{H_{m+1,n}}{H_{m,n}}$$

$$w_{m,n}^{(\mathrm{ii})} = w(z; \alpha_{m,n}^{(\mathrm{ii})}, \beta_{m,n}^{(\mathrm{ii})}) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{H_{m,n}}{H_{m,n+1}}$$

$$w_{m,n}^{(\mathrm{iii})} = w(z; \alpha_{m,n}^{(\mathrm{iii})}, \beta_{m,n}^{(\mathrm{iii})}) = -2z + \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{H_{m,n+1}}{H_{m+1,n}}$$

are respectively solutions of $\mathrm{P_{IV}}$ for

$$(\alpha_{m,n}^{(\mathrm{i})}, \beta_{m,n}^{(\mathrm{i})}) = (2m + n + 1, -2n^2)$$

$$(\alpha_{m,n}^{(\mathrm{ii})}, \beta_{m,n}^{(\mathrm{ii})}) = (-m - 2n - 1, -2m^2)$$

$$(\alpha_{m,n}^{(\mathrm{iii})}, \beta_{m,n}^{(\mathrm{iii})}) = (n - m, -2(m + n + 1)^2)$$

$\mathrm{P_{IV}}$ — Generalized Okamoto Polynomials

Theorem (**Kajiwara & Ohta [1998], Noumi & Yamada [1998], PAC [2006]**)

Let $\varphi_k(z) = 3^{k/2}\mathrm{e}^{-k\pi\mathrm{i}/2}H_k\left(\tfrac{1}{3}\sqrt{3}\,\mathrm{i}z\right)$, *with* $H_k(\zeta)$ *the* k^{th} *Hermite polynomial, then define the generalized Okamoto polynomial* $Q_{m,n}(z)$ *by*

$$Q_{m,n}(z) = \mathcal{W}(\varphi_1, \varphi_4, \ldots, \varphi_{3m+3n-5}; \varphi_2, \varphi_5, \ldots, \varphi_{3n-4})$$

with $m, n \geq 1$, *where* $\mathcal{W}(\varphi_1, \varphi_2, \ldots, \varphi_n)$ *is the Wronskian. Then*

$$\widetilde{w}^{(\mathrm{i})}_{m,n} = w(z; \widetilde{\alpha}^{(\mathrm{i})}_{m,n}, \widetilde{\beta}^{(\mathrm{i})}_{m,n}) = -\tfrac{2}{3}z + \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{Q_{m+1,n}}{Q_{m,n}}$$

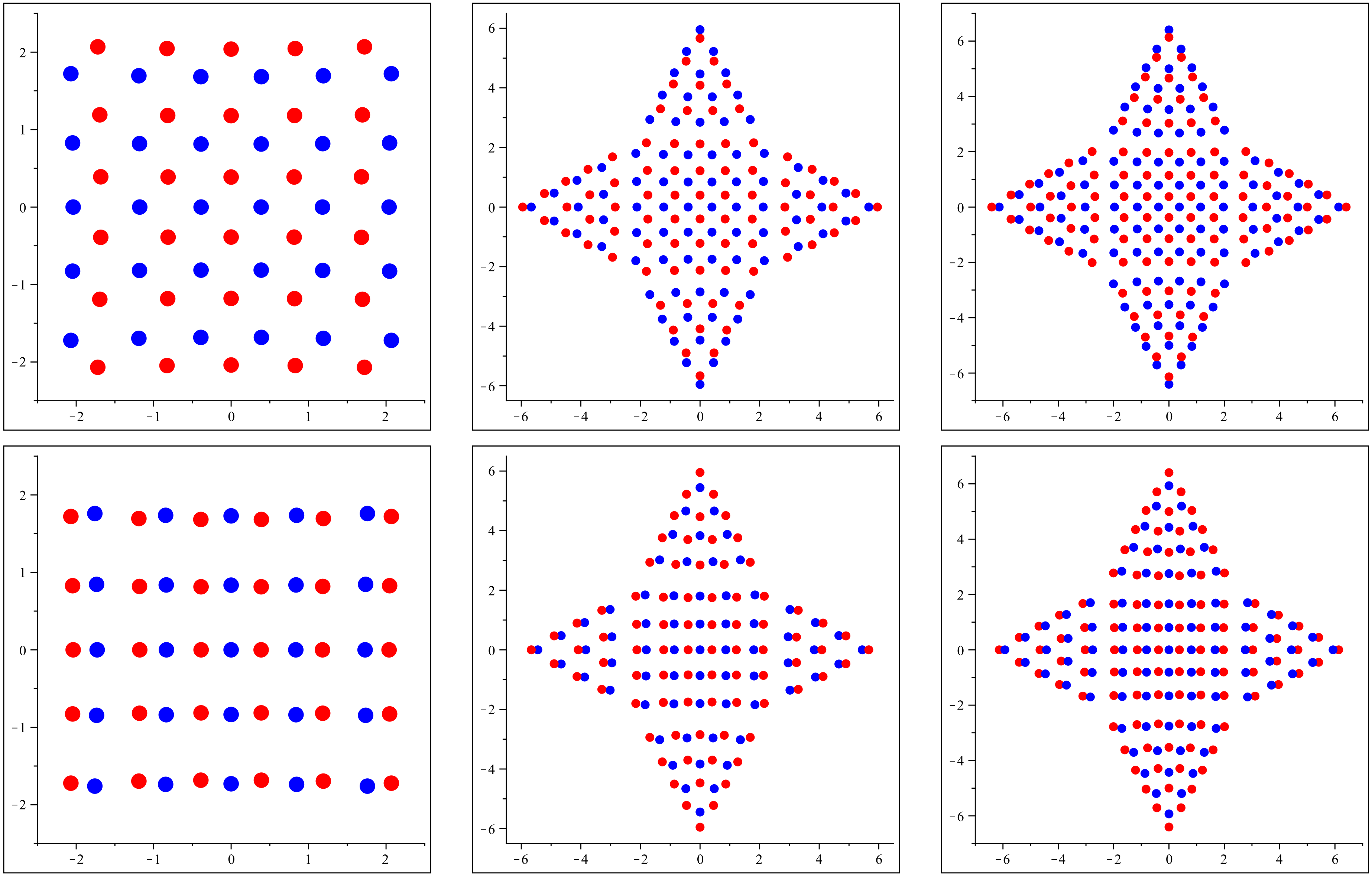
$$\widetilde{w}^{(\mathrm{ii})}_{m,n} = w(z; \widetilde{\alpha}^{(\mathrm{ii})}_{m,n}, \widetilde{\beta}^{(\mathrm{ii})}_{m,n}) = -\tfrac{2}{3}z + \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{Q_{m,n}}{Q_{m,n+1}}$$

$$\widetilde{w}^{(\mathrm{iii})}_{m,n} = w(z; \widetilde{\alpha}^{(\mathrm{iii})}_{m,n}, \widetilde{\beta}^{(\mathrm{iii})}_{m,n}) = -\tfrac{2}{3}z + \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{Q_{m,n+1}}{Q_{m+1,n}}$$

are respectively solutions of $\mathrm{P_{IV}}$ *for*

$$(\widetilde{\alpha}^{(\mathrm{i})}_{m,n}, \widetilde{\beta}^{(\mathrm{i})}_{m,n}) = (2m+n, -2(n-\tfrac{1}{3})^2)$$
$$(\widetilde{\alpha}^{(\mathrm{ii})}_{m,n}, \widetilde{\beta}^{(\mathrm{ii})}_{m,n}) = (-m-2n, -2(m-\tfrac{1}{3})^2)$$
$$(\widetilde{\alpha}^{(\mathrm{iii})}_{m,n}, \widetilde{\beta}^{(\mathrm{iii})}_{m,n}) = (n-m, -2(m+n+\tfrac{1}{3})^2)$$

