Lowering of order and general solutions of some classes of partial differential equations

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A procedure of lowering the order and construction of general solutions for some classes of partial differential equations (PDEs) are proposed. Some classes of general solutions of some linear and nonlinear equations of mathematical physics are constructed and a series of examples is presented.

The construction of the general solution of a definite partial differential equation is in a number cases an unsolved problem. In what follows, we propose an algorithm of lowering the order and constructing general solutions of specific partial differential equations.

Consider the following partial differential equation

$$L(D[u]) + F(D[u]) = 0, \tag{1}$$

where $u = u(x)$, $x = (x_0, x_1, \ldots, x_k)$; L is a first-order differential operator of the form

$$L \equiv a^i(x,u)\partial_{x_i}, \quad i = 0, 1, \ldots, k, \tag{2}$$

and $a^i(x,u)$ are arbitrary smooth functions which are not identically equal to zero simultaneously. $D[u]$ is an n-order differential expression

$$D[u] = D\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(n)}\right), \tag{3}$$

where $u_{(m)}$ is the collection of m-th order derivatives, $m = 1, \ldots, n$, and F is an arbitrary smooth function of $D[u]$. As a particular case, $D[u]$ may depend only on x and u. In this case we say that $D[u]$ is of order zero. In general, (1) is an $(n+1)$-th order partial diffrential equation.

For equations of the type (1), we propose a method of lowering the order and construction of solutions based on the local change of variables which reduces operators (2) to the operator of differentiation with respect to one of independent variables.

We introduce the change of variables

$$\tau = f^0(x,u), \quad \omega^a = f^a(x,u), \ a = 1, \ldots, k, \quad z = u, \tag{4}$$

where $z(\tau, \vec{\omega})$ is a new dependent variable, $\vec{\omega} = (\omega^1, \ldots, \omega^k)$.

We determine functions f^0, f^a from the conditions

$$L(f^0) = 1, \quad L(f^a) = 0, \quad a = 1, \ldots, k, \tag{5}$$

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and functions $f^1, \ldots, f^k$ and u must form a complete collection of functionally-independent invariant of operator (2). We choose f^0 as a particular solution of the equation $Ly = 1$.

Relations (5) determine the change of variables (4) such that operator L is reduced to the operator of differentiation with respect to the variable τ, i.e.,

$$L \Rightarrow \partial_\tau. \tag{6}$$

We obtain a new form of (3) in new variables (4) and rewrite the initial equation (1) in the form

$$\partial_\tau\big(\widetilde{D}[z]\big) + F\big(\widetilde{D}[z]\big) = 0, \tag{7}$$

where $\widetilde{D}[z]$ is $D[u]$ in the new variables (4).

Relation (7) is the first order ordinary differential equation with respect to the variable τ. We integrate it and obtain $\widetilde{D}[z]$. Thus, when we solve (7), we obtain an n-th order partial differential equation with respect to $z(\tau, \vec{\omega})$ with one arbitrary function depending on $\vec{\omega}$ which is a “constant” of integration of Eq. (7).

Remark. This algorithm is also effective in the case where Eq. (1) has the form

$$L(D[u]) + F(D[u], f^0, f^1, \ldots, f^k) = 0. \tag{8}$$

Here, functions $f^0, \ldots, f^k$ must satisfy relations (5). In this case, integrating the corresponding ordinary differential equation (an analog of equation (7)) we regard variables ω^a as parameters.

Example 1. Consider the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0. \tag{9}$$

Equation (9) can be written in the form (1), namely:

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\right) = 0. \tag{10}$$

After the change of variables

$$\tau = t, \quad \omega = x + t, \quad z = u,$$

Eq. (10) can be rewritten in the form

$$\partial_\tau\left(z_\tau + 2z_\omega\right) = 0.$$

We integrate this equation and obtain

$$z_\tau + 2z_\omega = g(\omega), \tag{11}$$

Since $g(\omega)$ is arbitrary, we set $g(\omega) = 2h'(\omega)$. Then characteristic system of for the inhomogeneous quasi-linear Eq. (11) has the form

$$\frac{d\tau}{1} = \frac{d\omega}{2} = \frac{dz}{2h'(\omega)}.$$

We find the first integrals of the characteristic system and we get the following solution of Eq. (11),

$$z - h(\omega) = f(\omega - 2\tau), \tag{12}$$

where h and f are arbitrary functions. Then we rewrite (12) in variables (t, x, u) and get the following well-known general solutions of Eq. (9)

$$u = h(x + t) + f(x - t).$$

Example 2. Consider the following equation proposed in [3] for description of motion of a liquid,

$$L(Lu) + \lambda Lu = 0, \quad L \equiv \partial_t + u\partial_x. \tag{13}$$

This equation can be regards as a generalization of the one-dimensional Newton–Euler equation (the equation of simple wave). In the explicit form, Eq. (13) has the form

$$\frac{\partial^2 u}{\partial t^2} + 2u\frac{\partial^2 u}{\partial t \partial x} + \frac{\partial u}{\partial t}\frac{\partial u}{\partial x} + u\left(\frac{\partial u}{\partial x}\right)^2 + u^2\frac{\partial^2 u}{\partial x^2} + \lambda\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = 0.$$

