On explicit integration of two non-holonomic problems

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Generalized Chaplygin systems

Equations of motion of the generalized Chaplygin system:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_1}\right) - \frac{\partial L}{\partial q_1} = \dot q_2 S, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_2}\right) - \frac{\partial L}{\partial q_2} = -\dot q_1 S, \\ S = a_1(\boldsymbol{q})\dot q_1 + a_2(\boldsymbol{q})\dot q_2 + b(\boldsymbol{q}), \tag{1}$$

where Lagrangian L is a function of generalized coordinates $\boldsymbol{q} = (q_1, q_2)$ and velocities $\dot{\boldsymbol{q}} = (\dot q_1, \dot q_2)$.
Equations of motion (1) are invariant to the change of time

$$\mathcal{N}(\boldsymbol{q})\,dt = d\tau,$$

if $\mathcal{N}$ does not depend on velocities. Indeed, in new time we get:

$$\frac{d}{d\tau}\left(\frac{\partial \bar L}{\partial q_1'}\right) - \frac{\partial \bar L}{\partial q_1} = q_2'\bar S, \quad \frac{d}{d\tau}\left(\frac{\partial \bar L}{\partial q_2'}\right) - \frac{\partial \bar L}{\partial q_2} = -q_1'\bar S, \\ \bar S = \mathcal{N} S + \frac{1}{\mathcal{N}}\left(\frac{\partial \mathcal{N}}{\partial q_2}\frac{\partial \bar L}{\partial q_1'} - \frac{\partial \mathcal{N}}{\partial q_1}\frac{\partial \bar L}{\partial q_2'}\right), \quad \bar L(q, q') = L(q, \mathcal{N} q'). \tag{2}$$

Generalized Chaplygin systems

Theorem

Let $\det\left\|\dfrac{\partial^2 L}{\partial\dot q_i\partial\dot q_j}\right\| \neq 0$ *and system* (1) *allow an invariant measure with density depending only on coordinates. Then there is a change of time* $\mathcal{N}(\boldsymbol{q})\,dt = d\tau$, *such that*

1. *function* $\bar S$ *defined by* (2) *depends only on coordinates:* $\bar S = \bar S(\boldsymbol{q})$,
2. *in new time equations of motion are Hamiltonian:*

$$\frac{dq_i}{d\tau} = \{q_i, \bar H\}, \quad \frac{dp_i}{d\tau} = \{p_i, \bar H\},$$

where

$$p_i = \frac{\partial \bar L}{\partial q_i'}, \quad \bar H = \sum_{k=1}^{2} p_k q_k' - \bar L\Big|_{q_i' \to p_i},$$

and the Poisson bracket is defined by relations

$$\{q_i, p_j\} = \delta_{ij}, \quad \{p_1, p_2\} = \bar S(\boldsymbol{q}), \quad \{q_1, q_2\} = 0. \tag{3}$$

Chaplygin ball on a sphere

Equations of motion

Equations of motion of a ball moving without slipping on a sphere

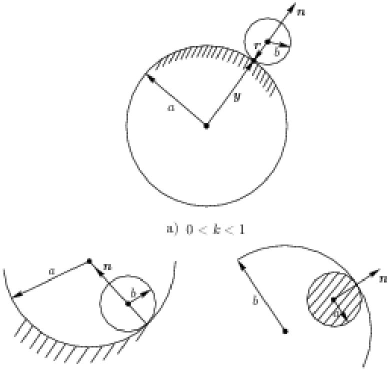
$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{n}} = k\boldsymbol{n} \times \boldsymbol{\omega}, \quad k = \frac{a}{a+b}, \tag{4}$$

where $\boldsymbol{\omega}$ — angular velocity of the body, $\boldsymbol{n}$ — the normal in the point of contact a — radius of the basic sphere, b — radius of the ball (fig. 1). The moment with respect to the point of contact $\boldsymbol{M}$ is related with angular velocity $\boldsymbol{\omega}$ by a linear equation

$$\boldsymbol{M} = \mathbf{I}\boldsymbol{\omega} + d\boldsymbol{n}(\boldsymbol{n} \times \boldsymbol{\omega}), \quad d = mb^2,$$

where m — mass of the body, $\mathbf{I} = \mathrm{diag}(I_1,\ I_2,\ I_3)$ — a central tensor of inertia. A coefficient k may be either positive or negative depending on possible cases (fig. 1).