Roots of the Generalized Hermite and Okamoto Polynomials



Scaling Reduction of the Nonlinear Schrödinger Equation

The de-focusing NLS equation

$$\mathrm{i}u_t = u_{xx} - 2|u|^2 u$$

has the scaling reduction

$$u(x,t) = t^{-1/2} R(\zeta) \exp\{\mathrm{i}\Theta(\zeta)\}, \qquad \zeta = x/t^{1/2}$$

where $V(\zeta) = \int^{\zeta} R^2(s)\,\mathrm{d}s$ satisfies

$$\left(\frac{\mathrm{d}^2 V}{\mathrm{d}\zeta^2}\right)^2 = -\frac{1}{4}\left(V - \zeta\frac{\mathrm{d}V}{\mathrm{d}\zeta}\right)^2 + 4\left(\frac{\mathrm{d}V}{\mathrm{d}\zeta}\right)^3 + K\frac{\mathrm{d}V}{\mathrm{d}\zeta} \qquad (1)$$

with K an arbitrary integration constant, which is solvable in terms of $\mathrm{P_{IV}}$ provided that

$$K = \tfrac{1}{9}(\alpha+1)^2, \qquad \beta = -\tfrac{2}{9}(\alpha+1)^2$$

Equation (1) has rational solutions

$$V_n(\zeta) = -\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln H_{n,n}(\tfrac{1}{2}\mathrm{e}^{-\pi\mathrm{i}/4}\zeta)$$

$$\widetilde{V}_n(\zeta) = \frac{\zeta^3}{108} - \frac{\mathrm{d}}{\mathrm{d}\zeta}\ln Q_{n,n}(\tfrac{1}{2}\mathrm{e}^{-\pi\mathrm{i}/4}\zeta)$$

for $K_n = n^2$ and $\widetilde{K}_n = (n-\frac{1}{3})^2$, respectively.

Rational and Rational-Oscillatory Solutions of the NLS Equation

Theorem (**PAC [2006]**)

The de-focusing NLS equation

$$\mathrm{i}u_t = u_{xx} - 2|u|^2 u \qquad (1)$$

has decaying rational solutions of the form

$$u_n(x,t) = \frac{n\mathrm{e}^{\pi\mathrm{i}/4}}{\sqrt{t}} \frac{H_{n+1,n-1}(z)}{H_{n,n}(z)}, \qquad z = \frac{x\,\mathrm{e}^{\pi\mathrm{i}/4}}{2\sqrt{t}} \qquad (2)$$

and non-decaying rational-oscillatory solutions of the forms

$$\widetilde{u}_n(x,t) = \frac{\mathrm{e}^{-\pi\mathrm{i}/4}}{3\sqrt{2t}} \frac{Q_{n+1,n-1}(z)}{Q_{n,n}(z)} \exp\left(-\frac{\mathrm{i}x^2}{6t}\right), \qquad z = \frac{x\,\mathrm{e}^{\pi\mathrm{i}/4}}{2\sqrt{t}} \qquad (3)$$

where $n \geq 1$.

Remarks:

- The rational solutions (2) generalize the results of **Hirota & Nakamura [1985]** (see also **Boiti & Pempinelli [1981], Hone [1996]**).
- The rational-oscillatory solutions (3) are new solutions of the NLS equation (1).

Generalized Rational Solutions of the NLS Equation

Theorem

(**PAC [2006]**)

Define the polynomials $\Phi_m(x,t;\boldsymbol{\kappa}_m)$, *with* $\boldsymbol{\kappa}_m=(\kappa_3,\kappa_4,\ldots,\kappa_m)$, *through*

$$\sum_{m=0}^{\infty}\frac{\Phi_m(x,t;\boldsymbol{\kappa}_m)\lambda^m}{m!}=\exp\left\{x\lambda-\mathrm{i}t\lambda^2+\mathrm{i}\sum_{j=3}^{\infty}\frac{\kappa_j(-\mathrm{i}\lambda)^j}{j!}\right\}$$

where $\boldsymbol{\kappa}_m=(\kappa_3,\kappa_4,\ldots,\kappa_m)$, *with* κ_j, *for* $j\geq 3$, *arbitrary constants, and then define*

$$G_n(x,t;\boldsymbol{\kappa}_{2n-1})=a_n\mathcal{W}(\Phi_{n-1},\Phi_n,\ldots,\Phi_{2n-1}),\qquad a_n=\prod_{m=1}^{n}\frac{1}{m!}$$
$$F_n(x,t;\boldsymbol{\kappa}_{2n-1})=a_{n-1}\mathcal{W}(\Phi_n,\Phi_{n+1},\ldots,\Phi_{2n-1}),$$

so $G_n(x,t;\boldsymbol{\kappa}_{2n-1})$ *and* $F_n(x,t;\boldsymbol{\kappa}_{2n-1})$ *are monic polynomials in* x *of degrees* n^2-1 *and* n^2, *respectively, with coefficients which are polynomials in* t *and the arbitrary parameters* $\boldsymbol{\kappa}_{2n-1}=(\kappa_3,\kappa_4,\ldots,\kappa_{2n-1})$. *Then the de-focusing NLS equation has rational solutions in the form*

$$u_n(x,t;\boldsymbol{\kappa}_{2n-1})=\frac{nG_n(x,t;\boldsymbol{\kappa}_{2n-1})}{F_n(x,t;\boldsymbol{\kappa}_{2n-1})}$$

Remark

- The first few of these generalized rational solutions were obtained by **Hone [1996]** by applying Crum transformations successively to the associated linear problem.

Scaling Reduction of the Boussinesq Equation

The Boussinesq equation

$$u_{tt} + (u^2)_{xx} + \tfrac{1}{3}u_{xxxx} = 0 \tag{1}$$

has the scaling reduction

$$u(x,t) = \frac{w(z)}{8\,t}, \qquad z = \frac{x}{(\frac{4}{3}t)^{1/2}}$$

where $v(z)$ satisfies

$$\frac{\mathrm{d}^4 w}{\mathrm{d}z^4} + w\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{4z^2}{3}\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \frac{28z}{3}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{32}{3}w = 0$$

which is solvable in terms of $\mathrm{P_{IV}}$. This has rational solutions

$$w_{m,n}(z) = -\frac{16z^2}{3} + 8(m-n) + 12\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln H_{m,n}(z)$$
$$\widetilde{w}_{m,n}(z) = 12\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln Q_{m,n}(z)$$

with $H_{m,n}(z)$ and $Q_{m,n}(z)$ the generalized Hermite and Okamoto polynomials, and so we obtain rational solutions of the Boussinesq equation (1) in the form

$$u_{m,n}(x,t) = -\frac{x^2}{2t^2} + \frac{m-n}{t} + \frac{3}{2t}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln H_{m,n}(z),$$
$$\widetilde{u}_{m,n}(x,t) = \frac{3}{2t}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln Q_{m,n}(z), \qquad z = \frac{x}{(\frac{4}{3}t)^{1/2}}$$

Rational Solutions of the Boussinesq Equation

$$u_{tt} + (u^2)_{xx} + \tfrac{1}{3}u_{xxxx} = 0 \tag{1}$$

Theorem (PAC [2008])

Define the polynomials

$$\sum_{n=0}^{\infty} \varphi_n(x,t)\lambda^n = \exp\left(x\lambda - \tfrac{1}{3}t\lambda^2\right), \qquad \psi_n(x,t) = \varphi_n(x,-3t)$$

and

$$\begin{aligned}\Phi_{m,n}(x,t) &= \mathcal{W}_x(\varphi_m, \varphi_{m+1}, \ldots, \varphi_{m+n-1})\\ \Psi_{m,n}(x,t) &= \mathcal{W}_x(\psi_1, \psi_4, \ldots, \psi_{3m+3n-5}, \psi_2, \psi_5, \ldots, \psi_{3n-4})\end{aligned}$$

for $m, n \geq 0$. *Then the Boussinesq equation (1) has rational solutions in the form*

$$\begin{aligned}u_{m,n}(x,t) &= -\frac{x^2}{2t^2} + \frac{m-n}{t} + 2\frac{\partial^2}{\partial x^2}\ln\Phi_{m,n}(x,t)\\ \widetilde{u}_{m,n}(x,t) &= 2\frac{\partial^2}{\partial x^2}\ln\Psi_{m,n}(x,t)\end{aligned}$$

which are obtained through the scaling reduction of the Boussinesq equation (1) to $\mathrm{P_{IV}}$.