Since Eq. (13) belongs to the class of (1), the change of variables

$$\tau = t, \quad \omega = x - ut, \quad z = u,$$

allows us to write it as

$$\partial_\tau\left(\frac{z_\tau}{1 + \tau z_\omega}\right) + \lambda\frac{z_\tau}{1 + \tau z_\omega} = 0. \tag{14}$$

Having integrated (14), e.g., for $\lambda = 0$, we obtain the parametric solution

$$z \pm \int \frac{d\omega}{\sqrt{h(\omega) + p}} = \varphi(p), \quad \tau^2 - h(\omega) = p, \tag{15}$$

where p is a parameter, h and φ are arbitrary functions.

Then we return to the initial variables and obtain a solution of Eq. (13). Below, we give several classes of solutions of Eq. (13) with one arbitrary function [1] (The fact that we have only one arbitrary function associated with the problem of integration of system of type (15)).

1. $L(Lu) = 0$:

1.1 $u \pm \ln(x - ut \mp t) = \varphi\left(t^2 - (x - ut)^2\right),$

1.2 $u + \dfrac{t(x - ut)^3}{t^2(x - ut)^2 - 1} = \varphi\left(t^2 - \dfrac{1}{(x - ut)^2}\right),$

1.3 $u = \varphi\left(\dfrac{x - ut}{\exp\left(t^2\right)}\right) - \dfrac{x - ut}{\exp\left(t^2\right)}\displaystyle\int \exp\left(t^2\right) dt.$

2. $L(Lu) = a$:

$$x - ut + \frac{a}{3}t^3 + \frac{C}{2}t^2 = \varphi\left(u - \frac{a}{2}t^2 - Ct\right).$$

3. $L(Lu)+Lu=a$

$$x-ut-C(t+1)\exp(-t)+\frac{a}{2}t^2=\varphi\left(u+C\exp(-t)-at\right).$$

Here, $C=\mathrm{const}$, φ is arbitrary function.

Example 3. The equation

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}+2\frac{\partial^2 u}{\partial x\partial y}=0 \tag{16}$$

can be written in the form (1) as follows:

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right)=0.$$

Using the change of variables

$$\tau=t,\quad \omega^1=t+x,\quad \omega^2=t-y,\quad z=u,$$

and applying the algorithm described earlier, we obtain the following solution of Eq. (16)

$$u=f(t+x,t-y)+g(t-x,t+y),$$

where f and g are arbitrary functions.

It is natural to generalize the described algorithm for equations of the form (1) to the classes of partial differential equations of the form

$$L^m(D[u])+b_{m-1}L^{m-1}(Du)+\cdots+b_1L(D[u])+b_0=0, \tag{17}$$

where

$$b_j=b_j(Du,f^0,f^1,\ldots,f^k),\quad j=\overline{0,m-1};\quad L^m=\underbrace{LLL\cdots LL}_{m};$$

L, $D[u]$, $f^0,f^1,\ldots,f^k$ are determined according to the relations (2)–(6).

After the change of variables (4)–(6), the problem lowering the order of Eq. (17) is reduced to the problem of integrating the m-th order ordinary differential equation.

Example 4. For

$$D^n(u)=0,\quad D\equiv x_\mu\partial_{x_\mu},\quad \mu=0,\ldots,k,$$

we use the change of variables

$$\tau=\ln x_0,\quad \omega^a=\frac{x_a}{x_0},\ a=\overline{1,k},\quad z=u,$$

and we obtain the solution

$$u=C_{n-1}(\ln x_0)^{n-1}+C_{n-2}(\ln x_0)^{n-2}+\cdots+C_1\ln x_0+C_0,$$

where $C_i=C_i\left(\frac{x_1}{x_0};\cdots;\frac{x_k}{x_0}\right)$, $i=\overline{0,n-1}$.

