What is the velocity of the electromagnetic field?

W.I. FUSHCHYCH

A new definition for the electromagnetic field velocity is proposed. The velocity depends on the physical fields.

The question posed by the title of this paper is, surprisingly, not yet answered uniquely today; not even by way of definition. According to modern assumptions the light is the electromagnetic field (with corresponding frequencies) and therefore it is obvious that the answer to the posed fundamental question is not obvious.

Today the following definitions of the velocity of light are used [1, 2]:

1) phase velocity,
2) group velocity,
3) velocity of energy transport.

The definition of phase- and group velocity is based on assumptions that the electromagnetic wave can be characterized by the function $\Psi(t,\vec{x})$, which has the following form [1, 2]

$$\Psi(t,\vec{x}) = A(\vec{x})\cos(\omega t - g(\vec{x})) \tag{1}$$

or

$$\Psi(t,\vec{x}) = \int_0^\infty A_\omega(\vec{x})\cos(\omega t - g_\omega(\vec{x}))d\omega, \tag{2}$$

where $A(\vec{x})$ is the wave amplitude and $g(\vec{x})$ is an arbitrary real function. The phase-velocity is defined by the following formula

$$v_1 = \omega/|\vec{\bigtriangledown} g(\vec{x})|. \tag{3}$$

By the above formulas it is clear that the definition of the phase- and group-velocity is based on the assumption that the electromagnetic wave has the structure (1) (or (2)) and its velocity does not depend on the amplitude A. Moreover, the equation which is to be satisfied by Ψ, has never been clearly stated. This is, in fact, a very important point since Ψ can satisfy the standard linear wave equation (d'Alembert equation) or, for example a nonlinear wave equation [3]. These two cases are essentially different and lead to principly different results. One should mention that the phase- and group-velocities cannot directly be defined in terms of the electromagnetic fields $\vec{E}$ and $\vec{H}$.

The velocity of electromagnetic energy transport is defined by the formula

$$\vec{v}_2 = \frac{\vec{s}}{W}, \quad \vec{s} = c(\vec{E}\times\vec{H}), \quad W = \vec{E}^2 + \vec{H}^2, \tag{4}$$

where $\vec{s}$ is the Poyting–Heaviside vector.

J. Nonlinear Math. Phys., 1998, **5**, № 2, P. 159–161 (translated by Marianna Euler and reprinted from Dopovidi of the Academy of Sciences of Ukraine, 1997, № 4, P. 51–53).

Formula (4) has the following disadvantage: Both E and H are invariant under the Lorentz transformation, whereas v_2 does not have this property.

The aim of the present paper is to give some new definitions of the electromagnetic field velocity.

If the electromagnetic field is some energy flow, then we define the velocity of such flow, in analogy with hydrodynamics [4], by the following equation

$$\begin{aligned}\frac{\partial\vec{v}}{\partial t}+v_l\frac{\partial\vec{v}}{\partial x_l}&=a_1(\vec{D},\vec{B}^2,\vec{E}^2,\vec{H}^2,\vec{D}\vec{E},\ldots)\vec{D}+a_2(\vec{D},\vec{B}^2,\vec{E}^2,\vec{H}^2,\vec{D}\vec{E},\ldots)\vec{B}+\\ &+a_3(\vec{D},\vec{B}^2,\vec{E}^2,\vec{H}^2,\vec{D}\vec{E},\ldots)\vec{E}+a_4(\vec{D},\vec{B}^2,\vec{E}^2,\vec{H}^2,\vec{D}\vec{E},\ldots)\vec{H}+\\ &+a_5(\vec{D},\vec{B}^2,\vec{E}^2,\vec{H}^2,\vec{D}\vec{E},\ldots)\left(c(\vec{\bigtriangledown}\times\vec{H})-\frac{\partial\vec{D}}{\partial t}-4\pi\vec{J}\right)+\\ &+a_6(\vec{D},\vec{B}^2,\vec{E}^2,\vec{H}^2,\vec{D}\vec{E},\ldots)\left(c(\vec{\bigtriangledown}\times\vec{H})+\frac{\partial\vec{B}}{\partial t}\right).\end{aligned}\tag{5}$$

The structure and explicit form of the coefficients $a_1,\ldots,a_6$ is defined by the demand that equation (5) should be invariant with respect to the Poincaré group if the fields are transformed according to the Lorentz transformation [5].

The main advantage of (5), in comparison with (1), (2), lies in the following:

1. The velocity of the electromagnetic field is directly defined by the observables $\vec{D}$, $\vec{B}$, $\vec{E}$, $\vec{H}$, $\vec{J}$, and their first derivatives.
2. For particular coefficients, eq. (5) is invariant under the Poincaré group.
3. In the case where $a_1=a_2=a_3=a_4=0$ and the fields $\vec{D}$, $\vec{B}$, $\vec{E}$, $\vec{H}$ satisfy Maxwell's equation
$$c(\vec{\bigtriangledown}\times\vec{H})-\frac{\partial\vec{D}}{\partial t}-4\pi\vec{J}=0,\quad c(\vec{\bigtriangledown}\times\vec{E})+\frac{\partial\vec{B}}{\partial t}=0,\tag{6}$$
then the velocity of the electromagnetic field is of constant value, with
$$\frac{\partial\vec{v}}{\partial t}+v_l\frac{\partial\vec{v}}{\partial x_l}=0.\tag{7}$$

In order to use eq. (5) one should concretely define the coefficients $a_1,\ldots,a_6$.

The explicitly-covariant definition of electromagnetic field velocity can be given the following equation [5]

$$v_\mu\frac{\partial v_\alpha}{\partial x^\mu}=a(\vec{E}^2,\vec{H}^2,\vec{E}\vec{H})F_{\alpha\beta}v^\beta.\tag{8}$$

Using Maxwell's equation in vacuum, one can obtain the following formula for the velocity of the electromagnetic field

$$|\vec{v}|=\left\{\frac{1}{2}\frac{(\partial\vec{E}/\partial t)^2+(\partial\vec{H}/\partial t)^2}{(\operatorname{rot}\vec{E})^2+(\operatorname{rot}\vec{H})^2}\right\}^{1/2}\tag{9}$$

From (7) it is clear that the velocity depends only on derivatives of the fields. $|\vec{v}|$ is a conditional invariant with respect to the Lorentz transformation, i.e., if $\vec{E}$ and $\vec{H}$ satisfy the full system of Maxwell's equations in vacuum, then $|\vec{v}|$ would be an invariant of the Lorentz group. In other words, the conditional invariant is a particular scalar combination of the fields, for which the fields satisfy some equations with nontrivial solutions. Well known invariants for the electromagnetic field $\vec{E}\vec{H}$ and $\vec{E}^2 - \vec{H}^2$ are absolute invariants with respect to the Lorentz group.

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