# On integrable Weingarten surfaces

**Hynek Baran** and Michal Marvan

Opava, Czech Republic

# Immersed surfaces and ZCRs

The Gauss–Mainardi–Codazzi equations describing surfaces immersed into three-dimensional Euclidean space are always of the form

$$A_y - B_x + [A, B] = 0, \tag{1}$$

being the compatibility condition of an auxiliary linear system

$$\Psi_x = A\Psi, \;\; \Psi_y = B\Psi, \tag{2}$$

identifiable with the Gauss–Weingarten equations.

## Gauge invariance

Equation $A_y - B_x + [A, B] = 0$ is called a *zero curvature representation (ZCR)* and is invariant under a group of *gauge transformations*

$$\begin{aligned} A' &= S_x S^{-1} + SAS^{-1}, \\ B' &= S_y S^{-1} + SBS^{-1}, \end{aligned} \tag{3}$$

resulting from the transformation $\Psi' = S\Psi$ of the linear system $\Psi_x = A\Psi, \; \Psi_y = B\Psi$ (2). Here $S$ is an invertible matrix.

## Coordinates and fundamental forms

Our coordinate system of choice will be the curvature coordinates, denoted by $x, y$. They are unique up to arbitrary changes $x = X(x)$, $y = Y(y)$.

The first and second fundamental forms are respectively given by

$$\mathrm{I} = u^2 \,\mathrm{d}x^2 + v^2 \,\mathrm{d}y^2,$$
$$\mathrm{II} = u^2 p \,\mathrm{d}x^2 + v^2 q \,\mathrm{d}y^2.$$

Consequently, $p, q$ are the principal curvatures.

## Gauss–Mainardi–Codazzi equations

The corresponding Gauss–Mainardi–Codazzi equations are

$$uu_{yy} + vv_{xx} - \frac{v}{u}u_x v_x - \frac{u}{v}u_y v_y + u^2v^2pq = 0, \qquad (4a)$$

$$(p - q)u_y + up_y = 0, \qquad (4b)$$

$$(q - p)v_x + vq_x = 0. \qquad (4c)$$

Such equations are always equivalent to a ZCR $A_y - B_x + [A, B] = 0$ with suitable matrices $A, B$.

## Parameter-free ZCR

The GMC equations possess an $\mathfrak{sl}(2,\mathbb{C})$-valued parameter-free ZCR

$$A = \begin{pmatrix} \dfrac{\mathrm{i}u_y}{2v} & -\dfrac{1}{2}\,up \\ \dfrac{1}{2}\,up & -\dfrac{\mathrm{i}u_y}{2v} \end{pmatrix},$$
$$B = \begin{pmatrix} -\dfrac{iv_x}{2u} & -\dfrac{1}{2}\,\mathrm{i}qv \\ -\dfrac{1}{2}\,\mathrm{i}qv & \dfrac{\mathrm{i}v_x}{2u} \end{pmatrix}. \tag{5}$$

## Geometric constraints, Weingarten surfaces

The functional relation

$$f(p, q) = 0 \Longleftrightarrow q = F(p), \tag{6}$$

between principal curvatures is a general algebraic constraint invariant w. r. to coordinate changes restricting us to the class of *Weingarten surfaces*. The Mainardi–Codazzi equations (4b), (4c) becomes

$$(\ln u)_y = \frac{u_y}{u} = \frac{p_y}{q - p}, \ (\ln v)_x = \frac{v_x}{v} = \frac{q_x}{p - q}.$$

# Weingarten surfaces II – reduction

Those equations have the solution

$$u = u_0 \int \frac{\mathrm{d}p}{q-p}, v = v_0 \int \frac{\mathrm{d}q}{p-q}$$

Substituting $q = F = 1/(\ln G)_p + p$, where $G = G(p)$ is an arbitrary function we obtain

$$u = u_0 G,\ v = v_0 \frac{G_p}{G^2}.$$

Without loss of generality, we can put $u_0 = 1$, $v_0 = 1$ (this can be always achieved by a change of curvature coordinates).