The obtained results can be easily generalized to the case of system of equations

$$L(\vec{D}[\vec{u}]) = \vec{F}\big(f^0, f^1, \dots, f^k, \vec{D}[\vec{u}]\big),$$

where $\vec{u} = (u^1(x), \dots, u^m(x))$, $x = (x_0, x_1, \dots, x_k)$; L, $f^0, f^1, \dots, f^k$ are determined according to relations (2), (4), (5) and (6). Here, $u \equiv \vec{u}$; $\vec{D}[\vec{u}] = (D^1, \dots, D^m)$, where $D^i = D^i\left(x, \vec{u}, \vec{u}_{(1)}, \vec{u}_{(2)}, \dots, \vec{u}_{(n)}\right)$, $i = 1, \dots, m$, $\vec{u}_{(i)}$ is a collection of i-th order derivatives for each component of the vector $\vec{u}$; and $\vec{F} = (F^1, \dots, F^m)$. In particular, the components of the vector $\vec{D}[\vec{u}]$ can dependent only on x and $\vec{u}$.

Example 5. Consider the system of Euler equations

$$\frac{\partial \vec{v}}{\partial x_0} + v^k \frac{\partial \vec{v}}{\partial x_k} = \vec{0}, \tag{18}$$

where $\vec{v} = (v^1, v^2, v^3)$, $v^l = v^l(x_0, x_1, x_2, x_3)$, $l = 1, 2, 3$.

The system (18) can be written as follows:

$$\left(\partial_0 + v^k \partial_k\right) v^l = 0, \quad l = 1, 2, 3. \tag{19}$$

After the change of variables

$$\begin{aligned}
&\tau = x_0, \\
&\omega^a = x_a - v^a x_0, \quad a = 1, 2, 3, \\
&z^l = v^l, \quad l = 1, 2, 3
\end{aligned}$$

the system (19) takes the form

$$\partial_\tau z^l = 0, \quad l = 1, 2, 3. \tag{20}$$

Then we integrate Eq. (20), apply the inverse change of variables, and obtain a solution of system (18) in an implicit form (compare this solutions with one from [2])

$$v^l = g^l(x_1 - v^1 x_0, x_2 - v^2 x_0, x_3 - v^3 x_0).$$

where g^l are arbitrary functions.

Example 6. Consider the following system of equation for vector-potential A^μ,

$$A^\nu \frac{\partial A^\mu}{\partial x_\nu} = 0, \quad \mu = 0, \dots, 3. \tag{21}$$

Assume that $A^0 \neq 0$. By the change of variables

$$\begin{aligned}
&\tau = \frac{x_0}{A^0}, \\
&\omega^a = x_a A^0 - x_0 A^a, \quad a = 1, 2, 3, \\
&A^\mu = A^\mu, \quad \mu = 0, 1, 2, 3
\end{aligned}$$

we obtain the following solutions of system (21)

$$A^\mu = g^\mu(x_1 A^0 - x_0 A^1, x_2 A^0 - x_0 A^2, x_3 A^0 - x_0 A^3),$$

where g^μ are arbitrary functions.

Consider a system of partial differential equations determined by the collection of operators $L^1, \dots, L^r$ of the form (2) ($u \equiv \vec{u}$), and the number of operators must not exceed the number of independent variables, i.e., $r \leq k+1$. In other words, consider the system of partial differential equations which consists of m equations of the form (8), where L is one of the operators $L^1, \ldots, L^r$ and $D[u] \equiv \vec{D}[\vec{u}]$. If these operators form a commutative algebra Lie and the rank of the matrix consisting of the coefficients of the operators $L^1, \ldots, L^r$ is equal to r, then there exists a local change of variables which transforms these operators to r operators of differentiation with respect to r first independent variables. Thus, if the above conditions are satisfied for a system, we can lower its order and in some cases construct its solutions (at least in principle).

Example 7. Consider the system

$$\begin{aligned} &\Big(\partial_t + v\partial_x\Big)u = 0, \\ &\Big(\partial_t + u\partial_x\Big)v = 0, \end{aligned} \tag{22}$$

where $u = u(t,x), v = v(t,x), u \neq v$.

After the change of variables

$$\tau = \frac{x - ut}{v - u}, \quad \omega = \frac{x - vt}{u - v}, \quad U = u, \quad V = v \tag{23}$$

the system (22) takes the simple form

$$\begin{aligned} &\partial_\tau U = 0, \\ &\partial_\omega V = 0. \end{aligned} \tag{24}$$

Integrating (24) and performing the change of variable inverse to (23), we obtain a solution of (22) in the form

$$u = f\left(\frac{x - vt}{u - v}\right), \quad v = g\left(\frac{x - ut}{v - u}\right),$$

where f and g are arbitrary functions.

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