Generalized Rational Solutions of the Boussinesq Equation

$$u_{tt} + (u^2)_{xx} + \tfrac{1}{3}u_{xxxx} = 0 \qquad (1)$$

Theorem

(**PAC [2008]**)

Define the polynomials $\vartheta_n(x,t;\boldsymbol{\kappa}_n)$ *by*

$$\sum_{n=0}^{\infty} \vartheta_n(x,t;\boldsymbol{\kappa}_n)\lambda^n = \exp\left(x\lambda + t\lambda^2 + \sum_{j=3}^{\infty} \kappa_j \lambda^j \right)$$

with $\boldsymbol{\kappa}_n = (\kappa_3, \kappa_4, \ldots, \kappa_n)$, *and then define*

$$\Theta_{m,n}(x,t;\boldsymbol{\kappa}_{m,n}) = \mathcal{W}_x\left(\vartheta_1, \vartheta_4, \ldots, \vartheta_{3m+3n-5}, \vartheta_2, \vartheta_5, \ldots, \vartheta_{3n-4}\right)$$

Then the Boussinesq equation (1) has decaying real rational solutions in the form

$$u_{m,n}(x,t;\boldsymbol{\kappa}_{m,n}) = 2\frac{\partial^2}{\partial x^2} \ln \Theta_{m,n}(x,t;\boldsymbol{\kappa}_{m,n})$$

However there are other rational solutions of the Boussinesq equation (1), e.g.

$$u(x,t) = -\tfrac{1}{2} - \frac{4(x^2+t^2-1)}{(x^2-t^2+1)^2}$$

which was obtained by **Ablowitz & Satsuma [1978]** by taking a long-wave limit of the known 2-soliton solution.

Rational Solutions of the modified Boussinesq Equation

The modified Boussinesq equation

$$3u_{tt} + 6u_t u_{xx} - 6u_x^2 u_{xx} + u_{xxxx} = 0 \tag{1}$$

has a scaling reduction which is solvable in terms of $\mathrm{P_{IV}}$.

Theorem **(Thomas & PAC [2008])**

Define the polynomials

$$\sum_{n=0}^{\infty} \varphi_n(x,t)\lambda^n = \exp\left(x\lambda + \tfrac{1}{3}t\lambda^2\right), \qquad \psi_n(x,t) = \varphi_n(x,-3t)$$

and

$$\Phi_{m,n}(x,t) = \mathcal{W}_x(\varphi_m, \varphi_{m+1}, \ldots, \varphi_{m+n-1})$$
$$\Psi_{m,n}(x,t) = \mathcal{W}_x\left(\psi_1, \psi_4, \ldots, \psi_{3m+3n-5}, \psi_2, \psi_5, \ldots, \psi_{3n-5}\right)$$

for $m, n \geq 0$. *Then the modified Boussinesq equation (1) has the rational solutions*

$$u_{m,n}^{(\mathrm{i})}(x,t) = -\ln\frac{\Phi_{m+1,n}(x,t)}{\Phi_{m,n}(x,t)} + \frac{x^2}{4t} - (m+\tfrac{1}{2})\ln t, \qquad \widetilde{u}_{m,n}^{(\mathrm{i})}(x,t) = \ln\frac{\Psi_{m+1,n}(x,t)}{\Psi_{m,n}(x,t)}$$

$$u_{m,n}^{(\mathrm{ii})}(x,t) = -\ln\frac{\Phi_{m,n}(x,t)}{\Phi_{m,n+1}(x,t)} + \frac{x^2}{4t} + (n+\tfrac{1}{2})\ln t, \qquad \widetilde{u}_{m,n}^{(\mathrm{ii})}(x,t) = \ln\frac{\Psi_{m,n}(x,t)}{\Psi_{m,n+1}(x,t)}$$

$$u_{m,n}^{(\mathrm{iii})}(x,t) = -\ln\frac{\Phi_{m,n+1}(x,t)}{\Phi_{m+1,n}(x,t)} - \frac{x^2}{2t} + (m-n)\ln t, \qquad \widetilde{u}_{m,n}^{(\mathrm{iii})}(x,t) = \ln\frac{\Psi_{m,n+1}(x,t)}{\Psi_{m+1,n}(x,t)}$$

Generalized Rational Solutions of the Modified Boussinesq Equation

$$u_{tt} + 2u_t u_{xx} - 2u_x^2 u_{xx} + \tfrac{1}{3} u_{xxxx} = 0 \tag{1}$$

Theorem

(Thomas & PAC [2008])

Define the polynomials $\vartheta_n(x,t;\boldsymbol{\kappa}_n)$ *by*

$$\sum_{n=0}^{\infty} \vartheta_n(x,t;\boldsymbol{\kappa}_n)\lambda^n = \exp\left(x\lambda + t\lambda^2 + \sum_{j=3}^{\infty} \kappa_j \lambda^j\right)$$

with $\boldsymbol{\kappa}_n = (\kappa_3, \kappa_4, \ldots, \kappa_n)$*, and then define*

$$\mathit{\Theta}_{m,n}(x,t;\boldsymbol{\kappa}_{m,n}) = \mathcal{W}_x\left(\vartheta_1, \vartheta_4, \ldots, \vartheta_{3m+3n-5}, \vartheta_2, \vartheta_5, \ldots, \vartheta_{3n-4}\right)$$

Then the modified Boussinesq equation (1) has decaying real rational solutions in the form

$$\widetilde{u}^{(\mathrm{i})}_{m,n}(x,t;\boldsymbol{\kappa}_{m+1,n}) = \ln \frac{\mathit{\Theta}_{m+1,n}(x,t;\boldsymbol{\kappa}_{m+1,n})}{\mathit{\Theta}_{m,n}(x,t;\boldsymbol{\kappa}_{m,n})}$$

$$\widetilde{u}^{(\mathrm{ii})}_{m,n}(x,t;\boldsymbol{\kappa}_{m,n+1}) = \ln \frac{\mathit{\Theta}_{m,n}(x,t;\boldsymbol{\kappa}_{m,n})}{\mathit{\Theta}_{m,n+1}(x,t;\boldsymbol{\kappa}_{m,n+1})}$$

$$\widetilde{u}^{(\mathrm{iii})}_{m,n}(x,t;\boldsymbol{\kappa}_{m+1,n+1}) = \ln \frac{\mathit{\Theta}_{m,n+1}(x,t;\boldsymbol{\kappa}_{m,n+1})}{\mathit{\Theta}_{m+1,n}(x,t;\boldsymbol{\kappa}_{m+1,n})}$$

Complex Sine-Gordon equation

(Thomas & PAC [2008])