## Weingarten surfaces III – reduced GMC eqs

The GMC equations (4a), (4b), (4c) then reduce to the single equation

$$
\begin{aligned}
p_{yy} = & -\frac{G_p p^2}{G^3} - \frac{p}{G^2} - 2\,\frac{G_p p_y^2}{G} \\
& -\frac{(-2\,G_p^2 + G_{pp}G)p_{xx}}{G^6} \\
& -\frac{(8\,G_p^3 - 7\,G_{pp}G_pG + G_{ppp}G^2)p_x^2}{G^7}.
\end{aligned}
\tag{7}
$$

## Weingarten surfaces IV – ZCR

The ZCR (5) becomes

$$A_0 = \begin{pmatrix} \frac{1}{2}\,\mathrm{i}G^2 p_y & -\frac{1}{2}\,Gp \\ \frac{1}{2}\,Gp & -\frac{1}{2}\,\mathrm{i}G^2 p_y \end{pmatrix},$$

$$B_0 = \begin{pmatrix} -\dfrac{\mathrm{i}(GG_{pp} - 2G_p^2)p_x}{2G^4} & -\dfrac{\mathrm{i}(G_p p + G)}{2G^2} \\ -\dfrac{\mathrm{i}(G_p p + G)}{2G^2} & \dfrac{\mathrm{i}(GG_{pp} - 2G_p^2)p_x}{2G^4} \end{pmatrix}. \quad (8)$$

## Jets language

In the context of reduced GMC equation (7) it is more appropriate to rewrite (1) as

$$D_y A_0 - D_x B_0 + [A_0, B_0] = 0, \tag{9}$$

where $D_x, D_y$ denote the total derivatives

$$D_x = \frac{\partial}{\partial x} + \sum_I p_{xI} \frac{\partial}{\partial p_I}, \quad D_y = \frac{\partial}{\partial y} + \sum_I p_{yI} \frac{\partial}{\partial p_I}.$$

# More on inserting the spectral parameter: restriction procedure

Consider the formal Taylor expansions

$$A(\lambda) = \sum_{i=0}^{\infty} A_i \lambda^i, \;\; B(\lambda) = \sum_{i=0}^{\infty} B_i \lambda^i. \tag{10}$$

Plugging (10) into $A_y - B_x + [A, B] = 0$ yields

$$D_y A_k - D_x B_k + \sum_{i+j=k} [A_i, B_j] = 0, \tag{11}$$

for all $k \geq 0$. For $k = 0$ we obtain an identity once $A_0, B_0$ is a ZCR.

## Gauge cohomology

For $k = 1$, eq.(11) is linear in $A_1, B_1$ and reads

$$D_y A_1 - D_x B_1 + [A_1, B_0] + [A_0, B_1] = 0. \qquad (12)$$

Solutions $A_1, B_1$ are called *cocycles*. This equation always has solutions of the form

$$A_1 = D_x C + [A_0, C], \quad B_1 = D_y C + [B_0, C], \qquad (13)$$

where $C$ is an arbitrary matrix. These are called *coboundaries*. Cocycles that differ by a coboundary are called *cohomological*.

Cocycles modulo coboundaries are in one-to-one correspondence with ZCRs of the form

$$A^{[1]} = \begin{pmatrix} A_0 & 0 \\ A_1 & A_0 \end{pmatrix}, \quad B^{[1]} = \begin{pmatrix} B_0 & 0 \\ B_1 & B_0 \end{pmatrix}, \tag{14}$$

modulo gauge equivalence with respect to gauge matrices of the form

$$S^{[1]} = \begin{pmatrix} E & 0 \\ S & E \end{pmatrix},$$

where $E$ is the unit matrix.

It is easily verified that the subsystem of equations (11)

$$D_y A_k - D_x B_k + \sum_{i+j=k} [A_i, B_j] = 0, \ \ k = 0, \ldots, m$$

is equivalent to the same zero curvature condition

$$D_y A^{[m]} - D_x B^{[m]} + [A^{[m]}, B^{[m]}] = 0,$$

where

$$D_y A^{[m]} - D_x B^{[m]} + [A^{[m]}, B^{[m]}] = 0, \tag{15}$$

where

$$A^{[m]} = \begin{pmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A_m & \cdots & A_1 & A_0 \end{pmatrix}, \quad B^{[m]} = \begin{pmatrix} B_0 & 0 & \cdots & 0 \\ B_1 & B_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ B_m & \cdots & B_1 & B_0 \end{pmatrix}.$$

If $A_k, B_k$ are already known for all $k < m$, then (15) says whether expansions (10) can be extended one step further.

If, for some $m$, eq. (15) has no solution, then there is no possibility to extend the expansions (10) beyond the first $m$ terms.

