Symmetry classification of multi-component scale-invariant wave equations

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We describe systems of nonlinear wave equations of the form $\Box u_j = F_j(u_1,\ldots,u_4)$, $j=1,\ldots,4$ invariant under the extended Poincaré group $\widetilde{P}(1,3)$. As a result, we have obtained twenty inequivalent classes of nonlinear $\widetilde{P}(1,3)$-invariant systems of partial differential equations.

It is well-known that the maximal symmetry group admitted by the nonlinear wave equation

$$\Box u \equiv u_{x_0x_0} - \triangle_3 u = F(u) \tag{1}$$

with an arbitrary smooth function $F(u)$ is the 10-parameter Poincaré group $P(1,3)$ having the following generators:

$$P_\mu = \partial_\mu, \quad J_{\mu\nu} = g_{\mu\alpha}x_\alpha\partial_\nu - g_{\nu\alpha}x_\alpha\partial_\mu, \tag{2}$$

where $\partial_\mu = \partial/\partial x_\mu$, $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, $\mu,\nu,\alpha = 0,\ldots,3$. Hereafter, the summation over the repeated indices from 0 to 3 is understood.

As established in [1], equation (1) admits the wider symmetry group in two cases

$$1.\ F(u) = \lambda u^k, \quad k \neq 1, \tag{3}$$

$$2.\ F(u) = \lambda e^{ku}, \quad k \neq 0, \tag{4}$$

where λ, k are arbitrary constants, only.

Equations (1) with nonlinearities (3), (4) admit the one-parameter groups of scale transformations $D(1)$ having the following generators:

$$\begin{aligned} &1.\quad D = x_\mu\partial_\mu + \frac{2}{1-k}u\partial_u, \\ &2.\quad D = x_\mu\partial_\mu - \frac{2}{k}\partial_u. \end{aligned} \tag{5}$$

The Lie transformation group generated by the operators (2), (5) is called the extended Poincaré group $\widetilde{P}(1,3)$ [2].

Let us note that in [3] a partial symmetry classification of $\widetilde{P}(1,3)$-invariant partial differential equations (PDEs) of the form

$$\Box u = F(u, u^*) \tag{6}$$

have been performed and two classes of $\widetilde{P}(1,3)$-invariant PDEs have been constructed. A complete solution of the problem of classifying two-component wave equations (6) admitting the extended Poincaré group has been obtained in [4].

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In the present paper following an approach suggested in [4] we classify systems of four PDEs

$$\Box u_j = F_j(u_1, u_2, u_3, u_4), \quad j = 1, \ldots, 4, \tag{7}$$

for real-valued functions $u_i = u_i(x_0, x_1, x_2, x_3)$, $i = 1, \ldots, 4$ admitting the extended Poincaré group $\widetilde{P}(1,3)$ and the conformal group $C(1,3)$.

Before formulating the principal assertions we make an important remark. As a direct check shows, the class of equations (7) is invariant under the linear transformations of dependent variables

$$u_j \to u'_j = \sum_{k=1}^{4} \alpha_{jk} u_k + \beta_j, \quad j = 1, \ldots, 4, \tag{8}$$

where α_{jk}, β_j, $j = 1, 2, 3, 4$ are arbitrary constants and what is more $\det \|\alpha_{jk}\| \neq 0$.

That is why, we carry out symmetry classification of equations (7) within the equivalence transformations (8).

Theorem 1. *Let generators of the Poincaré group be of the form (2). Then system of partial differential equations (7) is invariant under the extended Poincaré group $\widetilde{P}(1,3)$ if and only if it is equivalent to one of the following systems (for all cases $F_j = F_j(\Omega_1, \Omega_2, \Omega_3)$, $j = 1, \ldots, 4$):*

