# On a new conformal symmetry for a complex scalar field

*P. BASARAB-HORWATH, W.I. FUSHCHYCH, O.V. ROMAN*

We exhibit a new nonlinear representation of the conformal algebra which is the symmetry algebra of a nonlinear hyperbolic wave equation. The equation is the only one of its type invariant under the conformal algebra in this nonlinear representation. We also give a list of some nonlinear hyperbolic equations which are invariant under the conformal algebra in the standard representation.

In this note we examine a nonlinear wave equation for a complex field, having the following structure

$$\Box u = F(u, u^*, \nabla u, \nabla u^*, \nabla|u|\nabla|u|, \Box|u|)u, \tag{1}$$

where $u = u(x) = u(x_0, x_1, \dots, x_n)$, $\nabla u = (u_{x_0}, \dots, u_{x_n})$, $\nabla u^* = (u^*_{x_0}, \dots, u^*_{x_n})$, $\nabla|u|\nabla|u| = |u|_\mu |u|^\mu = g^{\mu\nu}\frac{\partial|u|}{\partial x^\mu}\frac{\partial|u|}{\partial x^\nu}$, $g^{\mu\nu} = \mathrm{diag}(1, -1, \dots, -1)$, and we use the usual summation convention. Here, $F$ is an arbitrary real-valued function.

Examples of equations such as (1) can be found in the literature, the most common being the nonlinear Klein–Gordon type [2, 3],

$$\Box u = F(|u|,\ |u|_\mu |u|^\mu)u. \tag{2}$$

Another such equation is that proposed (independently of each other) by Guéret and Vigier [9] and by Guerra and Pusterla [10],

$$\Box u = \frac{\Box|u|}{|u|}u - \frac{m^2c^2}{\hbar^2}u. \tag{3}$$

This equation arose in the modelling of an equation for de Broglie's theory of the double solution [1]. Guéret and Vigier were able to show that a solution to this problem, obtained by Mackinnon [11] satisfied Eq. (3). Guerra and Pusterla obtained (3) as a relativistic version of a nonlinear Schrödinger equation they had found by applying stochastic methods to quantum mechanics.

Eq. (3) is from our point of view (namely, the symmetry view) a remarkable nonlinear equation, since it is invariant under the conformal algebra $AC(1, n+1)$ in an unusual representation.

It is well-known (see, for instance, Refs. [3, 7]) that the free wave equation $\Box u = 0$ is invariant under the conformal group $AC(1, n)$ with infinitesimal operators

$$P_\mu = \frac{\partial}{\partial x^\mu}, \quad J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu, \tag{4}$$

$$D = x^\mu P_\mu - \frac{n-1}{2}(u\partial_u + u^*\partial_{u^*}), \quad K_\mu = 2x_\mu - x^2 P_\mu, \tag{5}$$

Physics Letters A, 1997, **226**, P. 150–154.

with $x^2 = x_\mu x^\mu$. The wave equation is also invariant under the operators

$$I = i(u\partial_u - u^*\partial_{u^*}), \quad Q = u\partial_u + u^*\partial_{u^*},$$
$$L_1 = u^*\partial_u + u\partial_{u^*}, \quad L_2 = i(u^*\partial_u - u\partial_{u^*}),$$

which are important in reducing the wave equation to the Schrödinger and heat equations (see Refs. [4, 5, 6]).

The conformal operators $K_\mu$ generate the finite conformal transformations

$$x_\mu \to x'_\mu = \frac{x_\mu - x^2 c_\mu}{1 - 2c_\alpha x^\alpha + c^2 x^2}, \tag{6}$$

$$u \to u' = (1 - 2c_\alpha x^\alpha + c^2 x^2)^{(n-1)/2} u, \tag{7}$$

where $c_\mu$ are parameters.

All equations of the form (2) invariant under the conformal group with infinitesimal generators given in the representation (4), (5) were classified in Ref. [2]. In particular, it was shown there that when the function $F$ is independent of the derivatives of $u$, then the equation is conformally invariant under (4), (5) if and only if

$$F(u) = \lambda |u|^{4/(n-1)}, \tag{8}$$

where $n \geq 2$ and $\lambda$ is an arbitrary parameter. Thus, Eq. (1), when the right-hand side does not depend on the derivatives of $u$, has the same conformal invariance as the free wave equation if and only if $F$ is given by (8).