The 2-dimensional complex Sine-Gordon equation

$$\psi_{xx}+\psi_{yy}+\frac{(\psi_x^2+\psi_y^2)\,\overline{\psi}}{2-|\psi|^2}+\tfrac{1}{2}\psi(1-|\psi|^2)(2-|\psi|^2)=0 \qquad (1)$$

has a separable solution in polar coordinates

$$\psi(r,\varphi)=Q_n^{1/2}(r)\,\mathrm{e}^{in\varphi}$$

a so-called ***n*-vortex configuration**, where Q_n satisfies

$$\frac{\mathrm{d}^2Q_n}{\mathrm{d}r^2}=\frac{Q_n-1}{Q_n(Q_n-2)}\left(\frac{\mathrm{d}Q_n}{\mathrm{d}r}\right)^2-\frac{1}{r}\frac{\mathrm{d}Q_n}{\mathrm{d}r}-Q_n(Q_n-1)(Q_n-2)-\frac{4n^2Q_n}{r^2(Q_n-2)} \qquad (2)$$

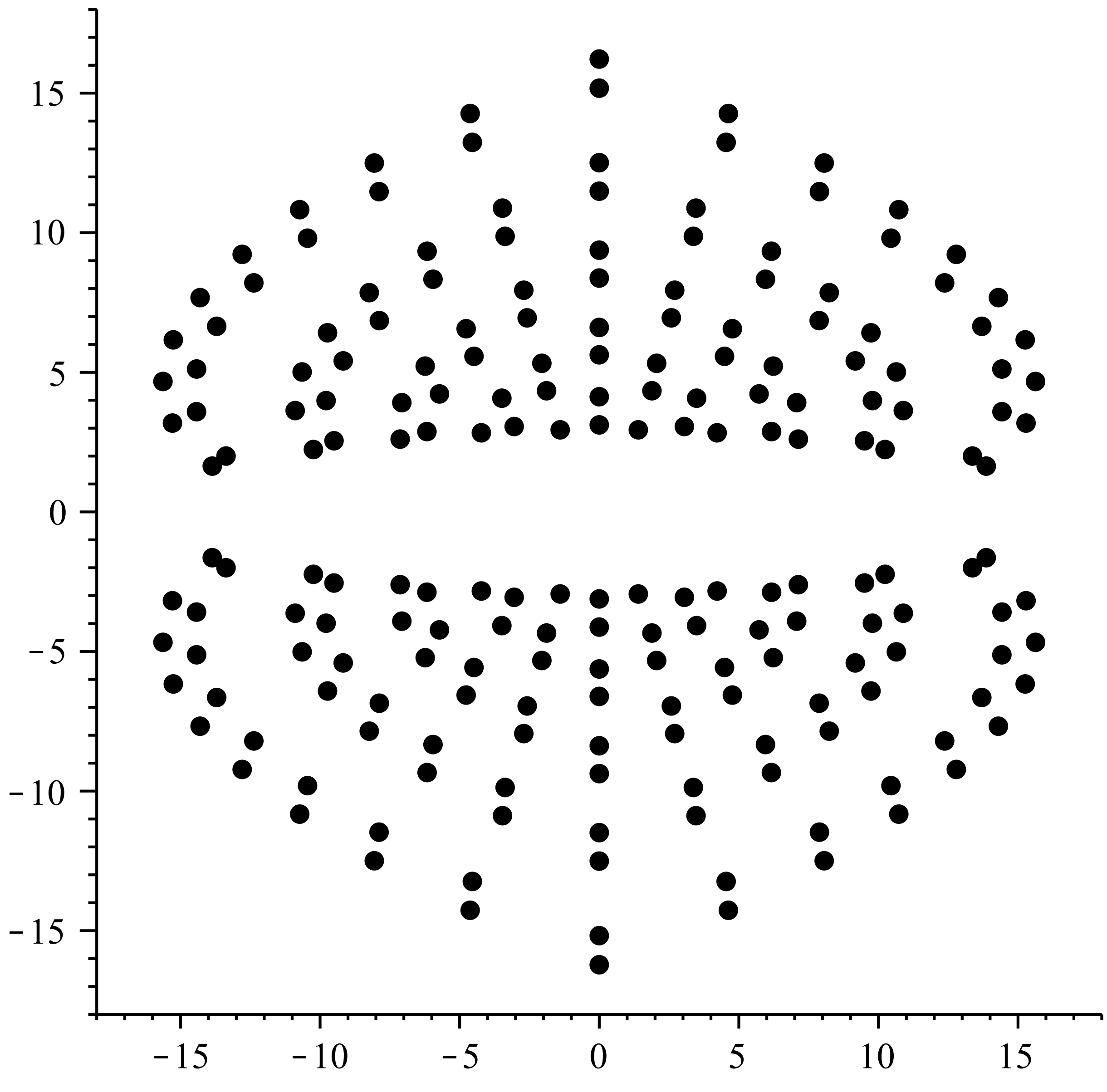
which is solvable in terms of $\mathrm{P_V}$. Rational solutions of $\mathrm{P_V}$, and so also of equation (2), can be expressed in terms of Wronskians involving **associated Laguerre polynomials**

$$L_n^{(\alpha)}(x)=\frac{x^{-\alpha}\,\mathrm{e}^{x}}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^{n+\alpha}\,\mathrm{e}^{-x}\right), \qquad k\geq 0$$

Earlier **Barashenkov & Pelinovsky [1998]** and **Olver & Barashenkov [2005]** derived a sequence of 4 Schlesinger maps to obtain Q_{n+1} from Q_n.

$$Q_0=1\rightarrow Q_1\rightarrow Q_2\rightarrow Q_3\rightarrow Q_4\rightarrow\ldots$$

Poles of Q_{10}



Vortices of the Same Strength and Mixed Signs

The equations of motion for $m+n$ point vortices with circulations Γ_j at positions z_j, are

$$\frac{\mathrm{d}z_j^*}{\mathrm{d}t} = \frac{1}{2\pi\mathrm{i}} \sum_{k=1}^{m+n}{}' \frac{\Gamma_k}{z_j - z_k}, \qquad j = 1, 2, \ldots, n$$

Suppose that the vortices at $z_1, z_2, \ldots, z_m$ have strength $\Gamma > 0$ (i.e. m positive vortices), the vortices at $z_{m+1}, z_{m+2}, \ldots, z_{m+n}$ have strength $-\Gamma$ (i.e. n negative vortices) and then define the polynomials

$$P(z) = \prod_{j=1}^{m} (z - z_j), \qquad Q(z) = \prod_{k=1}^{n} (z - z_{k+m})$$

- **Stationary vortex patterns** arise when $\dfrac{\mathrm{d}z_j^*}{\mathrm{d}t} = 0$.
- **Translating vortex patterns** arise when $\dfrac{\mathrm{d}z_j^*}{\mathrm{d}t} = v^*$, with v^* a (complex) constant.
- Both these cases are solved in terms of the **Adler–Moser polynomials**, which arose in the description of rational solutions of the KdV equation.