1. $\Box u_1 = F_1 u_1^{\frac{\lambda_1 - 2}{\lambda_1}}, \quad \Box u_2 = F_2 u_2^{\frac{\lambda_2 - 2}{\lambda_2}}, \quad \Box u_3 = F_3 u_3^{\frac{\lambda_3 - 2}{\lambda_3}}, \quad \Box u_4 = F_4 u_4^{\frac{\lambda_4 - 2}{\lambda_4}},$
   $\Omega_1 = \dfrac{u_1^{\lambda_2}}{u_2^{\lambda_1}}, \quad \Omega_2 = \dfrac{u_1^{\lambda_3}}{u_3^{\lambda_1}}, \quad \Omega_3 = \dfrac{u_1^{\lambda_4}}{u_4^{\lambda_1}};$
2. $\Box u_1 = F_1 \exp\left(-\dfrac{2}{b} u_1\right), \quad \Box u_2 = F_2 \exp\left\{(\lambda_2 - 2)\dfrac{u_1}{b}\right\},$
   $\Box u_3 = F_3 \exp\left\{(\lambda_3 - 2)\dfrac{u_1}{b}\right\}, \quad \Box u_4 = F_4 \exp\left\{(\lambda_4 - 2)\dfrac{u_1}{b}\right\},$
   $\Omega_1 = \lambda_2 u_1 - b \ln u_2, \quad \Omega_2 = \lambda_3 u_1 - b \ln u_3, \quad \Omega_3 = \lambda_4 u_1 - b \ln u_4;$
3. $\Box u_1 = \left\{F_1 + \dfrac{u_1}{u_2} F_2\right\} \exp\left\{(\lambda_1 - 2)\dfrac{u_1}{u_2}\right\}, \quad \Box u_2 = F_2 \exp\left\{(\lambda_1 - 2)\dfrac{u_1}{u_2}\right\},$
   $\Box u_3 = F_3 \exp\left\{(\lambda_2 - 2)\dfrac{u_1}{u_2}\right\}, \quad \Box u_4 = F_4 \exp\left\{(\lambda_3 - 2)\dfrac{u_1}{u_2}\right\},$
   $\Omega_1 = \dfrac{\exp\left(\lambda_1 \frac{u_1}{u_2}\right)}{u_3}, \quad \Omega_2 = \dfrac{\exp\left(\lambda_2 \frac{u_1}{u_2}\right)}{u_2}, \quad \Omega_3 = \dfrac{\exp\left(\lambda_3 \frac{u_1}{u_2}\right)}{u_4};$
4. $\Box u_1 = (F_1 + F_2 u_2) \exp\left(-\dfrac{2}{b} u_2\right), \quad \Box u_2 = b F_2 \exp\left(-\dfrac{2}{b} u_2\right),$
   $\Box u_3 = F_3 \exp\left\{(\lambda_1 - 2)\dfrac{u_2}{b}\right\}, \quad \Box u_4 = F_4 \exp\left\{(\lambda_2 - 2)\dfrac{u_2}{b}\right\},$
   $\Omega_1 = 2bu_1 - u_2^2, \quad \Omega_2 = \lambda_1 u_2 - b \ln u_3, \quad \Omega_3 = \lambda_2 u_2 - b \ln u_4;$
5. $\Box u_1 = (F_1 + F_2 u_3) \exp\left\{(\lambda_1 - 2)\dfrac{u_3}{b}\right\}, \quad \Box u_2 = b F_2 \exp\left\{(\lambda_1 - 2)\dfrac{u_3}{b}\right\},$
   $\Box u_3 = F_3 \exp\left(-\dfrac{2}{b} u_3\right), \quad \Box u_4 = F_4 \exp\left\{(\lambda_2 - 2)\dfrac{u_3}{b}\right\},$
   $\Omega_1 = b \ln u_2 - \lambda_1 u_3, \quad \Omega_2 = b \dfrac{u_1}{u_2} - u_3, \quad \Omega_3 = b \ln u_4 - \lambda_2 u_3;$

6. $\Box u_1 = \left(F_1 + F_2\dfrac{u_2}{u_3} + F_3\dfrac{u_1}{u_3}\right)\exp\left\{(\lambda_1-2)\dfrac{u_2}{u_3}\right\},$
$\Box u_2 = \left(F_2 + F_3\dfrac{u_2}{u_3}\right)\exp\left\{(\lambda_1-2)\dfrac{u_2}{u_3}\right\},$
$\Box u_3 = F_3\exp\left\{(\lambda_1-2)\dfrac{u_2}{u_3}\right\}, \quad \Box u_4 = F_4\exp\left\{(\lambda_2-2)\dfrac{u_2}{u_3}\right\},$
$\Omega_1 = \lambda_1\dfrac{u_2}{u_3} - \ln u_3, \quad \Omega_2 = 2\dfrac{u_1}{u_3} - \left(\dfrac{u_2}{u_3}\right)^2, \quad \Omega_3 = \lambda_2\dfrac{u_2}{u_3} - \ln u_4;$

