Time-dependent symmetries of variable-coefficient evolution equations and graded Lie algebras

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Polynomial-in-time dependent symmetries are analysed for polynomial-in-time dependent evolution equations. Graded Lie algebras, especially Virasoro algebras, are used to construct nonlinear variable-coefficient evolution equations, both in $1+1$ dimensions and in $2+1$ dimensions, which possess higher-degree polynomial-in-time dependent symmetries. The theory also provides a kind of new realisation of graded Lie algebras. Some illustrative examples are given.

It is well known that the usual family of KdV equations has polynomial-in-time dependent symmetries (ptd-symmetries) which are only of the first-degree. This is because only master symmetries of first degree are so far found. Moreover there are usually[1] no higher-degree ptd-symmetries for time-independent integrable equations in $1+1$ dimensions; but this may not be so in $2+1$ dimensions.

However a form of special graded Lie algebras, namely centreless Virasoro symmetry algebras is apparently common to all time-independent integrable equations in whatever dimensions both in the continuous case and in the discrete case. This feature would therefore seem to be an important one in the discussion of integrability and integrable nonlinear equations. For the higher dimensional integrable equations, there may also exist still more general graded symmetry Lie algebras.

The purpose of the present paper is to discuss ptd-symmetries for evolution equations with polynomial-in-time dependent coefficients (conveniently expressed in terms of monomials in t as in equation (4) below). We provide a purely algebraic structure for constructing such integrable equations with these forms of symmetries. This way we show there do exist integrable equations in $1+1$ dimensions which possess these forms of symmetries and we construct actual examples. Graded Lie algebras, and especially centreless Virasoro algebras, are used for these constructions. In consequence new features are extracted from the graded Lie algebras which provide new realisations of these algebras and most particularly of the centreless Virasoro algebras.

We first define a symmetry for an evolution equation, linear and nonlinear [1–5]. For a given evolution equation $u_t = K(u)$, a vector field $\sigma(u)$ is called its symmetry if $\sigma(u)$ satisfies its linearized equation

$$\frac{d\sigma(u)}{dt} = K'[\sigma], \quad \text{i.e.} \quad \frac{\partial\sigma}{\partial t} = [K,\sigma], \tag{1}$$

where the prime and $[\cdot,\cdot]$ denote the Gateaux derivative and the Lie product

$$K'[S] = \frac{\partial}{\partial\varepsilon} K(u+\varepsilon S)|_{\varepsilon=0}, \quad [K,\sigma] = K'[\sigma] - \sigma'[K], \tag{2}$$

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[1]The Benjamin–Ono equation is a counter-example.

respectively. Of course, a symmetry σ may also depend explicitly on the time variable t. For example, σ may be of polynomial type in t, i.e.

$$\sigma(t,u) = \sum_{j=0}^{n} \frac{t^j}{j!} S_j(u) = S_0 + tS_1 + \cdots + \frac{t^n}{n!} S_n, \tag{3}$$

where the vector fields $S_j(u)$, $0 \le j \le n$, do not depend explicitly on the time variable t.

If we consider a variable-coefficient evolution equation $u_t = K(t,u)$ of the form

$$u_t = K(t,u) = \sum_{i=0}^{m} \frac{t^i}{i!} T_i(u) = T_0 + tT_1 + \cdots + \frac{t^m}{m!} T_m, \tag{4}$$

where the vector fields $T_i(u)$, $0 \le i \le m$, do not depend explicitly on the time variable t, either, then a precise result may be obtained which states (3) is a symmetry of (4). At this stage, we can have

$$\frac{\partial \sigma}{\partial t} = \sum_{i=0}^{n} \frac{t^{i-1}}{(i-1)!} S_i(u) = \sum_{k=0}^{n-1} \frac{t^k}{k!} S_{k+1}(u),$$

$$[K,\sigma] = \left[\sum_{i=0}^{m} \frac{t^i}{i!} T_i(u), \sum_{j=0}^{n} \frac{t^j}{j!} S_j(u) \right] = \sum_{k=0}^{m+n} \frac{t^k}{k!} \sum_{\substack{i+j=k \\ 0\le i\le m \\ 0\le j\le n}} \binom{k}{i} [T_i, S_j].$$