Quadrupole Background Flow

Lemma

(Kadtke & Campbell [1987])

The equations of motion for $m+n$ point vortices with circulations Γ_j at positions z_j in a background flow $w(z)$ are

$$\frac{\mathrm{d}z_j^*}{\mathrm{d}t} = \frac{1}{2\pi\mathrm{i}}\sum_{k=1}^{m+n}{}' \frac{\Gamma_k}{z_j - z_k} + \frac{w^*(z_j)}{2\pi\mathrm{i}}, \qquad j = 1, 2, \ldots, m+n$$

When $\dfrac{\mathrm{d}z_j^*}{\mathrm{d}t} = 0$*,* $w(z) = \Gamma\mu^* z^*$*, with* μ^* *a (complex) constant,* $\Gamma_k = \Gamma$ *for* $k = 1, 2, \ldots, m$ *and* $\Gamma_k = -\Gamma$ *for* $k = m+1, m+2, \ldots, m+n$*, then the polynomials*

$$P(z) = \prod_{j=1}^{m}(z - z_j), \qquad Q(z) = \prod_{j=1}^{n}(z - z_{j+m})$$

satisfy

$$\frac{\mathrm{d}^2P}{\mathrm{d}z^2}Q - 2\frac{\mathrm{d}P}{\mathrm{d}z}\frac{\mathrm{d}Q}{\mathrm{d}z} + P\frac{\mathrm{d}^2Q}{\mathrm{d}z^2} + 2\mu z\left(\frac{\mathrm{d}P}{\mathrm{d}z}Q - P\frac{\mathrm{d}Q}{\mathrm{d}z}\right) = 2\mu(m-n)PQ$$

Remark: If $Q = 1$ and $\mu = -1$ then P satisfies

$$\frac{\mathrm{d}^2P}{\mathrm{d}z^2} - 2z\frac{\mathrm{d}P}{\mathrm{d}z} + 2mP = 0$$

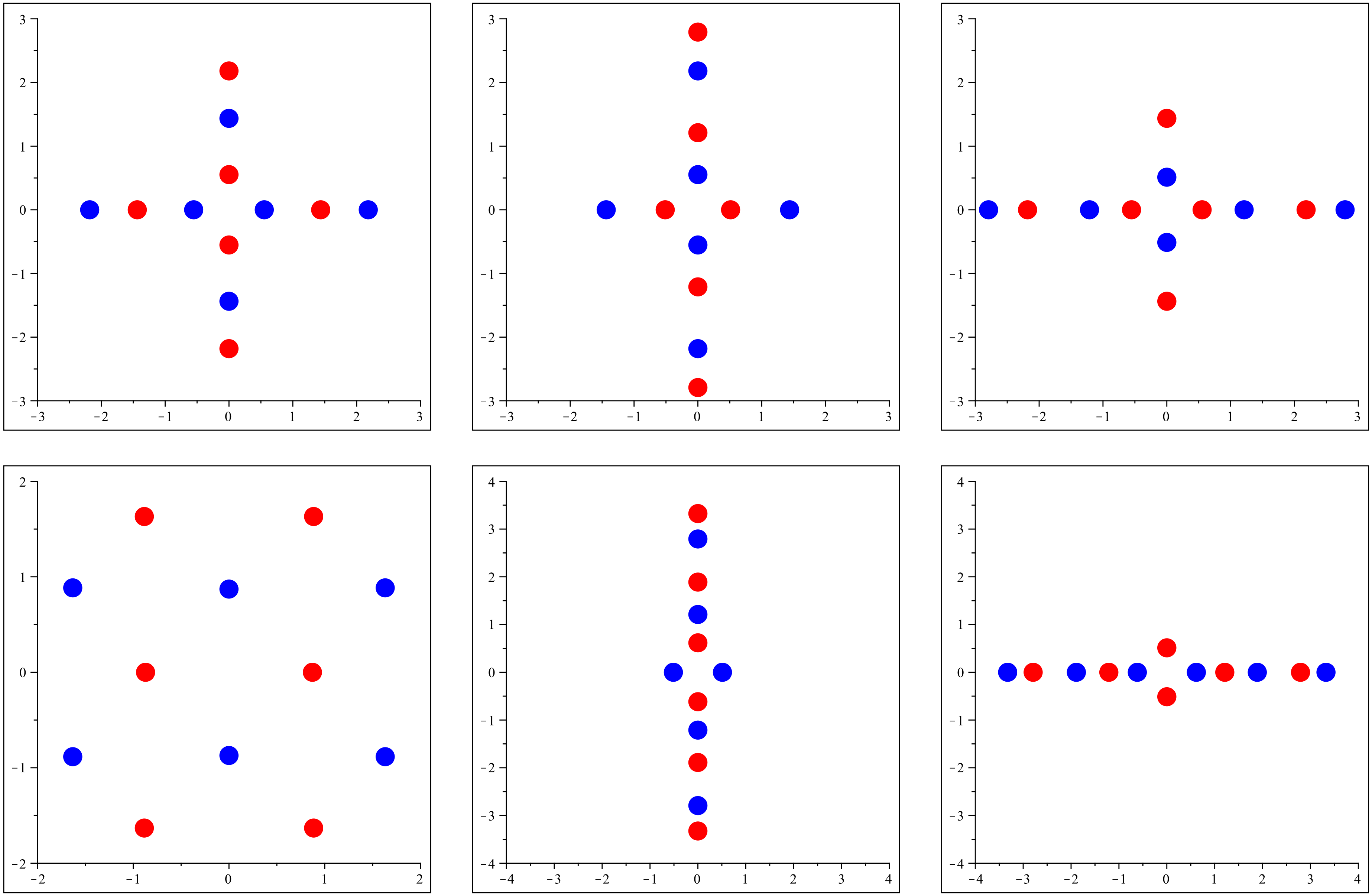
which is the equation for the m^{th} Hermite polynomial $H_m(z)$.

$$\frac{\mathrm{d}^2P}{\mathrm{d}z^2}Q - 2\frac{\mathrm{d}P}{\mathrm{d}z}\frac{\mathrm{d}Q}{\mathrm{d}z} + P\frac{\mathrm{d}^2Q}{\mathrm{d}z^2} + 2\mu z\left(\frac{\mathrm{d}P}{\mathrm{d}z}Q - P\frac{\mathrm{d}Q}{\mathrm{d}z}\right) = 2\mu(m-n)PQ$$

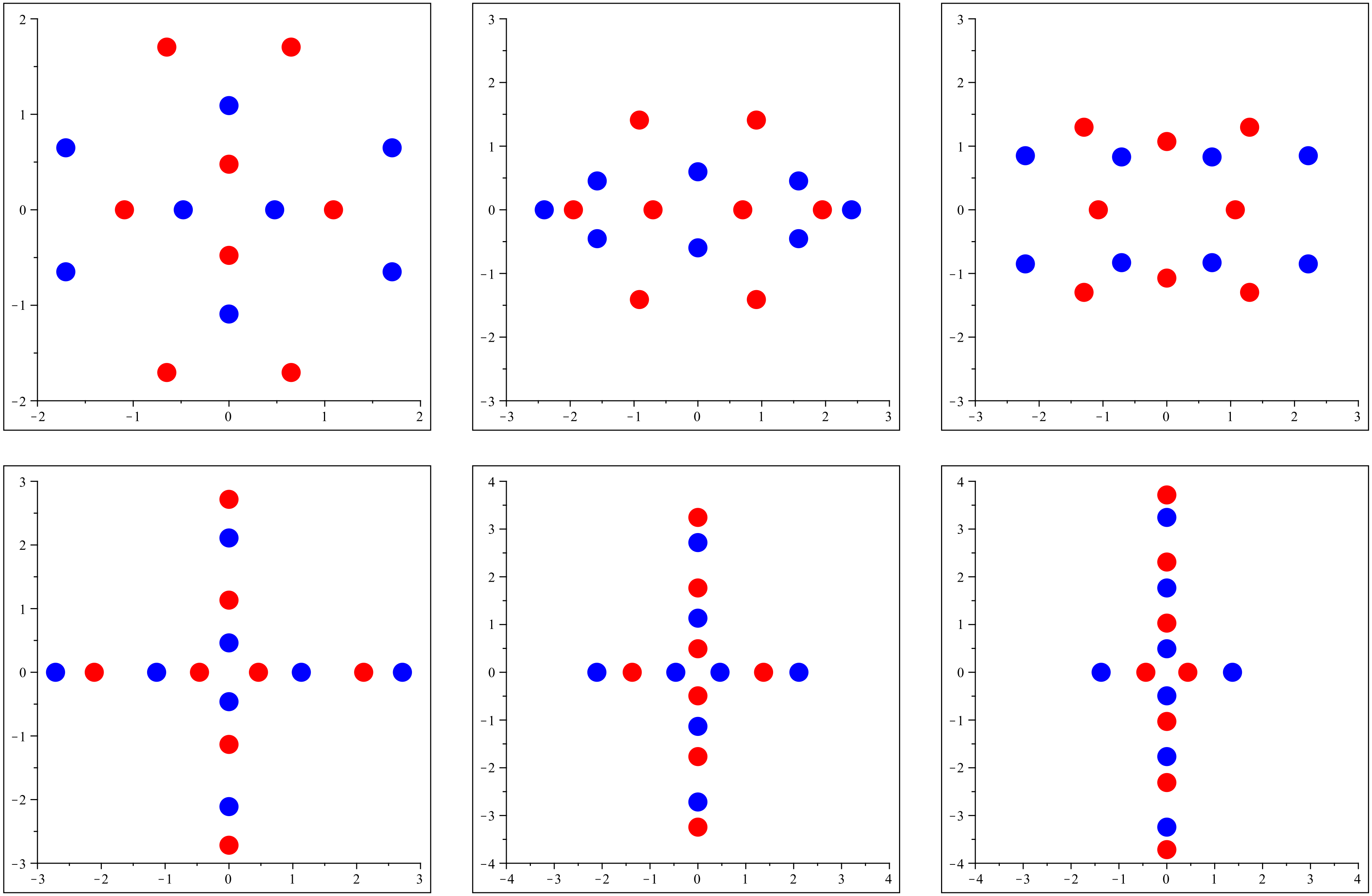
Kadtke & Campbell [1987] obtained some polynomial solutions of this equation when $m = n$, though they claimed that there were no solutions when $m = n = 6$. However, using MAPLE, it can be shown that there are solutions when $m = n = 6$.