An analysis of the proof of this statement shows that two things are fixed at the outset: the independence of $F$ of the derivatives; and the representation of the algebra $AC(1, n)$. One then sees that the following natural question arises: does there exist a representation of $AC(1, n)$ different from (4), (5)? That is, are there operators $K_\mu$, $D$ which are not equivalent to those given in (5)? Our answer to this question is that there *exists* such a representation.

To this end, we have calculated the Lie point symmetry algebra of the equation (see, for instance, Ref. [12, 3])

$$\Box u = \frac{\Box |u|}{|u|} u + \lambda u, \tag{9}$$

with $\lambda$ an arbitrary parameter. It is evident that this equation is Poincaré invariant with respect to the operators (4). On the other hand, it is definitely not invariant under the conformal operators given in (5). However, this does not mean that it is not at all conformally invariant, as we see from the following result.

**Theorem 1.** *Eq. (9) with $\lambda < 0$ has maximal point-symmetry algebra $AC(1, n+1) \oplus Q$ generated by operators*

$$P_\mu, \ J_{\mu\nu}, \ P_{n+1}, \ J_{\mu n+1}, \ D^{(1)}, \ K_\mu^{(1)}, \ K_{n+1}^{(1)}, \ Q,$$

*where*

$$P_\mu = \frac{\partial}{\partial x^\mu}, \quad J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu, \quad P_{n+1} = \frac{\partial}{\partial x^{n+1}} = i(u\partial_u - u^*\partial_{u^*}),$$

$$J_{\mu n+1} = x_\mu P_{n+1} - x_{n+1} P_\mu, \quad D^{(1)} = x^\mu P_\mu + x^{n+1} P_{n+1} - \frac{n}{2}(u\partial_u + u^*\partial_{u^*}),$$
$$K_\mu^{(1)} = 2x_\mu D^{(1)} - (x_\mu x^\mu + x_{n+1} x^{n+1}) P_\mu,$$
$$K_{n+1}^{(1)} = 2x_{n+1} D^{(1)} - (x_\mu x^\mu + x_{n+1} x^{n+1}) P_{n+1}, \quad Q = u\partial_u + u^*\partial_{u^*},$$

*where the additional variable $x^{n+1}$ is defined as*

$$x^{n+1} = -x_{n+1} = \frac{i}{2\sqrt{-\lambda}} \ln \frac{u^*}{u}, \quad \lambda < 0.$$

*For $\lambda > 0$ the maximal symmetry algebra of (9) is $AC(2,n) \oplus Q$ generated by the same operators above, but with the additional variable*

$$x^{n+1} = x_{n+1} = \frac{i}{2\sqrt{\lambda}} \ln \frac{u^*}{u}, \quad \lambda > 0.$$

**Remark 1.** In this theorem we have introduced a new metric tensor

$$g_{AB} = \operatorname{diag}(1, -1, \ldots, -1, g_{n+1\,n+1})$$

with $g_{n+1\,n+1} = 1$ when $\lambda > 0$ and $g_{n+1\,n+1} = -1$ when $\lambda < 0$.

Direct verification shows that the above operators satisfy the commutation relations of the conformal algebra $AC(1, n+1) \oplus Q$ when $\lambda < 0$ and $AC(2,n) \oplus Q$ when $\lambda > 0$.

The meaning of the new operators $P_{n+1}$, $J_{\mu n+1}$, $K_\mu^{(1)}$, $K_{n+1}^{(1)}$ is best understood when Eq. (9) is rewritten in the amplitude-phase representation, namely, on putting $u = Re^{i\theta}$ with $R$ and $\theta$ being real functions. Then equation (9) becomes the system

$$g^{\mu\nu}\theta_\mu\theta_\nu = -\lambda, \tag{10}$$

$$R\Box\theta + 2g^{\mu\nu} R_\mu \theta_\nu = 0. \tag{11}$$

The symmetry algebra of Eq. (9) is actually obtained by first calculating the symmetry algebra of the system (10), (11). Then we have, in the amplitude-phase representation

$$P_{n+1} = \frac{\partial}{\partial\theta}, \quad J_{\mu n+1} = \left(x_\mu \frac{\partial}{\partial\theta}\right) - \theta \frac{\partial}{x^\mu}, \tag{12}$$

$$D^{(1)} = x^\mu \frac{\partial}{\partial x^\mu} + \theta \frac{\partial}{\partial\theta} - \frac{n}{2} R \frac{\partial}{\partial R}, \tag{13}$$

$$K_\mu^{(1)} = 2x_\mu D^{(1)} - (x_\mu x^\mu + g_{n+1\,n+1}\theta^2) \frac{\partial}{\partial x^\mu}, \tag{14}$$

$$K_{n+1}^{(1)} = 2g_{n+1\,n+1}\theta D^{(1)} - (x_\mu x^\mu + g_{n+1\,n+1}\theta^2) \frac{\partial}{\partial\theta}. \tag{15}$$