Above considerations reduce the spectral parameter problem to that of computation of ZCRs modulo a gauge group. The substantial benefit is that the problem is linear in all unknowns $A_k, B_k$, $k \geq 1$ (except $A_0, B_0$, which are not unknowns).

To solve the system (15)

$$D_y A^{[m]} - D_x B^{[m]} + [A^{[m]}, B^{[m]}] = 0,$$

we apply the method of Marvan 1993,1997, based on the use of characteristic elements and their normal forms (Marvan 1997, Sebestyén 2005 and 2008).

# Application to Weingarten surfaces I

We start with the already known ZCR (8)

$$A_0 = \begin{pmatrix} \frac{1}{2}\,\mathrm{i}G^2 p_y & -\frac{1}{2}\,Gp \\ \frac{1}{2}\,Gp & -\frac{1}{2}\,\mathrm{i}G^2 p_y \end{pmatrix},$$

$$B_0 = \begin{pmatrix} -\dfrac{\mathrm{i}(GG_{pp} - 2G_p^2)p_x}{2G^4} & -\dfrac{\mathrm{i}(G_p p + G)}{2G^2} \\ -\dfrac{\mathrm{i}(G_p p + G)}{2G^2} & \dfrac{\mathrm{i}(GG_{pp} - 2G_p^2)p_x}{2G^4} \end{pmatrix}$$

# Application to Weingarten surfaces II

A routine computation shows that for every function $G$ satisfying

$$G_{pp} + 2\frac{G_p}{p} + \frac{G}{p^2} - \frac{G_p^2}{G} = 0, \tag{16}$$

solutions of equation (12)

$$D_y A_1 - D_x B_1 + [A_1, B_0] + [A_0, B_1] = 0.$$

are gauge equivalent to a particular solution of the following form:

## Application to Weingarten surfaces III

The particular solution in question reads

$$A_1 = \begin{pmatrix} -\dfrac{a_1(G_p p + G)p_x}{G^2 p^2} & a_1 \\ a_1 & \dfrac{a_1(G_p p + G)p_x}{G^2 p^2} \end{pmatrix},$$

$$B_1 = \begin{pmatrix} -\dfrac{a_1(G_p p + G)p_y}{G^2 p^2} & 0 \\ 0 & \dfrac{a_1(G_p p + G)p_y}{G^2 p^2} \end{pmatrix}.$$

## Application to Weingarten surfaces IV

Here $a_1 = a_1(p)$ a function subject to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}p} a_1 = \frac{a_1 G_p}{G} + \frac{a_1}{p}.$$

# Application to Weingarten surfaces V

The second step reveals that the matrices $A_2, B_2$ exist and are of the form

$$A_2 = \begin{pmatrix} -\frac{a_2(G_p p + G)p_x}{G^2p^2} - \frac{\mathrm{i}a_1^2 p_y}{p^2} & \cdots \\ a_2 & \cdots \end{pmatrix},$$

$$B_2 = \begin{pmatrix} -\frac{\mathrm{i}a_1^2(G_p^2p^2 + 2G_pGp + G^2)p_x}{p^4G^6} - \frac{a_2(G_p p + G)p_y}{G^2p^2} & \cdot \\ \frac{\mathrm{i}a_1^2(G_p p + G)}{G^4p^2} & \cdot\cdot \end{pmatrix}$$

## Application to Weingarten surfaces VI

Here the functions $a_2 = a_2(p)$ are subject to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}p} a_2 = \frac{a_2 G_p}{G} + \frac{a_2}{p}.$$

## Application to Weingarten surfaces VII

An analogous third step gives

$$A_3 = \begin{pmatrix} -\frac{a_3(G_p p + G)p_x}{G^2 p^2} - \frac{2ia_1 a_2 p_y}{p^2} & \cdots \\ a_3 & \cdots \end{pmatrix},$$

$$B_3 = \begin{pmatrix} -\frac{2ia_1 a_2 (pG_p + G)^2 p_x}{p^4 G^6} - \frac{a_3(pG_p + G)p_y}{p^2 G^2} & \cdots \\ \frac{2ia_1 a_2 (G_p p + G)}{G^4 p^2} & \cdots \end{pmatrix}$$

under the same condition on $G$.