Solutions for $\mu = \frac{1}{2}$ and $m = n$

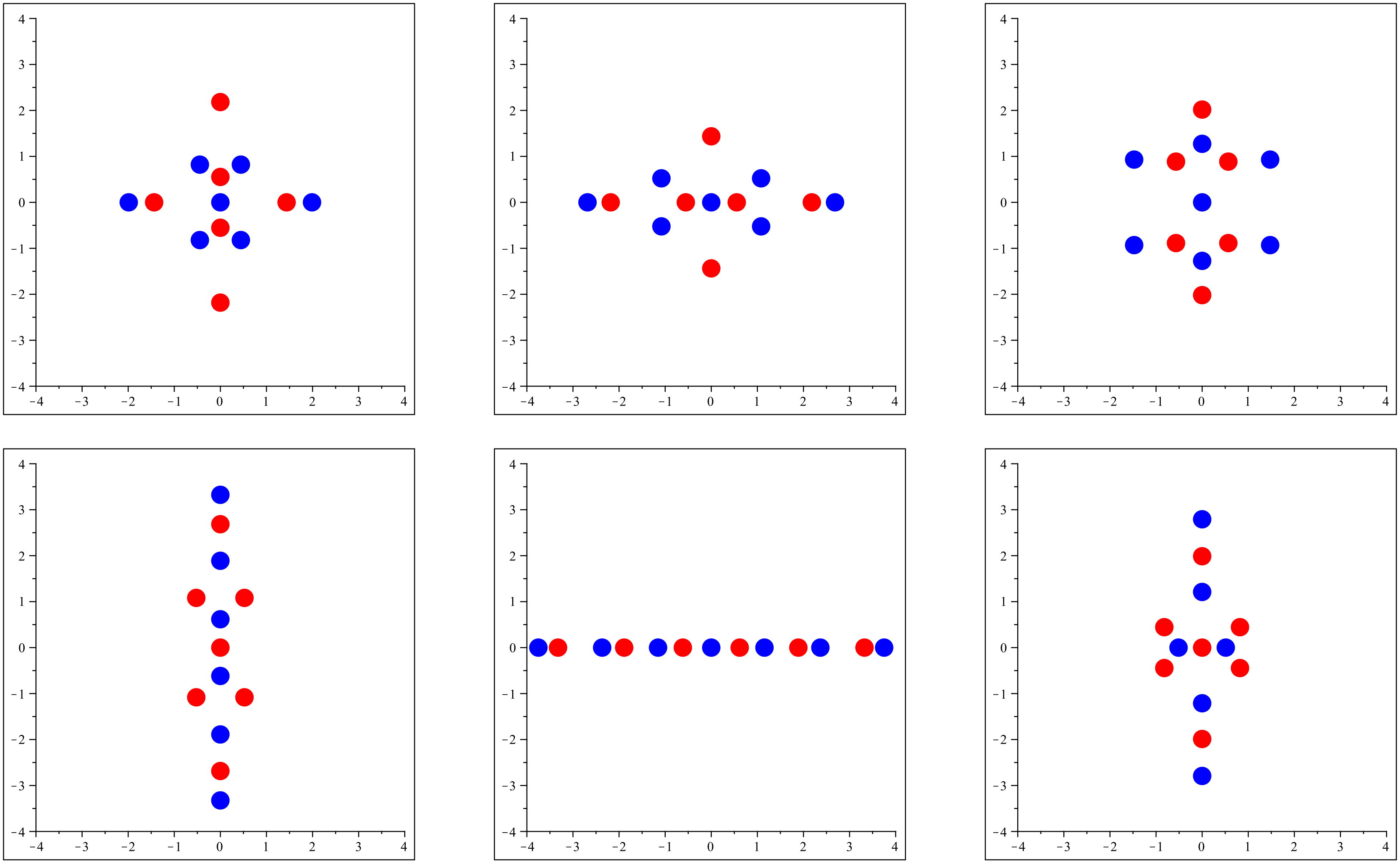
$m = n = 2$	$P = z^2 - 1$	$Q = z^2 + 1$
$m = n = 4$	$P = z^4 + 2z^2 - 1$	$Q = z^4 + 6z^2 + 3$
	$P = z^4 - 2z^2 - 1$	$Q = z^4 + 2z^2 - 1$
	$P = z^4 - 6z^2 + 3$	$Q = z^4 - 2z^2 - 1$
$m = n = 6$	$P = z^6 - 3z^4 - 9z^2 + 3$	$Q = z^6 + 3z^4 - 9z^2 + 3$
	$P = z^6 - 3z^4 + 9z^2 + 9$	$Q = z^6 + 3z^4 + 9z^2 - 9$
	$P = z^6 - 15z^4 + 45z^2 - 15$	$Q = z^6 - 9z^4 + 9z^2 + 3$
	$P = z^6 - 9z^4 + 9z^2 + 3$	$Q = z^6 - 3z^4 - 9z^2 + 3$
	$P = z^6 + 3z^4 - 9z^2 - 3$	$Q = z^6 + 9z^4 + 9z^2 - 3$
	$P = z^6 + 9z^4 + 9z^2 - 3$	$Q = z^6 + 15z^4 + 45z^2 + 15$