Chaplygin ball on a sphere



a) $0 < k < 1$

b) $k > 1$

c) $k < 0$

. Rolling of a body with a spherical part on the sphere. Fixed surface is marked by shading.

Chaplygin ball on a sphere

Integrals of motion and integrable cases

Under any value of k, system (4) has three integrals of motion

$$F_0 = (\boldsymbol{n}, \boldsymbol{n}) = 1, \quad H = \frac{1}{2}(\boldsymbol{M}, \omega), \quad F_1 = (\boldsymbol{M}, \boldsymbol{M}) \tag{5}$$

and invariant measure $\rho \, d\boldsymbol{\omega} \, d\boldsymbol{n}$ with density

$$\rho^2 = (\boldsymbol{n}, \boldsymbol{n}) - d(\boldsymbol{n}, (\mathbf{I} + d)^{-1}\boldsymbol{n}).$$

Integrable cases

1. $k = 1$ $(a \to \infty)$.
   Rolling on a plane.
2. $k = -1$ $(\frac{b}{a} = \frac{1}{2})$.
   Additional integral of motion

$$F_2 = (\mathbf{A}\boldsymbol{M}, \boldsymbol{n}), \tag{6}$$

where $\mathbf{A} = \mathrm{diag}(\frac{1}{2}(-I_1 + I_2 + I_3), \frac{1}{2}(I_1 - I_2 + I_3), \frac{1}{2}(I_1 + I_2 - I_3))$

Chaplygin ball on a sphere

Equations of motion of Chaplygin ball at $k = -1$ and $F_2 = 0$ as Chaplygin system

We can write the equation of motion (4) as a Chaplygin system

$$\frac{d}{dt}\frac{\partial T}{\partial \dot u} - \frac{\partial T}{\partial u} = \dot u \Phi, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot v} - \frac{\partial T}{\partial v} = -\dot v \Phi \tag{7}$$

where $T = \frac{1}{2}(b_{uu}\dot u^2 + b_{uv}\dot u \dot v + b_{vv}\dot v^2)$, and $\Phi = (a_u \dot u + a_v \dot v)$ — a linear homogeneous in velocity function, and u и v — sphero-conical coordinates on the sphere $\boldsymbol{n}^2 = 1$

$$n_i^2 = \frac{(J_i - u)(J_i - v)}{(J_i - J_j)(J_i - J_k)}, \quad i \neq j \neq k \neq i, \quad J_i = I_i + d. \tag{8}$$

Chaplygin ball on a sphere

Equations of motion of Chaplygin's ball for $k=-1$ and $F_2=0$ are a conformal Hamiltonian

After change of variables $\mathcal{N}(u,\,v)\,dt=d\tau$ system (7) has a Lagrangian form

$$\frac{d}{d\tau}\frac{\partial T}{\partial u'}-\frac{\partial T}{\partial u}=0,\quad \frac{d}{d\tau}\frac{\partial T}{\partial v'}-\frac{\partial T}{\partial v}=0,\quad u'=\frac{du}{d\tau},\quad v'=\frac{dv}{d\tau}.$$

The reducing multiplier:

$$\mathcal{N}=\frac{2uv+(u+v)(2d+\alpha_1)+\alpha_2-d\alpha_1}{\sqrt{\det(\mathbf{I}+d-d\boldsymbol{n}\otimes\boldsymbol{n})}}(4\alpha_3+2\alpha_1\alpha_2-\alpha_1^3-d\alpha_1^2+ \\ +(\alpha_1^2-2\alpha_2+4d\alpha_1)(u+v)-4d(u+v)^2)^{-1}$$

where $\alpha_1=\sum J_i$, $\alpha_2=\sum J_i^2, \alpha_3=J_1J_2J_3$.