7. $\Box u_1 = \left(F_1 + F_2\Omega_0 + F_3\Omega_0^2\right)\exp(-2\Omega_0),$
$\Box u_2 = (F_2 + 2F_3\Omega_0)\exp(-2\Omega_0),$
$\Box u_3 = 2F_3\exp(-2\Omega_0), \quad \Box u_4 = F_4\exp\{(\lambda-2)\Omega_0\},$
$\Omega_0 = \dfrac{u_3}{b}, \quad \Omega_1 = 2u_2 - \dfrac{u_3^2}{b}, \quad \Omega_2 = u_1 - \dfrac{u_2u_3}{b} + \dfrac{u_3^3}{3b^2},$
$\Omega_3 = \lambda u_3 - b\ln u_4;$

8. $\Box u_1 = (u_1^2+u_2^2)^{-\frac{1}{2}}(F_1u_2 + F_2u_1)\exp\left(\dfrac{a-2}{b}\arctan\dfrac{u_1}{u_2}\right),$
$\Box u_2 = (u_1^2+u_2^2)^{-\frac{1}{2}}(F_2u_2 - F_1u_1)\exp\left(\dfrac{a-2}{b}\arctan\dfrac{u_1}{u_2}\right),$
$\Box u_3 = F_3\exp\left(\dfrac{\lambda_1-2}{b}\arctan\dfrac{u_1}{u_2}\right), \quad \Box u_4 = F_4\exp\left(\dfrac{\lambda_2-2}{b}\arctan\dfrac{u_1}{u_2}\right),$
$\Omega_1 = \dfrac{(u_1^2+u_2^2)^{\lambda_1}}{u_3^{2a}}, \quad \Omega_2 = \dfrac{\exp\left(\frac{\lambda_1}{b}\arctan\frac{u_1}{u_2}\right)}{u_3}, \quad \Omega_3 = \dfrac{\exp\left(\frac{\lambda_2}{b}\arctan\frac{u_1}{u_2}\right)}{u_4};$

9. $\Box u_1 = \left(F_1\cos\left(\dfrac{b}{c}u_3\right) + F_2\sin\left(\dfrac{b}{c}u_3\right)\right)\exp\left(\dfrac{a-2}{c}u_3\right),$
$\Box u_2 = \left(F_2\cos\left(\dfrac{b}{c}u_3\right) - F_1\sin\left(\dfrac{b}{c}u_3\right)\right)\exp\left(\dfrac{a-2}{c}u_3\right),$
$\Box u_3 = F_3\exp\left(-\dfrac{2}{c}u_3\right), \quad \Box u_4 = F_4\exp\left\{(\lambda-2)\dfrac{u_3}{c}\right\},$
$\Omega_1 = \ln(u_1^2+u_2^2) - 2a\dfrac{u_3}{c}, \quad \Omega_2 = \arctan\dfrac{u_1}{u_2} - b\dfrac{u_3}{c}, \quad \Omega_3 = \lambda u_3 - c\ln u_4;$

10. $\Box u_1 = \left(F_1 + \dfrac{u_1}{u_2}F_2\right)\exp\left\{(\lambda_1-2)\dfrac{u_1}{u_2}\right\}, \quad \Box u_2 = F_2\exp\left\{(\lambda_1-2)\dfrac{u_1}{u_2}\right\},$
$\Box u_3 = \left(F_3 + \dfrac{u_3}{u_4}F_4\right)\exp\left\{(\lambda_2-2)\dfrac{u_3}{u_4}\right\}, \quad \Box u_4 = F_4\exp\left\{(\lambda_2-2)\dfrac{u_3}{u_4}\right\},$
$\Omega_1 = \dfrac{\exp\left(\lambda_1\frac{u_1}{u_2}\right)}{u_2}, \quad \Omega_2 = \dfrac{\exp\left(\lambda_2\frac{u_3}{u_4}\right)}{u_4}, \quad \Omega_3 = \dfrac{u_1}{u_2} - \dfrac{u_3}{u_4};$