Therefore a simple comparison of each power of t in (1) leads to

$$S_{k+1} = \sum_{\substack{i+j=k \\ 0\le i\le m \\ 0\le j\le n}} \binom{k}{i} [T_i, S_j], \quad 0 \le k \le n-1, \tag{5}$$

$$\sum_{\substack{i+j=k \\ 0\le i\le m \\ 0\le j\le n}} \binom{k}{i} [T_i, S_j] = 0, \quad n \le k \le m+n. \tag{6}$$

These equalities in (5) and (6) constitute a necessary and sufficient condition to state that (3) is a symmetry of (4). If we look at them a little more, it may be seen that

$$\begin{aligned} &S_1 = [T_0, S_0], \\ &S_2 = [T_0, S_1] + [T_1, S_0], \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &S_n = \tbinom{n-1}{0}[T_0, S_{n-1}] + \tbinom{n-1}{1}[T_1, S_{n-2}] + \cdots + \tbinom{n-1}{n-1}[T_{n-1}, S_0], \end{aligned}$$

where $T_i = 0$, $i \ge m+1$, and so a higher-degree ptd-symmetry $\sigma(t,u)$ defined by (3) is determined completely by a vector field S_0. However this vector field S_0 needs to satisfy (6). This kind of vector field S_0 is a generalisation of the master symmetries defined in [2] which here we still call a master symmetry of degree n for the more general evolution equation, equation (4). We conclude the discussion above as a theorem.

Theorem 1. *Let ρ be a vector field not depending explicitly on the time variable t. Define*

$$S_0(\rho) = \rho, \quad S_{k+1}(\rho) = \sum_{j=0}^{k} \binom{k}{j} [T_j, S_{k-j}(\rho)], \quad k \geq 0, \tag{7}$$

where we assume $T_i = 0$, $i \geq m+1$. If there exists $n \in N$ so that $S_j(\rho) = 0$, $j \geq n+1$, then

$$\sigma(\rho) = \sum_{j=0}^{n} \frac{t^j}{j!} S_j(\rho) \tag{8}$$

is a polynomial-in-time dependent symmetry of the evolution equation (4).

We shall go on to construct variable-coefficient integrable equations which possess higher-degree ptd-symmetries as defined by (3). We need to start from the centreless Virasoro algebra

$$\begin{aligned} &[K_{l_1}, K_{l_2}] = 0, \quad l_1, l_2 \geq 0, \\ &[K_{l_1}, \rho_{l_2}] = (l_1 + \gamma) K_{l_1+l_2}, \quad l_1, l_2 \geq 0, \\ &[\rho_{l_1}, \rho_{l_2}] = (l_1 - l_2)\rho_{l_1+l_2}, \quad l_1, l_2 \geq 0 \end{aligned} \tag{9}$$

in which the vector fields $K_{l_1} = K_{l_1}(u)$, $\rho_{l_2} = \rho_{l_2}(u)$, $l_1, l_2 \geq 0$, do not depend explicitly on the time variable t and γ is a fixed constant. Although the vector fields ρ_l, $l \geq 0$, are not symmetries of any equations that we want to discuss, an algebra isomorphic to this kind of Lie algebra commonly arises as a symmetry algebra for many well-known continuous and discrete integrable equations [3–5]. In equation (9), the vector fields ρ_l, $l \geq 0$, may provide the generators of Galilean invariance [6] and invariance under scale transformations for any standard equation $u_t = K_k(u)$. Let us choose a set of specific vector fields

$$T_j = K_{i_j}, \quad 0 \leq j \leq m, \tag{10}$$

which yields the following variable-coefficient evolution equation

$$u_t = K_{i_0} + tK_{i_1} + \frac{t^2}{2!} K_{i_2} + \cdots + \frac{t^m}{m!} K_{i_m}. \tag{11}$$