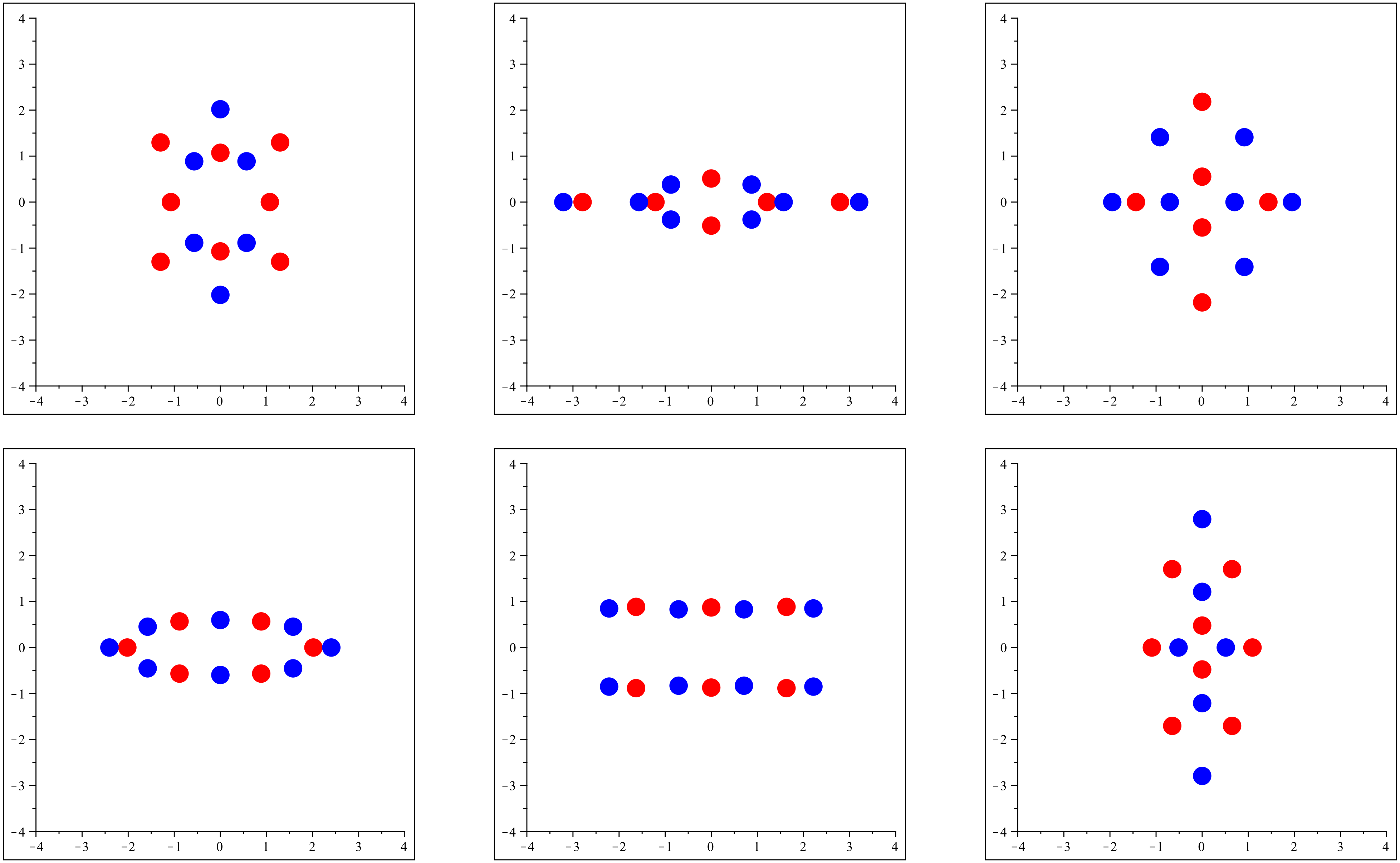
$$m = n = 6$$



$m = n = 8$



$$m = 7, n = 6$$



$$m = 8, n = 6$$

Solutions of

$$\frac{\mathrm{d}^2P}{\mathrm{d}z^2}Q - 2\frac{\mathrm{d}P}{\mathrm{d}z}\frac{\mathrm{d}Q}{\mathrm{d}z} + P\frac{\mathrm{d}^2Q}{\mathrm{d}z^2} + 2\mu z\left(\frac{\mathrm{d}P}{\mathrm{d}z}Q - P\frac{\mathrm{d}Q}{\mathrm{d}z}\right) = 2\mu(m-n)PQ$$

for $\mu = 1$ in terms of the **generalized Hermite polynomials** $H_{j,k}(z)$, and for $\mu = -\frac{1}{3}$ the **generalized Okamoto polynomials** $Q_{j,k}(z)$ are

$P(z)$	$Q(z)$	m	n	$m-n$
$H_{j+1,k}$	$H_{j,k}$	$(j+1)k$	jk	k
$H_{j,k}$	$H_{j,k+1}$	jk	$j(k+1)$	$-j$
$H_{j,k+1}$	$H_{j+1,k}$	$j(k+1)$	$(j+1)k$	$j-k$
$Q_{j+1,k}$	$Q_{j,k}$	j^2+k^2+jk+j	$j^2+k^2+jk-j-k$	$2j+k$
$Q_{j,k}$	$Q_{j,k+1}$	$j^2+k^2+jk-j-k$	j^2+k^2+jk+k	$-j-2k$
$Q_{j,k+1}$	$Q_{j+1,k}$	j^2+k^2+jk+k	j^2+k^2+jk+j	$k-j$

Question What is the form of polynomial solutions of the bilinear equation

$$\frac{\mathrm{d}^2 P}{\mathrm{d}z^2}Q - 2\frac{\mathrm{d}P}{\mathrm{d}z}\frac{\mathrm{d}Q}{\mathrm{d}z} + P\frac{\mathrm{d}^2 Q}{\mathrm{d}z^2} + 2\mu z\left(\frac{\mathrm{d}P}{\mathrm{d}z}Q - P\frac{\mathrm{d}Q}{\mathrm{d}z}\right) = 2\mu(m-n)PQ$$

Definition Consider the Schrödinger operator

$$\mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}z^2} + u(z)$$

with a potential $u(z)$ which is meromorphic in $\mathbb{C}$. Then the operator $\mathcal{L}$ has **trivial monodromy** if all the solutions of the corresponding Schrödinger equation

$$\mathcal{L}\psi = -\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + u\psi = \lambda\psi$$

are also meromorphic in $\mathbb{C}$ for all λ. Such an operator is said to be **monodromy-free**.

Theorem (**Oblomkov [1999]**)

Every Schrödinger operator $\mathcal{L}$ with trivial monodromy, and with a quadratically increasing rational potential, i.e. $u(z) = z^2 + R(z)$, with $\lim_{|z|\to\infty} R(z) = 0$, has the form

$$\mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}z^2} + z^2 - 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln \mathcal{W}\left(H_{k_1}, H_{k_2}, \ldots, H_{k_n}\right)$$

*where $H_k(z)$ is the k^{th} Hermite polynomial, $\mathcal{W}(\phi_1, \phi_2, \ldots, \phi_n)$ is the Wronskian and $k_1, k_2, \ldots, k_n$ are a sequence of **distinct** positive integers.*

A corollary of results due to **Crum [1955]** (see also **Oblomkov [1999]**, **Veselov [2001]**)

Theorem

The Schrödinger equation

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + u\psi = \lambda\psi \qquad (*)$$

with potential

$$u = z^2 - 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln\mathcal{W}\left(H_{k_1}, H_{k_2}, \ldots, H_{k_N}\right)$$

where $H_k(z)$ is the k^{th} Hermite polynomial, $\mathcal{W}(\phi_1, \phi_2, \ldots, \phi_n)$ is the Wronskian and $k_1, k_2, \ldots, k_n$ are a sequence of ***distinct*** *positive integers, has the solutions*

$$\psi(z) = \frac{\mathcal{W}\left(H_{k_1}, H_{k_2}, \ldots, H_{k_N}, H_{k_{N+1}}\right)}{\mathcal{W}\left(H_{k_1}, H_{k_2}, \ldots, H_{k_N}\right)}\exp\left(-\tfrac{1}{2}z^2\right)$$

$$\psi(z) = \frac{\mathcal{W}\left(H_{k_1}, H_{k_2}, \ldots, H_{k_{N-1}}\right)}{\mathcal{W}\left(H_{k_1}, H_{k_2}, \ldots, H_{k_N}\right)}\exp\left(\tfrac{1}{2}z^2\right)$$

with k_{N+1} another different positive integer for the eigenvalues $\lambda = 1 + 2(k_{N+1} - N)$ and $\lambda = 2(N - k_{N-1}) - 1$, respectively.