## Application to Weingarten surfaces VIII

The condition (16) has the general solution

$$G = \frac{\mathrm{e}^{1+c/p}}{bp}$$

with the numbers $b, c$ as parameters; after substituting $G$ back into $F$ we obtain

$$F = \frac{pc}{p+c}.$$

Thus, the class of surfaces we arrived at is characterized by the nonlinear condition

$$pq = c(p-q).$$

The reduced GMC equations (7) now can be rewritten in the following simple form:

$$p_{yy} = \frac{c^2 p_{xx}}{\mathrm{e}^{4(p+c)/p}} - 2\frac{c^2(p-c)p_x^2}{\mathrm{e}^{4(p+c)/p}p^2} + 2\frac{(p+c)p_y^2}{p^2} + \frac{cp^2}{\mathrm{e}^{2(p+c)/p}}. \tag{17}$$

# Application to Weingarten surfaces X

Using the already known first four terms of the Taylor expansion, the general form of the rest is rather obvious. The ZCR of reduced GMC (17) with spectral parameter $\lambda$ is then easily found to be

$$A = \begin{pmatrix} \dfrac{c(\lambda+\frac{1}{2})p_x + \sqrt{\lambda^2+\lambda}\,\mathrm{e}^{2+2c/p}p_y}{p^2} & \lambda \mathrm{e}^{1+c/p} \\ (\lambda+1)\mathrm{e}^{1+c/p} & -\dfrac{c(\lambda+\frac{1}{2})p_x + \sqrt{\lambda^2+\lambda}\,\mathrm{e}^{2+2c/p}p_y}{p^2} \end{pmatrix},$$

$$B = \begin{pmatrix} \dfrac{c^2\sqrt{\lambda^2+\lambda}\,p_x}{\mathrm{e}^{2+2c/p}p^2} + \dfrac{c(\lambda+\frac{1}{2})p_y}{p^2} & \dfrac{c\sqrt{\lambda^2+\lambda}}{\mathrm{e}^{1+c/p}} \\ \dfrac{c\sqrt{\lambda^2+\lambda}}{\mathrm{e}^{1+c/p}} & -\dfrac{c^2\sqrt{\lambda^2+\lambda}\,p_x}{\mathrm{e}^{2+2c/p}p^2} - \dfrac{c(\lambda+\frac{1}{2})p_y}{p^2} \end{pmatrix}.$$

# Application to Weingarten surfaces XI

Alternatively, the above ZCR can be written in the gauge equivalent form

$$A = \begin{pmatrix} \dfrac{\sqrt{\lambda^2+\lambda}\,\mathrm{e}^{2(p+c)/p}\,p_y}{p^2} & \lambda \mathrm{e}^{(p+2c\lambda+2c)/p} \\ (\lambda+1)\mathrm{e}^{(p-2c\lambda)/p} & -\dfrac{\sqrt{\lambda^2+\lambda}\,\mathrm{e}^{2(p+c)/p}\,p_y}{p^2} \end{pmatrix},$$

$$B = \begin{pmatrix} \dfrac{c^2\sqrt{\lambda^2+\lambda}\;p_x}{\mathrm{e}^{2(p+c)/p}\,p^2} & \dfrac{c\sqrt{\lambda^2+\lambda}}{\mathrm{e}^{(p-2c\lambda)/p}} \\ \dfrac{c\sqrt{\lambda^2+\lambda}}{\mathrm{e}^{(p+2c\lambda+2c)/p}} & -\dfrac{c^2\sqrt{\lambda^2+\lambda}\;p_x}{\mathrm{e}^{2(p+c)/p}\,p^2} \end{pmatrix}.$$

## Link to an evolution system

Consider an evolution system

$$u_y = v_x, \;\; v_y = -c\frac{\sqrt{c}\,x^2 + 1}{u_x}, \tag{18}$$

and set

$$p = \frac{2c}{\ln(c^2 u_x) - 2},$$

Then so defined $p$ satisfies the reduced GMC(17), i.e., (17) is a differential consequence of (18).