This equation still has a hierarchy of time-independent symmetries K_l, $l \geq 0$, and therefore it is integrable in the sense of symmetries [7]. What is more, it will inherit many integrable properties of $u_t = K_l$, $l \geq 0$. For example, if $u_t = K_l$, $l \geq 0$, have Hamiltonian structures of the form

$$u_t = K_l = J \frac{\delta H_l}{\delta u}, \quad l \geq 0,$$

where J is a symplectic operator and H_l, $l \geq 0$, do not depend explicitly on t, then the H_l are still conserved densities of equation (11) and equation (11) is then completely integrable in the commonly used sense for pdes. In what follows, we need to prove that ρ_l is a master symmetry (as explained above) of degree $m+1$ of equation (11). In fact, according to (7), we have

$$S_0(\rho_l) = \rho_l, \quad S_{k+1}(\rho_l) = [T_k, S_0(\rho_l)] = [K_{i_k}, \rho_l] = (i_k + \gamma) K_{i_k+l}, \quad 0 \leq k \leq m,$$

and further we can prove that $S_j(\rho_l) = 0$ when $j \geq m+2$, which shows that ρ_l is a master symmetry of degree $m+1$ of equation (11). Therefore we obtain a hierarchy of ptd-symmetries of the form

$$\begin{aligned}\sigma_l(t,u) &= \sum_{j=0}^{m+1} \frac{t^j}{j!} S_j(\rho_l) = \sum_{j=1}^{m+1} \frac{i_{j-1}+\gamma}{j!} t^j K_{i_{j-1}+l} + \rho_l = \\ &= \sum_{j=0}^{m} \frac{i_j+\gamma}{(j+1)!} t^{j+1} K_{i_j+l} + \rho_l, \quad l \geq 0,\end{aligned} \tag{12}$$

for the variable-coefficient and integrable equation (11). Moreover these higher-degree ptd-symmetries together with time-independent symmetries K_l, $l \geq 0$, constitute the same centreless Virasoro algebra as (9), namely

$$\begin{aligned}&[K_{l_1}, K_{l_2}] = 0, \quad l_1, l_2 \geq 0,\\ &[K_{l_1}, \sigma_{l_2}] = (l_1+\gamma) K_{l_1+l_2}, \quad l_1, l_2 \geq 0,\\ &[\sigma_{l_1}, \sigma_{l_2}] = (l_1-l_2)\sigma_{l_1+l_2}, \ l_1, l_2 \geq 0.\end{aligned} \tag{13}$$

For example, we can calculate that

$$\begin{aligned}&[\sigma_{l_1}, \sigma_{l_2}] = \left[\sum_{j=0}^{m} \frac{i_j+\gamma}{(j+1)!} t^{j+1} K_{i_j+l_1} + \rho_{l_1}, \sum_{j=0}^{m} \frac{i_j+\gamma}{(j+1)!} t^{j+1} K_{i_j+l_2} + \rho_{l_2}\right] = \\ &= \left[\sum_{j=0}^{m} \frac{i_j+\gamma}{(j+1)!} t^{j+1} K_{i_j+l_1}, \rho_{l_2}\right] + \left[\rho_{l_1}, \sum_{j=0}^{m} \frac{i_j+\gamma}{(j+1)!} t^{j+1} K_{i_j+l_2}\right] + [\rho_{l_1}, \rho_{l_2}] = \\ &= \sum_{j=0}^{m} \frac{(l_1-l_2)(i_j+\gamma)}{(j+1)!} t^{j+1} K_{i_j+l_1+l_2} + (l_1-l_2)\rho_{l_1+l_2} = (l_1-l_2)\sigma_{l_1+l_2}.\end{aligned}$$

The algebra (13) also gives us a new realisation of centreless Virasoro algebras. By now we may very much see that there exist higher-degree ptd-symmetries for some evolution equations in $1+1$ dimensions. Moreover our derivation does not refer to any particular choices of dimensions and space variables. Hence the evolution equation (11) may be not only both continuous and discrete, but also both $1+1$ and $2+1$ dimensional.