Chaplygin ball on a sphere

Equations of motion of Chaplygin's ball for $k = -1$ and $F_2 = 0$ are a conformal Hamiltonian

After change of variables we get a Hamiltonian system on two-dimensional sphere S^2, which can be presented as equations on a special (null) orbit of co-algebra $e(3)$

$$\begin{gathered}
H = \frac{\delta \det \mathbf{J}}{8(\boldsymbol{\gamma},\, \mathbf{B}\boldsymbol{\gamma})^2} \sum_{i=1}^{3} c_i m_i^2, \quad F_2 = \frac{\rho^2}{4(\boldsymbol{\gamma},\, \mathbf{B}\boldsymbol{\gamma})^2} (\delta^2 \boldsymbol{m}^2 - 4 \sum_{i=1}^{3} d_i m_i^2), \\
\delta = (\boldsymbol{\gamma},\, \mathbf{J}\bar{\mathbf{A}}\boldsymbol{\gamma}) - d(\boldsymbol{\gamma},\, \bar{\mathbf{A}}\boldsymbol{\gamma})^2, \\
c_i = \frac{\rho^2 \delta}{J_i} - 4 \prod_{k \neq i} (J_i - J_k) \gamma_i^2 (\rho^2 - \frac{d\delta}{4 J_i \det \mathbf{J}}), \\
d_i = \prod_{k \neq i} (J_i - J_k) \gamma_i^2 (\delta(J_i + d) - (\boldsymbol{\gamma},\, \mathbf{J}(\mathbf{J} + d)\bar{\mathbf{A}}^2 \boldsymbol{\gamma}) + 2d(\boldsymbol{\gamma},\, \mathbf{J}\bar{\mathbf{A}}\boldsymbol{\gamma})(\boldsymbol{\gamma},\, \bar{\mathbf{A}}\boldsymbol{\gamma})).
\end{gathered} \tag{9}$$

where $\bar{\mathbf{A}} = 2\mathbf{A}$. Poisson brackets are given by the following expressions

$$\{m_i,\, m_j\} = \varepsilon_{ijk} m_k, \quad \{m_i,\, \gamma_j\} = \varepsilon_{ijk} \gamma_k, \quad \{\gamma_i,\, \gamma_j\} = 0,$$

and the orbit is determined by integrals

$$\gamma^2 = 1, \quad (\boldsymbol{m},\, \gamma) = 0.$$

Chaplygin ball on a sphere

Case $F_2 \neq 0$

Remark

We can show that if $k = -1$ *and* $F_2 \neq 0$ *then equations of motion* (7) *have the form of generalized Chaplygin system* (1)*. Therefore, using above Theorem, we can write these equations as a Hamiltonian system with gyroscopic forces. But since the equations are rather complicated and general methods of integration in quadratures are not known for such systems, we do not present them here.*

Chaplygin ball on a sphere

Separation of variables

We use a canonical presentation of algebra $e(3)$ (Darboux coordinates)

$$m_1 = p_1(x^2-1)+p_2(y^2-1), \quad m_2 = ip_1(x^2+1)+ip_2(y^2+1),$$
$$m_3 = 2p_1x+2p_2y, \quad \gamma_1 = \frac{xy-1}{x-y}, \quad \gamma_2 = i\frac{xy-1}{x-y}, \quad \gamma_3 = \frac{x+y}{x-y} \tag{10}$$

Using (10) we can write the pair (H, F_2) in canonical form

$$\begin{aligned} H &= a(x,\,y)p_1^2+2b(x,\,y)p_1p_2+c(x,\,y)p_2^2, \\ F_2 &= A(x,\,y)p_1^2+2B(x,\,y)p_1p_2+C(x,\,y)p_2^2 \end{aligned} \tag{11}$$