From the expressions (12)–(15), we see that the phase variable $\theta$ has been added to the $n+1$-dimensional geometric space of the $x^\mu$. This is the same effect we see for the eikonal equation [3], and it is not surprising, since the first equation of system (10), (11) is indeed the eikonal equation for the phase function $\theta$. What is novel here is that equation (11), which is the equation of continuity, does not reduce the symmetry of equation (10). On using an appropriate ansatz (see Ref. [5]) for $\theta$ and $A$ one can reduce system (10), (11) to another system consisting of the Hamilton–Jacobi equation and the non-relativistic continuity equation. This second system also exhibits surprising symmetry properties [8]: it is again conformally invariant.

Let us remark that the operators $D^{(1)}$, $K_\mu^{(1)}$, $K_{n+1}^{(1)}$ are a nonlinear representation of the dilatation and conformal translation operators. They generate the following finite transformations:

$$
\begin{aligned}
D^{(1)}: \quad & x_\mu \to x'_\mu = \exp(b)x_\mu, \quad \theta \to \theta' = \exp(b)\theta,\\
& R \to R' = \exp(-bn/2)R;\\
K_\mu^{(1)}: \quad & x_\mu \to x'_\mu = \frac{x_\mu - c_\mu(x_\alpha x^\alpha + g_{n+1\,n+1}\theta^2)}{1 - 2c_\nu x^\nu - 2c_{n+1}\theta + c^2(x_\alpha x^\alpha + g_{n+1\,n+1}\theta^2)},\\
& \theta \to \theta' = \frac{\theta}{1 - 2c_\nu x^\nu - 2c_{n+1}\theta + c^2(x_\alpha x^\alpha + g_{n+1\,n+1}\theta^2)},\\
& R \to R' = \left(1 - 2c_\nu x^\nu - 2c_{n+1}\theta + c^2(x_\alpha x^\alpha + g_{n+1\,n+1}\theta^2)\right)^{n/2} R;\\
K_{n+1}^{(1)}: \quad & x_\mu \to x'_\mu = \frac{x_\mu}{1 - 2c_\nu x^\nu - 2c_{n+1}\theta + c^2(x_\alpha x^\alpha + g_{n+1\,n+1}\theta^2)},\\
& \theta \to \theta = \frac{\theta - c^{n+1}(x_\alpha x^\alpha + g_{n+1\,n+1}\theta^2)}{1 - 2c_\nu x^\nu - 2c_{n+1}\theta + c^2(x_\alpha x^\alpha + g_{n+1\,n+1}\theta^2)},\\
& R \to R' = \left(1 - 2c_\nu x^\nu - 2c_{n+1}\theta + c^2(x_\alpha x^\alpha + g_{n+1\,n+1}\theta^2)\right)^{n/2} R.
\end{aligned}
$$

where $b$, $c_\nu$, $c_{n+1}$ are the group parameters and $c^2 = c_\nu c^\nu + c_{n+1}c^{n+1}$ with the usual lowering and raising of indices using the metric $g_{AB}$ used in Theorem 1. The expressions for these finite transformations can be compared with those given in (6), (7). The form is exactly the same, but the new feature is that $\theta$ is considered as a geometrical variable on the same footing as the $x^\mu$, and it is the amplitude $R$ which transforms as the dependent variable, just as $u$ does in (7).