11. $\Box u_1 = \left(F_1 + F_2\dfrac{u_3}{u_4} + F_3\dfrac{u_2}{u_4} + F_4\dfrac{u_1}{u_4}\right)\exp\left\{(\lambda-2)\dfrac{u_3}{u_4}\right\},$
$\Box u_2 = \left(F_2 + F_3\dfrac{u_3}{u_4} + F_4\dfrac{u_2}{u_4}\right)\exp\left\{(\lambda-2)\dfrac{u_3}{u_4}\right\},$
$\Box u_3 = \left(F_3 + F_4\dfrac{u_3}{u_4}\right)\exp\left\{(\lambda-2)\dfrac{u_3}{u_4}\right\}, \quad \Box u_4 = F_4\exp\left\{(\lambda-2)\dfrac{u_3}{u_4}\right\},$

$$\Omega_1 = \frac{\exp\left(\lambda \frac{u_3}{u_4}\right)}{u_4}, \quad \Omega_2 = 2\frac{u_2}{u_4} - \left(\frac{u_3}{u_4}\right)^2, \quad \Omega_3 = 3\frac{u_1}{u_4} + \left(\frac{u_3}{u_4}\right)^3 - 3\frac{u_2 u_3}{u_4^2};$$

12. $$\Box u_1 = (u_1^2 + u_2^2)^{-\frac{1}{2}}(F_1 u_2 + F_2 u_1) \exp\left(\frac{a_1 - 2}{b_1} \arctan \frac{u_1}{u_2}\right),$$
$$\Box u_2 = (u_1^2 + u_2^2)^{-\frac{1}{2}}(F_2 u_2 - F_1 u_1) \exp\left(\frac{a_1 - 2}{b_1} \arctan \frac{u_1}{u_2}\right),$$
$$\Box u_3 = \left(F_3 + \frac{u_3}{u_4} F_4\right) \exp\left\{(\lambda - 2)\frac{u_3}{u_4}\right\}, \quad \Box u_4 = F_4 \exp\left\{(\lambda - 2)\frac{u_3}{u_4}\right\},$$
$$\Omega_1 = \arctan \frac{u_1}{u_2} - b\frac{u_3}{u_4}, \quad \Omega_2 = \frac{\exp\left(\lambda \frac{u_3}{u_4}\right)}{u_4},$$
$$\Omega_3 = (u_1^2 + u_2^2)^{\frac{1}{2}} \exp\left(-a\frac{u_3}{u_4}\right);$$

13. $$\Box u_1 = (u_1^2 + u_2^2)^{-\frac{1}{2}}(F_1 u_2 + F_2 u_1) \exp\left(\frac{a_1 - 2}{b_1} \arctan \frac{u_1}{u_2}\right),$$
$$\Box u_2 = (u_1^2 + u_2^2)^{-\frac{1}{2}}(F_2 u_2 - F_1 u_1) \exp\left(\frac{a_1 - 2}{b_1} \arctan \frac{u_1}{u_2}\right),$$
$$\Box u_3 = (u_3^2 + u_4^2)^{-\frac{1}{2}}(F_3 u_4 + F_4 u_3) \exp\left(\frac{a_2 - 2}{b_2} \arctan \frac{u_3}{u_4}\right),$$
$$\Box u_4 = (u_3^2 + u_4^2)^{-\frac{1}{2}}(F_4 u_4 - F_3 u_3) \exp\left(\frac{a_2 - 2}{b_2} \arctan \frac{u_3}{u_4}\right),$$
$$\Omega_1 = b_2 \arctan \frac{u_1}{u_2} - b_1 \arctan \frac{u_3}{u_4}, \quad \Omega_2 = \frac{\exp\left(\arctan \frac{a_1 u_1}{b_1 u_2}\right)}{(u_1^2 + u_2^2)^{\frac{1}{2}}},$$
$$\Omega_3 = \frac{\exp\left(\arctan \frac{a_2 u_3}{b_2 u_4}\right)}{(u_3^2 + u_4^2)^{\frac{1}{2}}};$$