We can show that separating variables are the roots of the equation

$$(B-bs)^2 = (A-as)(C-cs) \tag{12}$$

Chaplygin ball on a sphere

Separation of variables

In new coordinates functions H and F have the Liouville form

$$H = \frac{S_1(s_1)}{s_1 - s_2} P_1^2 - \frac{S_2(s_2)}{s_1 - s_2} P_2^2, \quad F = \frac{s_2 S_1(s_1)}{s_1 - s_2} P_1^2 - \frac{s_1 S_2(s_2)}{s_1 - s_2} P_2^2, \qquad (13)$$

where

$$S(x) = \frac{2(8x^3 + 8(d - \epsilon)x^2 + (2\epsilon^2\beta - 4d\epsilon)x - 4\gamma - d\beta + \sqrt{\triangle})}{\gamma(2x - \epsilon + 2d)^2} \qquad (14)$$

and

$$\begin{gathered}
\alpha = (J_2 + J_1 - J_3)(-J_2 + J_2 - J_3)(-J_2 + J_2 + J - 3) \\
\beta = J_1^2 + J_2^2 + J_3^2 - 2J_1 J_2 - 2J_2 J_3 - 2J_3 J_1 \\
\gamma = J_1 J_2 J_3, \quad \epsilon = J_1 + J_2 + J_3, \\
\triangle = x^2(\beta^2 + 8\alpha d) + 2x(4\beta\gamma + d\beta^2 - 2d\alpha\epsilon + 4\alpha d^2) + (4\gamma + d\beta)^2
\end{gathered}$$

Rubber ball moving on a sphere

Equations of motion

Let us assume an additional constraint, which does not let the ball twist

$$(\boldsymbol{\omega}, \boldsymbol{n}) = 0. \tag{15}$$

Equations of motion can be given in the form

$$\begin{gathered}\mathbf{J}\dot{\boldsymbol{\omega}} = \mathbf{J}\boldsymbol{\omega} \times \boldsymbol{\omega} + \lambda \boldsymbol{n} + \boldsymbol{M}_Q, \quad \dot{\boldsymbol{n}} = k\boldsymbol{n} \times \boldsymbol{\omega}, \\ \mathbf{J} = \mathbf{I} + mb^2\mathbf{E}, \quad \mathbf{E} = \|\delta_{ij}\|,\end{gathered} \tag{16}$$

where

$$\lambda = -\frac{(\mathbf{J}\boldsymbol{\omega} \times \boldsymbol{\omega}, \mathbf{J}^{-1}\boldsymbol{n}) + (\boldsymbol{M}_Q, \mathbf{J}\boldsymbol{n})}{(\boldsymbol{n}, \mathbf{J}^{-1}\boldsymbol{n})}.$$

and $\boldsymbol{M}_Q$ — moment of external forces.

Rubber ball on a sphere

Conservation laws and integrable cases

Equations (16) have integral of energy and geometric integral

$$H = \frac{1}{2}(\mathbf{J}\boldsymbol{\omega}, \boldsymbol{\omega}), \quad (\boldsymbol{n}, \boldsymbol{n}) = 1,$$

and have invariant measure

$$(\boldsymbol{n}, \mathbf{J}^{-1}\boldsymbol{n})^{\frac{1}{2k}} d\boldsymbol{\omega}\, d\boldsymbol{n}. \tag{17}$$

Integrable cases

1. $k = 1$ $(a = \infty)$ — rolling of a ball on a horizontal plane. Additional integral $F = (\mathbf{J}\boldsymbol{\omega} \times \boldsymbol{n}, \mathbf{J}\boldsymbol{\omega} \times \boldsymbol{n})$. The obtained system is equivalent to Veselova's system and can be integrated with the help of sphero-conical coordinates.
2. $k = -1$ $(b = -2a)$ — rolling of dynamically asymmetric sphere by its inner surface on a fixed ball. Additional integral:

$$F = \frac{(\mathbf{J}\boldsymbol{\omega}, \mathbf{J}\boldsymbol{\omega})\boldsymbol{n}^2 + \det \mathbf{J}(\boldsymbol{\omega}, \mathbf{J}\boldsymbol{\omega})(\mathbf{J}^{-1}\boldsymbol{n}, \mathbf{J}^{-1}\boldsymbol{n})}{(\boldsymbol{n}, \mathbf{J}^{-1}\boldsymbol{n})}. \tag{18}$$