Actually there are many integrable equations which possess a centreless Virasoro algebra (9) (see [3–5, 8, 9] for example). Among the most famous examples are the KdV hierarchy in the continuous case and the Toda lattice hierarchy in the discrete case. Through the theory above, we can say that a KdV-type equation

$$u_t = tK_0 + K_1 = tu_x + u_{xxx} + 6uu_x \tag{14}$$

possesses a hierarchy of second-degree time-polynomial-dependent symmetries

$$\sigma_l = \frac{3}{2} t K_{l+1} + \frac{1}{4} t^2 K_l + \rho_l, \quad l \geq 0, \tag{15}$$

where the vector fields K_l, σ_l, $l \geq 0$, are defined by

$$K_l = \Phi^l u_x, \quad \rho_l = \Phi^l (u + \frac{1}{2} x u_x), \quad \Phi = \partial^2 + 4u + 2u_x \partial^{-1}, \quad l \geq 0.$$

They constitute a centreless Virasoro algebra (9) with $\gamma = \frac{1}{2}$ [8, 10] and thus so do the symmetries K_l, σ_l, $l \geq 0$. We can also conclude that a Toda-type lattice equation

$$\begin{aligned} (u(n))_t = \begin{pmatrix} p(n) \\ v(n) \end{pmatrix}_t &= K_0 + tK_1 + \frac{t^2}{2!}K_0 = \\ &= \left(1 + \frac{1}{2}t^2\right)\begin{pmatrix} v(n) - v(n-1) \\ v(n)(p(n) - p(n-1)) \end{pmatrix} + \\ &\quad + t\begin{pmatrix} p(n)(v(n) - v(n-1)) + v(n)(p(n+1) - p(n-1)) \\ v(n)(v(n-1) - v(n+1)) + v(n)(p(n)^2 - p(n-1)^2) \end{pmatrix} \end{aligned} \tag{16}$$

possesses a hierarchy of third-degree time-polynomial-dependent symmetries

$$\sigma_l = tK_l + t^2K_{l+1} + \frac{1}{6}t^3K_l + \rho_l, \quad l \geq 0, \tag{17}$$

where the corresponding vector fields are defined by

$$K_l = \Phi^l K_0, \quad K_0 = \begin{pmatrix} v - v^{(1)} \\ v(p - p^{(-1)}) \end{pmatrix}, \quad l \geq 0,$$

$$\rho_l = \Phi^l \rho_0, \quad \rho_0 = \begin{pmatrix} p \\ 2v \end{pmatrix}, \quad l \geq 0,$$

in which the hereditary operator Φ is defined by

$$\Phi = \begin{pmatrix} p & (v^{(1)}E^2 - v)(E-1)^{-1}v^{-1} \\ v(E^{-1} + 1) & v(pE - p^{(-1)})(E-1)^{-1}v^{-1} \end{pmatrix}.$$

Here we have used a normal shift operator E: $(Eu)(n) = u(n+1)$ and $u^{(m)} = E^m u$. These discrete vector fields K_l, $l \geq 0$, (see [11] for more information) together with the discrete vector fields ρ_l, $l \geq 0$, constitute a centreless Virasoro algebra (9) with $\gamma = 1$ [4] and the symmetry Lie algebra of σ_l, $l \geq 0$ and K_l, $l \geq 0$, has the same commutation relations as that Virasoro algebra.

More generally, we can consider further algebraic structures by starting from a more general graded Lie algebra. In keeping with the notation in [12], let us write a graded Lie algebra consisting of vector fields not depending explicitly on the time variable t as follows:

$$E(R) = \sum_{i=0}^{\infty} E(R_i), \quad [E(R_i), E(R_j)] \subseteq E(R_{i+j-1}), \quad i, j \geq 0, \tag{18}$$

where $E(R_{-1}) = 0$. Note that such a graded Lie algebra is called a master Lie algebra in [12] since it is actually not a graded Lie algebra as defined in [13]. However we still call it a graded Lie algebra because it is very similar. Choose

$$T_i = K_i \in E(R_0), \quad 0 \leq i \leq m, \tag{19}$$

and consider a variable-coefficient evolution equation

$$u_t = \sum_{i=0}^{m} \frac{t^i}{i!}T_i = K_0 + tK_1 + \frac{t^2}{2!}K_2 + \cdots + \frac{t^m}{m!}K_m. \tag{20}$$

Before we state the main result, we derive two properties of the generating vector fields S_j, $j \ge 0$.