14. $$\Box u_1 = \left(F_1 + F_2 \frac{u_1}{u_2}\right) \Omega_0, \quad \Box u_2 = F_2 \Omega_0,$$
$$\Box u_3 = \left(F_3 + F_4 \frac{u_3}{u_4}\right) \Omega_0, \quad \Box u_4 = F_4 \Omega_0,$$
$$\Omega_0 = \exp\left\{(a - 2)\frac{u_1}{u_2}\right\} \sec \frac{b u_1}{u_2}, \quad \Omega_1 = \frac{\Omega_0 \exp\left(\frac{a u_1}{u_2}\right)}{u_2},$$
$$\Omega_2 = \frac{\Omega_0 \exp\left(\frac{a u_3}{u_4}\right)}{u_4}, \quad \Omega_3 = \frac{u_1}{u_2} - \frac{u_3}{u_4};$$

15. $$\Box u_1 = \left(F_1 + F_2 \frac{u_2}{u_3} + F_3 \frac{u_1}{u_3}\right) \exp\left\{(\lambda - 2)\frac{u_2}{u_3}\right\},$$
$$\Box u_2 = \left(F_2 + F_3 \frac{u_2}{u_3}\right) \exp\left\{(\lambda - 2)\frac{u_2}{u_3}\right\},$$
$$\Box u_3 = F_3 \exp\left\{(\lambda - 2)\frac{u_2}{u_3}\right\}, \quad \Box u_4 = F_4 \exp\left(-2\frac{u_2}{u_3}\right),$$
$$\Omega_1 = \lambda \frac{u_2}{u_3} - \ln u_3, \quad \Omega_2 = 2\frac{u_1}{u_3} - \left(\frac{u_2}{u_3}\right)^2, \quad \Omega_3 = b\frac{u_2}{u_3} - u_4;$$

16. $\Box u_1 = (F_1 + F_2 u_4) \exp\left\{(\lambda - 2)\dfrac{u_4}{b}\right\}, \quad \Box u_2 = bF_2 \exp\left\{(\lambda - 2)\dfrac{u_4}{b}\right\},$
$\Box u_3 = (F_3 + F_4 u_4) \exp\left(-\dfrac{2}{b}u_4\right), \quad \Box u_4 = bF_4 \exp\left(-\dfrac{2}{b}u_4\right),$
$\Omega_1 = \lambda\dfrac{u_1}{u_2} - \ln u_2, \quad \Omega_2 = b\dfrac{u_1}{u_2} - u_4, \quad \Omega_3 = 2bu_3 - u_4^2;$

17. $\Box u_1 = \left(F_1 + F_2\Omega_0 + F_3\dfrac{\Omega_0^2}{2} + F_4\dfrac{\Omega_0^3}{6}\right)\exp(-2\Omega_0),$
$\Box u_2 = \left(F_2 + F_3\Omega_0 + F_4\dfrac{\Omega_0^2}{2}\right)\exp(-2\Omega_0),$
$\Box u_3 = (F_3 + F_4\Omega_0)\exp(-2\Omega_0), \quad \Box u_4 = F_4\exp(-2\Omega_0),$
$\Omega_0 = \dfrac{u_4}{b}, \quad \Omega_1 = \dfrac{u_3}{b} - \dfrac{1}{2}\left(\dfrac{u_4}{b}\right)^2, \quad \Omega_2 = \dfrac{u_2}{b} - \dfrac{u_3 u_4}{b^2} + \dfrac{1}{3}\left(\dfrac{u_4}{b}\right)^3,$
$\Omega_3 = \dfrac{u_1}{b} - \dfrac{u_2 u_4}{b^2} + \dfrac{u_3 u_4^2}{2b^3} - \dfrac{u_4^4}{8b^4};$

18. $\Box u_1 = (F_1 + F_2 u_2) \exp\left(-\dfrac{2}{b}u_2\right), \quad \Box u_2 = bF_2 \exp\left(-\dfrac{2}{b}u_2\right),$
$\Box u_3 = (F_3 + F_4 u_4) \exp\left(-\dfrac{2}{c}u_4\right), \quad \Box u_4 = cF_4 \exp\left(-\dfrac{2}{c}u_4\right),$
$\Omega_1 = 2bu_1 - u_2^2, \quad \Omega_2 = 2cu_3 - u_4^2, \quad \Omega_3 = bu_3 + cu_1 - u_2 u_4;$