Equations of motion (16) can be represented in the form of Veselova's system

$$\frac{d}{dt}\frac{\partial T}{\partial \dot\xi} - \frac{\partial T}{\partial \xi} = \dot\eta S, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot\eta}\right) - \frac{\partial T}{\partial \eta} = -\dot\xi S,$$
$$S = \frac{2k-1}{8k^3}(\xi - \eta)\left(\frac{\dot\xi}{A(\xi)} + \frac{\dot\eta}{A(\eta)}\right). \tag{19}$$

Here (ξ, η) — sphero-conical coordinates on a sphere $|\boldsymbol{n}| = 1$

$$n_1^2 = \frac{(J_1 - \xi)(J_1 - \eta)}{(J_1 - J_2)(J_1 - J_3)}, \quad n_2^2 = \frac{(J_2 - \xi)(J_2 - \eta)}{(J_2 - J_1)(J_2 - J_3)}, \quad n_3^2 = \frac{(J_3 - \xi)(J_3 - \eta)}{(J_3 - J_1)(J_3 - J_2)}; \tag{20}$$

and kinetic energy $T = \frac{1}{2}(\boldsymbol{\omega}, \mathbf{J}\boldsymbol{\omega})$ in these variables is

$$T = \frac{\xi - \eta}{8k^2}\left(-\frac{\eta\dot\xi^2}{A(\xi)} + \frac{\xi\dot\eta^2}{A(\eta)}\right). \tag{21}$$

Rubber ball on a sphere
Hamiltonian structure and algebraization

By the above theorem, after the change of time $\mathcal{N}(u, v)\,dt = d\tau$ with reducing multiplier

$$\mathcal{N} = \left(\frac{\xi\eta}{\det \mathbf{J}}\right)^{-1+\frac{1}{2k}} \tag{22}$$

equations of motion become canonical

$$\xi' = \frac{\partial T}{\partial p_\xi}, \quad \eta' = \frac{\partial T}{\partial p_\eta}, \quad p_\xi' = -\frac{\partial T}{\partial \xi}, \quad p_\eta' = -\frac{\partial T}{\partial \eta}. \tag{23}$$

Therefore, system (16) *for* $k = \pm 1$ *is conform-Hamiltonian.*

Define isomorphism of systems (16), (19) with the problem of motion of point on a sphere. Introduce three-dimensional vectors

$$\boldsymbol{M} = \widetilde{\mathcal{N}}\mathbf{J}^{1/2}\boldsymbol{\omega}, \quad \boldsymbol{\gamma} = \frac{1}{\rho}\mathbf{J}^{-1/2}\boldsymbol{n}, \tag{24}$$

where

$$\widetilde{\mathcal{N}} = \frac{\sqrt{\det \mathbf{J}}}{k}\frac{N}{\rho^2} = \frac{\sqrt{\det \mathbf{J}}}{k}(\boldsymbol{n}, J^{-1}\boldsymbol{n})^{\frac{1}{2k}}.$$

Rubber ball on a sphere

Hamiltonian structure and algebraization

It is evident that

$$(\boldsymbol{\gamma},\boldsymbol{\gamma})=1,\quad (\boldsymbol{M},\boldsymbol{\gamma})=\frac{\widetilde{\mathcal{N}}}{\rho}(\boldsymbol{\omega},\boldsymbol{n})=0. \qquad (25)$$

Poisson brackets between $\boldsymbol{M}$ and $\boldsymbol{\gamma}$:

$$\{M_i,M_j\}=-\varepsilon_{ijk}M_k,\quad \{M_i,\gamma_j\}=-\varepsilon_{ijk}\gamma_k,\quad \{\gamma_i,\gamma_j\}=0. \qquad (26)$$

Hamiltonian

$$\mathscr{H}=T=\frac{1}{2}\widetilde{\mathcal{N}}^{-2}\boldsymbol{M}^2=\frac{1}{2}\frac{k^2}{\det\mathbf{J}}(\boldsymbol{\gamma},\mathbf{J}\boldsymbol{\gamma})^{1/k}\boldsymbol{M}^2=$$

$$=\frac{2k^2}{\xi-\eta}\left(\frac{\xi\eta}{\det\mathbf{J}}\right)^{(2k-1)/k}\left(\frac{A(\xi)}{\eta}p_\xi^2-\frac{A(\eta)}{\xi}p_\eta^2\right). \qquad (27)$$