Lemma 1. *Assume that T_i, $0 \le i \le m$, are defined by (19), and let $l \ge 0$ and $\rho_l \in E(R_l)$. Then the vector fields $S_j(\rho_l)$, $j \ge 0$, defined by (7) satisfy the following property*

$$S_{(\alpha-1)(m+1)+\beta}(\rho_l) \in \sum_{i=0}^{l-\alpha} E(R_i), \quad 1 \le \alpha \le l, \quad 1 \le \beta \le m+1, \tag{21}$$

$$S_j(\rho_l) = 0, \quad j \ge l(m+1)+1. \tag{22}$$

Proof. Note the definition (7) of $S_j(\rho_l)$, $j \ge 0$, and $T_i = K_i$, $0 \le i \le m$. We can calculate that

$$\begin{aligned} S_{\alpha(m+1)+\beta+1}(\rho_l) &= \sum_{\gamma=0}^{m} \binom{\alpha(m+1)+\beta}{\gamma} \left[K_\gamma, S_{\alpha(m+1)+\beta-\gamma}(\rho_l)\right] = \\ &= \sum_{\gamma=0}^{\beta-1} \binom{\alpha(m+1)+\beta}{\gamma} \left[K_\gamma, S_{\alpha(m+1)+\beta-\gamma}(\rho_l)\right] + \\ &+ \sum_{\gamma=\beta}^{m} \binom{\alpha(m+1)+\beta}{\gamma} \left[K_\gamma, S_{(\alpha-1)(m+1)+[(m+1)-(\gamma-\beta)]}(\rho_l)\right] \in \\ &\in \sum_{i=0}^{l-(\alpha+2)} E(R_i) + \sum_{i=0}^{l-(\alpha+1)} E(R_i) = \sum_{i=0}^{l-(\alpha+1)} E(R_i), \end{aligned}$$

where in the last but one step we have used the induction assumption. This result shows that (21) is true by mathematical induction. The proof of (22) is the same so that the proof of the Lemma is complete. ■

Lemma 2. *Assume that T_i, $0 \le i \le m$, are defined by (19), and let $l_1, l_2 \ge 0$ and $\rho_{l_1} \in E(R_{l_1})$, $\rho_{l_2} \in E(R_{l_2})$. Then we have*

$$S_k([\rho_{l_1}, \rho_{l_2}]) = \sum_{i+j=k} \binom{k}{i} [S_i(\rho_{l_1}), S_j(\rho_{l_2})], \quad k \ge 0, \tag{23}$$

where the $S_j(\rho)$, $j \ge 0$, are defined by (7).

Proof. We use mathematical induction to prove the required result. Noting that $T_i = K_i$, $0 \le i \le m$, we can calculate that

$$\begin{aligned} S_{k+1}([\rho_{l_1}, \rho_{l_2}]) &= \sum_{i+j=k} \binom{k}{i} \left[K_i, S_j([\rho_{l_1}, \rho_{l_2}])\right] = \\ &= \sum_{i+j=k} \binom{k}{i} \left[K_i, \sum_{\alpha+\beta=j} \binom{j}{\alpha} [S_\alpha(\rho_{l_1}), S_\beta(\rho_{l_2})]\right] \quad \text{(by the induction assumption)} = \\ &= \sum_{i+j=k} \binom{k}{i} \sum_{\alpha+\beta=j} \binom{j}{\alpha} [K_i, [S_\alpha(\rho_{l_1}), S_\beta(\rho_{l_2})]] = \\ &= \sum_{i+\alpha+\beta=k} \frac{k!}{i!\alpha!\beta!} [K_i, [S_\alpha(\rho_{l_1}), S_\beta(\rho_{l_2})]] = \end{aligned}$$