19. $\Box u_1 = \left\{F_1\cos\left(\dfrac{b}{c}u_4\right) + F_2\sin\left(\dfrac{b}{c}u_4\right)\right\}\exp\left(\dfrac{a-2}{c}u_4\right),$
$\Box u_2 = \left\{F_2\cos\left(\dfrac{b}{c}u_4\right) - F_1\sin\left(\dfrac{b}{c}u_4\right)\right\}\exp\left(\dfrac{a-2}{c}u_4\right),$
$\Box u_3 = (F_3 + F_4 u_4) \exp\left(-\dfrac{2}{c}u_4\right),$
$\Box u_4 = cF_4 \exp\left(-\dfrac{2}{c}u_4\right), \quad b \neq 0, \ c \neq 0,$
$\Omega_1 = \ln(u_1^2 + u_2^2) - 2a\dfrac{u_4}{c}, \quad \Omega_2 = \arctan\dfrac{u_1}{u_2} - b\dfrac{u_4}{c}, \quad \Omega_3 = 2cu_3 - u_4^2;$

20. $\Box u_j = 0, \quad j = 1, \dots, 4,$

where F_1, F_2, F_3, F_4 are arbitrary smooth functions and a, b, c are arbitrary constants.

Furthermore, the basis generators P_μ, $J_{\mu\nu}$ are given by formulae (2) and generators of corresponding groups of scale transformations are given by the following formulae:

1. $D = x_\mu\partial_\mu + \lambda_1 u_1\partial_{u_1} + \lambda_2 u_2\partial_{u_2} + \lambda_3 u_3\partial_{u_3} + \lambda_4 u_4\partial_{u_4}, \quad \lambda_1 \neq 0;$
2. $D = x_\mu\partial_\mu + b\partial_{u_1} + \lambda_2 u_2\partial_{u_2} + \lambda_3 u_3\partial_{u_3} + \lambda_4 u_4\partial_{u_4};$
3. $D = x_\mu\partial_\mu + \lambda_1(u_1\partial_{u_1} + u_2\partial_{u_2}) + u_2\partial_{u_1} + \lambda_2 u_3\partial_{u_3} + \lambda_3 u_4\partial_{u_4};$
4. $D = x_\mu\partial_\mu + u_2\partial_{u_1} + b\partial_{u_2} + \lambda_1 u_3\partial_{u_3} + \lambda_2 u_4\partial_{u_4};$ (9)
5. $D = x_\mu\partial_\mu + \lambda_1(u_1\partial_{u_1} + u_2\partial_{u_2}) + u_2\partial_{u_1} + b\partial_{u_3} + \lambda_2 u_4\partial_{u_4};$
6. $D = x_\mu\partial_\mu + \lambda_1(u_1\partial_{u_1} + u_2\partial_{u_2} + u_3\partial_{u_3}) + u_2\partial_{u_1} + u_3\partial_{u_2} + \lambda_2 u_4\partial_{u_4};$
7. $D = x_\mu\partial_\mu + u_2\partial_{u_1} + u_3\partial_{u_2} + b\partial_{u_3} + \lambda u_4\partial_{u_4};$
8. $D = x_\mu\partial_\mu + a_1(u_1\partial_{u_1} + u_2\partial_{u_2}) + b_1(u_2\partial_{u_1} - u_1\partial_{u_2}) + \lambda_1 u_3\partial_{u_3} + \lambda_2 u_4\partial_{u_4};$