It should be added that Eq. (9) is the only equation of type (1) which is invariant under $AC(1, n+1) \oplus Q$ in the representation given in Theorem 1. This is not the standard representation. However, if we keep the standard representation (4), (5) of the conformal algebra but allow dependence of the nonlinearity in (1) on the derivatives, then we find that there are other equations of this type which are invariant under the conformal algebra:

$$
\begin{aligned}
& \Box u = |u|^{4/(n-1)} F\left(|u|^{(3+n)/(1-n)} \Box |u|\right) u, \quad n \neq 1,\\
& \Box u = \Box |u| F\left(\frac{\Box |u|}{(\nabla |u|)^2}, |u|\right) u, \quad n = 1,\\
& 4\Box u = \left\{\frac{\Box |u|}{|u|} + \lambda \frac{(\Box |u|)^n}{|u|^{n+4}}\right\} u, \quad n \text{ arbitrary},\\
& \Box u = (1+\lambda)\frac{\Box |u|}{|u|} u,\\
& \Box u = \frac{\Box |u|}{|u|}\left(1 + \frac{\lambda}{|u|^4}\right) u,\\
& \Box u = \frac{\Box |u|}{|u|}\left(1 + \frac{\lambda}{1+\sigma |u|^4}\right) u.
\end{aligned}
$$

Thus, we see that wave equations which have a nonlinear quantum potential term $\Box|u|/|u|$ have an unusually wide symmetry. This is in sharp contrast with nonlinearities not containing derivatives. Moreover, we see that the representation of a given algebra plays a fundamental role in picking out certain equations which are invariant. This remark leads us to asking how one can construct all possible representations, linear and nonlinear. Linear representation theory is well-developed, but nonlinear representations are not at all well understood. Certainly, the equation dictates the symmetry and the representation of the symmetry, and both equation and representation are intimately tied together. From the symmetry point of view, we cannot truly distinguish between them as phenomena.

Finally, we remark that given an equation, its symmetry algebra can be exploited to construct ansatzes (see, for example, [3]) for the equation, which reduce the problem of solving the equation to one of solving an equation of lower order, even ordinary differential equations. We examine this question for some of the equations we have given above in a future article, and we hope that some of them will find some application in nonlinear quantum mechanics or optics, not least because of their beautiful symmetry properties and relation to nonlinear Schrödinger equations.

**Acknowledgements.** This work was supported in part by NFR grant R-RA 09423-315, and INTAS and DKNT of Ukraine. W.I. Fushchych thanks the Mathematics Department of Linköping University for its hospitality and for support for travel. P. Basarab-Horwath thanks the Wallenberg Fund of Linköping University the Tornby Fund and the Magnusson Fund of the Swedish Academy of Science for travel grants, and the Mathematics Institute of the Ukrainian Academy of Sciences in Kiev for its hospitality. Olena Roman thanks the International Soros Science Education Program for financial support through grant number PSU061088.

1. de Broglie L., Non-linear wave mechanics, Elsevier Publishing Company, 1960.
2. Fushchych W.I., Yehorchenko I.A., The symmetry and exact solutions of the nonlinear d'Alembert equation for complex fields, *J. Phys. A*, 1989, **22**, 2643–2652.
3. Fushchych W., Shtelen W., Serov M., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Dordrecht, Kluwer, 1993.
4. Basarab-Horwath P., Fushchych W., Barannyk L., Exact solutions of the wave equation by reduction to the heat equation, *J. Phys. A*, 1995, **28**, 5291–5304.
5. Basarab-Horwath P., Fushchych W., Barannyk L., Solutions of the relativistic nonlinear wave equation by solutions of the nonlinear Schrödinger equation, Preprint, Linköping University, 1996.
6. Basarab-Horwath P., A symmetry connection between hyperbolic and parabolic equations, *J. Nonlinear Math. Phys.*, 1996, **3**, 311–318.
7. Fushchych W., Nikitin A., Symmetries of the equations of quantum mechanics, New York, Allerton Press, 1994.
8. Basarab-Horwath P., Barannyk L.L., Fushchych W., Some exact solutions of a conformally invariant nonlinear Schrödinger equation, Preprint, Linköping University, 1997.
9. Geuéret Ph., Vigier J.-P., Relativistic wave equations with quantum potential nonlinearity, *Lett. Nuovo Cimento*, 1983, **38**, 125–128.
10. Guerra F., Pusterla M., Nonlinear Klein–Gordon equation carrying a nondispersive solitonlike singularity, *Lett. Nuovo Cimento*, 1982, **35**, 256–259.
11. Mackinnon L., A nondispersive de Broglie wave packet, *Foundations of Physics*, 1978, **8**, 157–176.
12. Bluman G.W., Kumei S., Symmetries and differential equations, New York, Springer, 1989.