$$= \sum_{i+\alpha+\beta=k} \frac{k!}{i!\alpha!\beta!} \{[[K_i, S_\alpha(\rho_{l_1})], S_\beta(\rho_{l_2})] + [S_\alpha(\rho_{l_1}), [K_i, S_\beta(\rho_{l_2})]]\} =$$

$$= \sum_{j+\beta=k} \binom{k}{j} \left[\sum_{i+\alpha=j} \binom{j}{i} [K_i, S_\alpha(\rho_{l_1})], S_\beta(\rho_{l_2}) \right] +$$

$$+ \sum_{\alpha+j=k} \binom{k}{j} \left[S_\alpha(\rho_{l_1}), \sum_{i+\beta=j} \binom{j}{i} [K_i, S_\beta(\rho_{l_2})] \right] =$$

$$= \sum_{j+\beta=k} \binom{k}{j} [S_{j+1}(\rho_{l_1}), S_\beta(\rho_{l_2})] + \sum_{\alpha+j=k} \binom{k}{j} [S_\alpha(\rho_{l_1}), S_{j+1}(\rho_{l_2})] =$$

$$= \sum_{i+j=k+1} \binom{k+1}{i} [S_i(\rho_{l_1}), S_j(\rho_{l_2})], \quad k \geq 0,$$

and this yields the key step in the mathematical induction. On the other hand, we easily see that

$$S_0([\rho_{l_1}, \rho_{l_2}]) = [\rho_{l_1}, \rho_{l_2}] = [S_0(\rho_{l_1}), S_0(\rho_{l_2})].$$

Therefore mathematical induction gives the proof of the equality (23). ■

Theorem 2. *Assume that T_i, $0 \leq i \leq m$, are defined by (19), and let $l \geq 0$ and $\rho_l \in E(R_l)$. Then the vector field*

$$\sigma(\rho_l) = \sum_{j=0}^{l(m+1)} \frac{t^j}{j!} S_j(\rho_l), \tag{24}$$

where the $S_j(\rho_l)$, $0 \leq j \leq l(m+1)$, are defined by (7), is a time-independent symmetry of (20) when $l = 0$ and a polynomial-in-time dependent symmetry of (20) when $l > 0$. Furthermore we have

$$[\sigma(\rho_{l_1}), \sigma(\rho_{l_2})] = \sigma([\rho_{l_1}, \rho_{l_1}]),\ \ \rho_{l_1} \in E(R_{l_1}),\ \ \rho_{l_2} \in E(R_{l_2}),\ \ l_1, l_2 \geq 0, \tag{25}$$

and thus all symmetries $\sigma(\rho_l)$ with $\rho_l \in E(R_l)$, $l \geq 0$, constitute the same graded Lie algebra as (18) and the map $\sigma : \rho_l \mapsto \sigma(\rho_l)$ is a Lie homomorphism between two graded Lie algebras $E(R)$ and $\sigma(E(R))$.

Proof. By Lemma 1, we can observe that $\sigma(\rho_l)$ defined by (24) is a symmetry of (20). We go on to prove (25). Assume that $\rho_{l_1} \in E(R_{l_1})$, $\rho_{l_2} \in E(R_{l_2})$, $l_1, l_2 \geq 0$. By Lemmas 1 and 2, we can make the following calculation

$$[\sigma(\rho_{l_1}), \sigma(\rho_{l_2})] = \left[\sum_{i=0}^{l_1(m+1)} \frac{t^i}{i!} S_i(\rho_{l_1}), \sum_{j=0}^{l_2(m+1)} \frac{t^j}{j!} S_j(\rho_{l_2}) \right] =$$

$$= \sum_{k=0}^{(l_1+l_2-1)(m+1)} \frac{t^k}{k!} \sum_{i+j=k} \binom{k}{i} [S_i(\rho_{l_1}), S_j(\rho_{l_2})] \quad \text{(by Lemma 1)} =$$