9. $D = x_\mu\partial_\mu + a_1(u_1\partial_{u_1} + u_2\partial_{u_2}) + b_1(u_2\partial_{u_1} - u_1\partial_{u_2}) + c\partial_{u_3} + \lambda u_4\partial_{u_4}$;
10. $D = x_\mu\partial_\mu + \lambda_1(u_1\partial_{u_1} + u_2\partial_{u_2}) + u_2\partial_{u_1} + \lambda_2(u_3\partial_{u_3} + u_4\partial_{u_4}) + u_4\partial_{u_3}$;
11. $D = x_\mu\partial_\mu + \lambda(u_1\partial_{u_1} + u_2\partial_{u_2} + u_3\partial_{u_3} + u_4\partial_{u_4}) + u_2\partial_{u_1} + u_3\partial_{u_2} + u_4\partial_{u_3}$;
12. $D = x_\mu\partial_\mu + a(u_1\partial_{u_1} + u_2\partial_{u_2}) + b(u_2\partial_{u_1} - u_1\partial_{u_2}) + {}$
$+ \lambda(u_3\partial_{u_3} + u_4\partial_{u_4}) + u_4\partial_{u_3}$;
13. $D = x_\mu\partial_\mu a_1(u_1\partial_{u_1} + u_2\partial_{u_2}) + b_1(u_2\partial_{u_1} - u_1\partial_{u_2}) + {}$
$+ a_2(u_3\partial_{u_3} + u_4\partial_{u_4}) + b_2(u_4\partial_{u_3} - u_3\partial_{u_4})$;
14. $D = x_\mu\partial_\mu + a_1(u_1\partial_{u_1} + u_2\partial_{u_2}) + b_1(u_2\partial_{u_1} - u_1\partial_{u_2}) + {}$
$+ a_2(u_3\partial_{u_3} + u_4\partial_{u_4}) + b_2(u_4\partial_{u_3} - u_3\partial_{u_4}) + u_3\partial_{u_1} + u_4\partial_{u_2}$;
15. $D = x_\mu\partial_\mu + \lambda(u_1\partial_{u_1} + u_2\partial_{u_2} + u_3\partial_{u_3}) + u_2\partial_{u_1} + u_3\partial_{u_2} + b\partial_{u_4}$;
16. $D = x_\mu\partial_\mu + u_4\partial_{u_3} + b\partial_{u_4} + \lambda(u_1\partial_{u_1} + u_2\partial_{u_2}) + u_2\partial_{u_1}, \quad b \neq 0$;
17. $D = x_\mu\partial_\mu + u_2\partial_{u_1} + u_3\partial_{u_2} + u_4\partial_{u_3} + b\partial_{u_4}$;
18. $D = x_\mu\partial_\mu + u_2\partial_{u_1} + b\partial_{u_2} + u_4\partial_{u_3} + c\partial_{u_4}, \quad b \neq 0, c \neq 0$;
19. $D = x_\mu\partial_\mu + a(u_1\partial_{u_1} + u_2\partial_{u_2}) + b(u_2\partial_{u_1} - u_1\partial_{u_2}) + u_4\partial_{u_3} + c\partial_{u_4}$;
20. $D = x_\mu\partial_\mu$.

Theorem 2. *System of PDEs (7) is invariant under the conformal group $C(1,3)$ iff it is equivalent to the following system:*

$$\Box u_j = u_1^3 \widetilde{F}_j\left(\frac{u_1}{u_2}, \frac{u_1}{u_3}, \frac{u_1}{u_4}\right), \quad j = 1, 2, 3, 4.$$

Proofs of Theorems 1, 2 are carried out with the help of the infinitesimal Lie algorithm (see, e.g. [2, 5, 6]). Here we present the scheme of the proof of Theorem 1 only.

Within the framework of the Lie method, a symmetry operator for system of PDEs (7) is looked for in the form

$$X = \xi_\mu(x,u)\partial_\mu + \eta_j(x,u)\partial_{u_j}, \quad j = 1, \ldots, 4, \tag{10}$$

where $\xi_\mu(x,u)$, $\eta_j(x,u)$ are some smooth functions.

The necessary and sufficient condition for system of PDEs (7) to be invariant under the group having the infinitesimal operator (10) reads

$$\widetilde{X}(\Box u_j + F_j)\Big|_{\Box u_i - F_i = 0,\ i=1,\ldots,4} = 0, \quad j = 1, \ldots, 4, \tag{11}$$

where $\widetilde{X}$ stands for the second prolongation of the operator X.