Remark: Substituting $u = z^2 - 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln Q$ and $\psi = \frac{P}{Q}\exp(-\frac{1}{2}z^2)$ into $(*)$ yields

$$\frac{\mathrm{d}^2P}{\mathrm{d}z^2}Q - 2\frac{\mathrm{d}P}{\mathrm{d}z}\frac{\mathrm{d}Q}{\mathrm{d}z} + P\frac{\mathrm{d}^2Q}{\mathrm{d}z^2} - 2z\left(\frac{\mathrm{d}P}{\mathrm{d}z}Q - P\frac{\mathrm{d}Q}{\mathrm{d}z}\right) + (\lambda - 1)PQ = 0$$

Consequently

Theorem (**Lousenko [2003]**)

The bilinear equation

$$\frac{\mathrm{d}^2P}{\mathrm{d}z^2}Q - 2\frac{\mathrm{d}P}{\mathrm{d}z}\frac{\mathrm{d}Q}{\mathrm{d}z} + P\frac{\mathrm{d}^2Q}{\mathrm{d}z^2} - 2z\left(\frac{\mathrm{d}P}{\mathrm{d}z}Q - P\frac{\mathrm{d}Q}{\mathrm{d}z}\right) + 2(m-n)PQ = 0$$

with $m, n \in \mathbb{Z}^+$, has polynomial solutions in the form

$$P(z) = \mathcal{W}\left(H_{k_1}(z), H_{k_2}(z), \ldots, H_{k_N}(z), H_{k_{N+1}}(z)\right),$$
$$Q(z) = \mathcal{W}\left(H_{k_1}(z), H_{k_2}(z), \ldots, H_{k_N}(z)\right)$$

*where $H_k(z)$ is the k^{th} Hermite polynomial, $\mathcal{W}(\phi_1, \phi_2, \ldots, \phi_n)$ is the Wronskian and $k_1, k_2, \ldots, k_N, k_{N+1}$ are a sequence of **distinct** positive integers. The degrees of the polynomials $P(z)$ and $Q(z)$, respectively m and n, are given by*

$$m = \sum_{j=1}^{N+1} k_j - \tfrac{1}{2}N(N+1), \qquad n = \sum_{j=1}^{N} k_j - \tfrac{1}{2}N(N-1)$$
$$\Rightarrow \quad m - n = k_{N+1} - N$$

However there are additional solutions of the equations

$$\frac{\mathrm{d}^2P}{\mathrm{d}z^2}Q - 2\frac{\mathrm{d}P}{\mathrm{d}z}\frac{\mathrm{d}Q}{\mathrm{d}z} + P\frac{\mathrm{d}^2Q}{\mathrm{d}z^2} - 2z\left(\frac{\mathrm{d}P}{\mathrm{d}z}Q - P\frac{\mathrm{d}Q}{\mathrm{d}z}\right) + (\lambda-1)PQ = 0 \qquad (1)$$

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + \left\{z^2 - 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln Q\right\}\psi = \lambda\psi \qquad (2)$$

in terms of the generalized Hermite polynomials and generalized Okamoto polynomials

$P(z)$	$Q(z)$	$\psi(z)\exp(\frac{1}{2}z^2) = P/Q$	$\lambda - 1$
$H_{j,k}$	$H_{j+1,k}$	$\dfrac{\mathcal{W}(H_j, H_{j+1}, \ldots, H_{j+k-1})}{\mathcal{W}(H_{j+1}, H_{j+2}, \ldots, H_{j+k})}$	$-2k$
$Q_{j,k}$	$Q_{j,k+1}$	$\dfrac{\mathcal{W}(H_1, H_4, \ldots, H_{3j+3k-5}, H_2, H_5, \ldots, H_{3k-4})}{\mathcal{W}(H_1, H_4, \ldots, H_{3j+3k-2}, H_2, H_5, \ldots, H_{3k-1})}$	$-2(j+2k)$

Further solutions of equations (1) and (2)

$$P(z) = (z^3 + 2\sqrt{3}\,z^2 + \tfrac{9}{2}z + \sqrt{3})(z+\sqrt{3}), \qquad Q(z) = (z+\sqrt{3})^3$$

$$\psi(z) = \frac{z^3 + 2\sqrt{3}\,z^2 + \frac{9}{2}z + \sqrt{3}}{(z+\sqrt{3})^2}\exp(-\tfrac{1}{2}z^2), \qquad \lambda = 3$$

Conclusions

- The roots of the special polynomials associated with rational solutions of P_{II} and P_{IV} have a very symmetric, regular structure. Similar symmetric structures arise for the roots of special polynomials associated with rational solutions of P_{III} and P_{V}.
- Rational solutions of various soliton equations can be expressed in terms of the special polynomials associated with rational solutions of P_{II} and P_{IV}. Generalized rational solutions, involving an infinite number of parameters, appear to **only** exist when the rational solutions decay as $|x| \to \infty$. Similar results hold for the dispersive water wave and modified Boussinesq equations and the classical Boussinesq system.
- This seems to be yet another remarkable property of "integrable" differential equations, in particular the soliton equations and the Painlevé equations.
- These polynomials have applications in the description of vortex dynamics. Further they demonstrate that there are more solutions of the Schrödinger equation for quadratically increasing rational potentials,

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + [z^2 + R(z)]\psi = \lambda\psi,$$

with $\lim_{|z|\to\infty} R(z) = 0$, than previously thought.

Open Problems

- Is there an analytical explanation and interpretation of these computational results?
- What is the structure of the roots of the special polynomials associated with rational and algebraic solutions of $\mathrm{P_{VI}}$ and discrete Painlevé equations?
- What is the structure of the roots of special polynomials associated with rational solutions and other special solutions of other soliton equations?
- Do these special polynomials have further applications, e.g. in numerical analysis?
- What is the general form of polynomial solutions of the bilinear equation

$$\frac{\mathrm{d}^2 P}{\mathrm{d}z^2}Q - 2\frac{\mathrm{d}P}{\mathrm{d}z}\frac{\mathrm{d}Q}{\mathrm{d}z} + P\frac{\mathrm{d}^2 Q}{\mathrm{d}z^2} + 2\mu z\left(\frac{\mathrm{d}P}{\mathrm{d}z}Q - P\frac{\mathrm{d}Q}{\mathrm{d}z}\right) = 2\mu(m-n)PQ$$

- What about polynomial solutions of the bilinear equations

$$\frac{\mathrm{d}^2 P}{\mathrm{d}z^2}Q - 2\ell\frac{\mathrm{d}P}{\mathrm{d}z}\frac{\mathrm{d}Q}{\mathrm{d}z} + \ell^2 P\frac{\mathrm{d}^2 Q}{\mathrm{d}z^2} + \mu\left(\frac{\mathrm{d}P}{\mathrm{d}z}Q - \ell P\frac{\mathrm{d}Q}{\mathrm{d}z}\right) = 0 \qquad (1)$$

$$\frac{\mathrm{d}^2 P}{\mathrm{d}z^2}Q - 2\ell\frac{\mathrm{d}P}{\mathrm{d}z}\frac{\mathrm{d}Q}{\mathrm{d}z} + \ell^2 P\frac{\mathrm{d}^2 Q}{\mathrm{d}z^2} + \mu z\left(\frac{\mathrm{d}P}{\mathrm{d}z}Q - \ell P\frac{\mathrm{d}Q}{\mathrm{d}z}\right) + \kappa PQ = 0 \qquad (2)$$

  where $\ell \neq 1$, μ and κ are constants? **Lousenko [2004]** has obtained some polynomial solutions of (1) in the case when $\ell = 2$ and $\mu = 0$, which corresponds to the case of vortices with two strengths of unequal magnitudes.