$$= \sum_{k=0}^{(l_1+l_2-1)(m+1)} \frac{t^k}{k!} S_k([\rho_{l_1}, \rho_{l_2}]) \quad \text{(by Lemma 2)} =$$

$$= \sigma([\rho_{l_1}, \rho_{l_2}]).$$

The rest is then obvious and the required result is obtained. ■

A graded Lie algebra has been exhibited for the time-independent KP hierarchy [14] in [2, 12], and it includes a centreless Virasoro algebra [5, 15]. The ordinary time-independent KP equation being considered here is the following

$$u_t = \partial_x^{-1} u_{yy} - u_{xxx} - 6uu_x. \tag{26}$$

From this we may now go on to generate the corresponding graded Lie algebra of ptd-symmetries for a resulting new set of variable-coefficient KP equations, but in this connection the reader must be referred to the comparable analysis in [16] mentioned below.

The idea of using graded Lie algebras as described in this paper is rather similar to the thinking used to extend the inverse scattering transform from $1+1$ to higher dimensions [17]. Moreover the resulting symmetry algebra consisting of the $\sigma(\rho_l)$, $l \geq 0$, provides a new realisation of a graded Lie algebra (18). The theory also shows us that more information can be extracted from graded Lie algebras, which is itself very interesting. What is more, we have shown here that there do exist various integrable equations in $1+1$ dimensions, such as KdV-type equations, possessing higher-degree polynomial-in-time dependent symmetries. We report a graded Lie algebra of ptd symmetries for a corresponding new set of variable coefficient *modified* KP equations in a second article [16]. In [16] we display this modified KP hierarchy explicitly, the time independent modified KP equation being, in comparison with (26), the equation

$$u_t = \frac{1}{4} u_{xxx} - \frac{3}{8} u^2 u_x - \frac{3}{4} u_x \partial_x^{-1} u_y + \frac{3}{4} \partial_x^{-1} u_{yy}. \tag{27}$$

In [16] we show also that this hierarchy actually has *two* Virasoro algebras and *two* graded Lie algebras.

We also hope to show elsewhere the connections between the rather general algebraic structure established in this paper and the specific representation of the W_∞ and $W_{1+\infty}$ algebras developed in connection with two-dimensional quantum gravity as described in in Refs. [18, 19]. (In [18, 19], these two infinite dimensional algebras were developed for the ordinary KP hierarchy and included the algebra, containing the centreless Virasoro algebra, of Ref. [5].) In this connection, we note already that if, for example, $E(R_i) = \mathrm{span}\{A_{im} | \, m \geq 1\}$, $i \geq 0$, and we impose

$$[A_{im}, A_{jn}] = \sum_{l=\min(i-1,j-1)}^{i+j-2} a_l(i-1, j-1, m-1, n-1) A_{l+1,m+n-1},$$

where the coefficients a_l are defined by

$$\left[x^{i+m+1} \frac{d^{i+1}}{dx^{i+1}}, x^{j+n+1} \frac{d^{j+1}}{dx^{j+1}} \right] = \sum_{l=\min(i,j)}^{i+j} a_l(i,j,m,n) x^{l+m+n+1} \frac{d^{l+1}}{dx^{l+1}},$$

then the $E(R) = \sum\limits_{i=0}^{\infty} E(R_i)$ is a sub-algebra of the $W_{1+\infty}$ algebra of Refs. [18, 19] by the identification $A_{im} = \tau_{i-1,m-1}$; here the $\tau_{i-1,m-1}$ are the elements forming the $W_{1+\infty}$ algebra [18, 19] and they may be realized by $x^{i+m-1} \frac{d^i}{dx^i}$.

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[2]With sadness we report the death of Professor Wilhelm I. Fushchych on Monday 7 April 1997 after a short illness. The remaining three authors of this paper, Wen-Xiu Ma, Robin Bullough and Philip Caudrey, dedicate this paper to his memory. Appreciations of W.I. Fushchych will appear in *J. Nonlinear Math. Phys.* of which he was Editor in Chief.