## Rescaling

Moreover, upon introducing rescaled independent variables $t = \sqrt{c}\,t$, $z = \sqrt{c}\,x$ we can transform (18) into

$$u_t = v_z, \;\; v_t = -\frac{1}{u_z} - z^2. \qquad (19)$$

# Symmetries of the evolution system

Define a nonlocal variable $w$ by the formulas $w_z = u, w_t = v$. Then we have the following symmetries of (19):

$$\begin{pmatrix} U_1 \\ V_1 \end{pmatrix} = \begin{pmatrix} t \\ z \end{pmatrix}, \quad \begin{pmatrix} U_2 \\ V_2 \end{pmatrix} = \begin{pmatrix} u_z \\ v_z \end{pmatrix}, \quad \begin{pmatrix} U_3 \\ V_3 \end{pmatrix} = \left( v_z, -\frac{1}{u_z} - z^2 \right)^T,$$

$$\begin{pmatrix} U_4 \\ V_4 \end{pmatrix} = \begin{pmatrix} t^2 \\ 2zt \end{pmatrix}, \quad \begin{pmatrix} U_5 \\ V_5 \end{pmatrix} = \begin{pmatrix} z u_z - t v_z + u \\ z v_z + t u_z^{-1} - v + z^2 t \end{pmatrix},$$

$$\begin{pmatrix} U_6 \\ V_6 \end{pmatrix} = \begin{pmatrix} v_z^2 + 2z^2 u_z - 2\ln(u_z) + 4zu - 4w \\ 2z^2 v_z - 2u_z^{-1} v_z - 4zv \end{pmatrix},$$

$$\begin{pmatrix} U_7 \\ V_7 \end{pmatrix} = \begin{pmatrix} 2u_z v_z + z t u_z - 2t^2 v_z + tu \\ v_z^2 + 4ztv_z + 2t^2 u_z^{-1} + 2\ln(u_z) - 4tv + 4w + 2z^2 t^2 \end{pmatrix}$$

...

# Integrability of the evolution system – ZCR

System (19) has a ZCR

$$A = \begin{pmatrix} \dfrac{(\lambda+\frac{1}{2})u_{zz}}{2u_z} + \dfrac{1}{2}\sqrt{\lambda^2+\lambda}\,v_{zz} & (\lambda+1)\sqrt{u_z} \\ \lambda\sqrt{u_z} & -\dfrac{(\lambda+\frac{1}{2})u_{zz}}{2u_z} - \dfrac{1}{2}\sqrt{\lambda^2+\lambda}\,v_{zz} \end{pmatrix},$$

$$B = \begin{pmatrix} \dfrac{\sqrt{\lambda^2+\lambda}\,u_{zz}}{2u_z^2} + \dfrac{(\lambda+\frac{1}{2})v_{zz}}{2u_z} & \dfrac{\sqrt{\lambda^2+\lambda}}{\sqrt{u_z}} \\ \dfrac{\sqrt{\lambda^2+\lambda}}{\sqrt{u_z}} & -\dfrac{\sqrt{\lambda^2+\lambda}\,u_{zz}}{2u_z^2} - \dfrac{(\lambda+\frac{1}{2})v_{zz}}{2u_z} \end{pmatrix}$$

with the spectral parameter $\lambda$.

# Recursion operator – preliminaries

Given a symmetry $(U,V)^T$ of (19), introduce

nonlocal variables $W, R, S$ given by the formulas

$$W_z = U,$$

$$W_t = V,$$

$$R_z = \frac{u_z v_z V_z + z^2 u_z U_z - 2W u_z + 2zU u_z - U_z}{2u_z}$$

$$R_t = \frac{z^2 u_z^2 V_z - 2zV u_z^2 - u_z V_z + v_z U_z}{2u_z^2}$$

$$S_z = \frac{1}{2}\, u_z V_z + \frac{1}{2}\, v_z U_z + ztU_z - \frac{1}{2}\, t^2 V_z + tU$$

$$S_t = \frac{u_z^2 v_z V_z + 2ztu_z^2 V_z - 2tV u_z^2 + 2W u_z^2 + u_z U_z - t^2 U_z}{2u_z^2}.$$

# Recursion operator

**Proposition 1** *Under the above assumptions*

$$\mathfrak{R}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} (v_z + 2zt)U - (u_z + t^2)V - 2S \\ -(u_z^{-1} + z^2)U - v_z V + 2R \end{pmatrix}$$

*is a new symmetry for (19), i.e.,* $\mathfrak{R}$ *is a recursion operator for (19).*

Thus, the system (19) has infinitely many symmetries. Indeed, applying $\mathfrak{R}$ to the above symmetries $(U_i, V_i)^T$ yields infinite hierarchies of (nonlocal) symmetries.

## Conclusions

- We disproved the conjecture that only linear Weingarten surfaces are integrable.
- We found a link from an evolution system to the hyperbolic equation describing the nonlinear integrable Weingarten surfaces and found a recursion operator for the evolution system in question.

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