Rubber ball on a sphere

Hamiltonian structure and algebraization

Hamiltonian (27) is the product of two functions, depending on $\boldsymbol{M}$ and $\boldsymbol{\gamma}$ respectively: $\mathscr{H} = G(\boldsymbol{\gamma})F(\boldsymbol{M})$. Equations of motion can be represented as:

$$\dot{\boldsymbol{M}} = G\left(\boldsymbol{M} \times \frac{\partial F}{\partial \boldsymbol{M}} - FG\boldsymbol{\gamma} \times \frac{\partial G^{-1}}{\partial \boldsymbol{\gamma}}\right), \quad \dot{\boldsymbol{\gamma}} = G\boldsymbol{\gamma} \times \frac{\partial F}{\partial \boldsymbol{M}}. \tag{28}$$

Make a change of time $G(\boldsymbol{\gamma})dt = ds$ and fix the level of integral $FG = h$. On this level we get a system:

$$\begin{gathered}\frac{d\boldsymbol{M}}{ds} = \boldsymbol{M} \times \frac{\partial \widetilde{\mathscr{H}}}{\partial \boldsymbol{M}} + \boldsymbol{\gamma} \times \frac{\partial \widetilde{\mathscr{H}}}{\partial \boldsymbol{\gamma}}, \quad \frac{d\boldsymbol{\gamma}}{ds} = \boldsymbol{\gamma} \times \frac{\partial \widetilde{\mathscr{H}}}{\partial \boldsymbol{s}}, \\ \widetilde{\mathscr{H}} = F(\boldsymbol{M}) - \frac{h}{G(\boldsymbol{\gamma})}.\end{gathered} \tag{29}$$

Rubber ball on a sphere

Trajectory isomorphysim

For Hamiltonian (27) we have

$$\widetilde{\mathscr{H}} = \frac{1}{2}\boldsymbol{M}^2 - h(\boldsymbol{\gamma}, \mathbf{J}\boldsymbol{\gamma})^{-1/k}, \quad h = \mathrm{const.} \tag{30}$$

Therefore, on a fixed level of the energy integral $\mathscr{H} = h$ the system (28) *is trajectory isomorphic to system* (30) *for $\widetilde{\mathscr{H}} = 0$.*

Hamiltonian (30) describes the motion of a particle on a surface (on condition that $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$) in potential field of forces with potential $V = h(\boldsymbol{\gamma}, \mathbf{J}\boldsymbol{\gamma})^{-1/k}$.

Integrable potentials correspond to cases

1. $k = 1$: the Braden system,
2. $k = -1$: the Neumann system.

Rubber ball on a sphere

Separation of variables for $k = -1$

Define sphero-conical coordinates u and v

$$n_1^2 = \rho^2 \frac{J_1(J_1 - u)(J_1 - v)}{(J_1 - J_2)(J_1 - J_3)}, \quad n_2^2 \quad = \rho^2 \frac{J_2(J_2 - u)(J_2 - v)}{(J_2 - J_1)(J_2 - J_3)}, \quad n_2^2 = \rho^2 \frac{J_3(J_3 - u)(J_3 - v)}{(J_3 - J_1)(J_3 - J_2)},$$

where $\rho^2 = (\boldsymbol{n}, \mathbf{J}^{-1}\boldsymbol{n}) = (\sum\limits_i J_i - u - v)^{-1}$. Kinetic energy (21):

$$T = \frac{\det \mathbf{J} \rho^4 (u - v)}{8} \left(\frac{\dot{u}^2}{A(u)} - \frac{\dot{v}^2}{A(v)} \right),$$

where $A(x) = \prod\limits_i (J_i - x)$, and reducing multiplier is

$$\mathcal{N} = \Big(\sum_i J_i - u - v\Big)^{-3/2}.$$

Thus, we receive canonical equations with Hamiltonian

$$\mathscr{H} = \frac{2}{\det \mathbf{J} \big(\sum\limits_i J_i - u - v\big)(u - v)} \left(A(u) p_u^2 - A(v) p_v^2\right).$$

Therefore, u and v are separating variables.