Splitting relations (11) by independent variables, we get the Killing-type system of PDEs for ξ_μ, η_k. Integrating it, we have:

$$\begin{aligned} &\xi_\mu = 2x_\mu g_{\alpha\beta}x_\alpha k_\beta - k_\mu g_{\alpha\beta}x_\alpha x_\beta + c_{\mu\alpha}g_{\alpha\beta}x_\beta + dx_\mu + e_\mu, \quad \mu = 0, \ldots, 3, \\ &\eta_k = \sum_{j=1}^{4} a_{kj}u_j + b_k(x) - 2g_{\alpha\beta}k_\alpha x_\beta u_k, \quad k = 1, \ldots, 4. \end{aligned} \tag{12}$$

Here k_α, $c_{\mu\nu} = -c_{\nu\mu}$, d, e_μ, a_{kj} are arbitrary constants, $b_k(x)$ are arbitrary functions satisfying the following relations:

$$\sum_{k=1}^{4}\left(\sum_{l=1}^{4} a_{kl}u_l + b_k(x) - 2g_{\alpha\beta}k_\alpha x_\beta u_k\right) F_{ju_k} + \Box b_j(x) + \\ + 2(d + 3g_{\alpha\beta}k_\alpha x_\beta)F_j - \sum_{l=1}^{4} a_{jl}F_l = 0, \quad j = 1,\dots,4. \tag{13}$$

From (12), (13) it follows that system of PDEs (7) is invariant under the Poincaré group $P(1,3)$ having the generators (2) with arbitrary F_1, F_2. To describe all functions F_1, F_2 such that system (7) admits the extended Poincaré group $\widetilde{P}(1,3)$, one has to solve two problems:

1) to describe all operators D of the form (10), (12) which together with operators (2) satisfy the commutation relations of the Lie algebra of the group $\widetilde{P}(1,3)$ (see, e.g. [2])

$$[P_\alpha, P_\beta] = 0, \quad [P_\alpha, J_{\beta\gamma}] = g_{\alpha\beta}P_\gamma - g_{\alpha\gamma}P_\beta,$$
$$[J_{\alpha\beta}, J_{\mu\nu}] = g_{\alpha\nu}J_{\beta\mu} + g_{\beta\mu}J_{\alpha\nu} - g_{\alpha\mu}J_{\beta\nu} - g_{\beta\nu}J_{\alpha\mu},$$
$$[D, J_{\alpha\beta}] = 0, \quad [P_\alpha, D] = P_\alpha, \quad \alpha,\beta,\gamma,\mu,\nu = 0,\dots,3;$$

2) to solve system of PDEs (13) for each operator D obtained.

On solving the first problem, we establish that the operator D has the form

$$D = x_\mu\partial_\mu + \sum_{i=1}^{4}\left(\sum_{j=1}^{4} A_{ij}u_j + B_i\right)\partial_{u_i}, \tag{14}$$

where A_{ij}, B_i are arbitrary constants.

As noted above, two operators D and D' connected by the transformation (8) (which does not alter the form of the operators P_μ, $J_{\mu\nu}$) are considered as equivalent. Using this fact we can simplify substantially the form of the operator (14).

On making in (14) the change of variables (8) with $\beta_j = 0$, we have

$$D' = x_\mu\partial_\mu + \sum_{i=1}^{4}\left(\sum_{j=1}^{4} \widetilde{A}_{ij}u'_j + \widetilde{B}_i\right)\partial_{u'_i},$$

where

$$\|\widetilde{A}_{ij}\| = \|\alpha_{ij}\|\|A_{ij}\|\|\alpha_{ij}\|^{-1}, \\ \widetilde{B}_i = \sum_{k=1}^{4}\alpha_{ik}B_k, \quad i = 1,2,3,4. \tag{15}$$

As an arbitrary (4×4)-matrix can be reduced to a Jordan form by transformation (15), we may assume without loss of generality that the matrix $\|A_{ij}\|$ is in the Jordan form. The further simplification of the form of operator (14) is achieved at the expense of transformation (8) with $\alpha_{ik} = 0$.

As a result, the set of operators (14) is split into twenty equivalence classes, whose representatives are adduced in (9).

Next, integrating corresponding system of PDEs (13), we get $\widetilde{P}(1,3)$-invariant systems of equations given above.

Note that when proving Theorem 1, we solve a standard problem of the representation theory, namely, we describe inequivalent representations of the extended Poincaré group which are realized on the set of solutions of system of PDEs (7). But the representation space (i.e., the set of solutions of system (7)) is not a linear vector space, whereas in the standard representation theory it is always the case. This fact makes impossible a direct application of the methods of the classical theory of linear group representations [7].

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