Symmetries and reductions of nonlinear Schrödinger equations of Doebner–Goldin type

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We compute symmetry algebras for nonlinear Schrödinger equations which contain an imaginary nonlinearity as derived by Doebner and Goldin and certain real nonlinearities not depending on the derivatives. In the three-dimensional case we find the maximal symmetry algebras for equations of this type. Admitting other imaginary nonlinearities does lead to similar symmetry algebras. These symmetries are used to obtain explicit solutions of these equations by means of reduction.

1. Introduction

Recently, a new nonlinear Schrödinger equation as the evolution equation of a quantum mechanical system on $\mathbb{R}^n$ has been derived from general principles by Doebner and Goldin [1–4]. Their derivation is based on the representation theory of the semidirect product of the group of diffeomorphisms with the smooth functions on $\mathbb{R}^n$ and results in the replacement of the usual continuity equation $\dot\rho = -\vec\nabla\vec j$ (where $\rho = \bar\psi\psi$ and $\vec j = \frac{\hbar}{2mi}(\bar\psi\vec\nabla\psi - \vec\nabla\bar\psi\psi))$ associated with the linear Schrödinger equation by the Fokker–Planck equation $\dot\rho = -\vec\nabla\vec j + d\Delta\rho$ describing diffusion of the probability density ρ. This Fokker–Planck equation for the probability density can be derived from a nonlinear Schrödinger equation which has to be of the form

$$i\hbar\dot\psi = \left(-\frac{\hbar}{2m}\Delta + V + i\frac{\hbar d}{2}\frac{\Delta\rho}{\rho}\psi + F[\psi,\bar\psi]\right)\psi, \tag{1}$$

where F is assumed to be an arbitrary real functional. Doebner and Goldin proceeded with the requirement that $F[\psi,\bar\psi]$ should have similar properties as the imaginary nonlinear functional, and were thus led to a five parameter functional including derivative terms [4]. Galilei-invariant nonlinear Schrödinger equations of type (1), where $d = 0$ and F depends on the wave function and its first order derivatives, were described by Fushchych and Cherniha [5].

On the other hand, equations similar to (1) have been considered in plasma physics [6] and for $d = 0$ and $F[\psi,\bar\psi] = a\rho$ it reduces to the usual nonlinear Schrödinger equation which appears in many subfields of physics. It seems therefore worthwhile to investigate the Lie symmetries for equations of this type and to use them to construct solutions. This is what we shall do in this paper.

Obviously, we shall have to restrict the functional F suitably since otherwise it would be impossible to say anything at all about the symmetries of this equation. Whereas the maximal Lie symmetry of the Doebner–Goldin equation has already been calculated [7], we shall restrict our considerations in this paper to another class of functional F given by (sufficiently smooth) functions f of a single real variable:

$$F[\psi,\bar\psi] := \hbar f(\rho), \tag{2}$$

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which includes many physically interesting models [8, 9]. Although we leave the framework set by Doebner and Goldin if f is not real, we will consider a slightly morel general case of *complex* valued functions f since calculations are similar. For $d = 0$ the Lie symmetry of this nonlinear Schrödinger equation has been discussed in [10, 11, 12].

In Section 2 we will determine the maximal Lie symmetries of the nonlinear Schrödinger equations (1) with functional of type (2). It turns out that the most prominent cases, i.e. $f(p) \equiv \rho^k$ and $f(\rho) \equiv \ln\rho$, admit the largest symmetry algebras.) Subalgebras of the maximal symmetry algebras will be used in Section 3 to reduce equation (1) and find exact solutions. We close this paper with some further remarks) on the equations and the solutions obtained.

2. Lie symmetry algebra

2.1. $n \le 3$. First, we shall treat the physically most interesting case of three space dimensions ($n = 3$) for which we will determine the *maximal* Lie symmetry algebra of equation (1) with the *complex* valued functional (2). In order to do so, we write ψ in terms of an amplitude function R and a phase function S:

$$\psi(\vec{x}, t) = R(\vec{x}, t)e^{iS(\vec{x},t)}.$$

With the decomposition of f into the real and imaginary parts, $f = u+iv$, equation (1) is thus equivalent to two real evolution equations:

$$\partial_t R + \frac{\hbar}{2m}\left(R\Delta S + 2\vec{\nabla}R\cdot\vec{\nabla}S\right) - d\left(\Delta R + \frac{(\vec{\nabla}R)^2}{R}\right) - Rv(R^2) = 0, \tag{3}$$

$$\partial_t S + \frac{\hbar}{2m}\left((\vec{\nabla}S)^2 - \frac{\Delta R}{R}\right) + u(R^2) = 0. \tag{4}$$

Vector fields acting on the space of independent (x_1, x_2, x_3, t) and dependent (R, S) variables

$$X = \xi_j\partial_{x_j} + \tau\partial_t + \phi\partial_R + \sigma\partial_S,$$

are generators of a Lie symmetry of the equations (3) and (4), if the coefficients ξ_j, τ, ϕ, σ satisfy the so-called determining equations. A detailed description of the theory can be found in the monographs [10, 13, 14]. Since the procedure is purely algorithmic, we use a **Mathematica** program [15] to obtain these equations. This leads to 62 determining equations among which only two contain the real and imaginary part of f. These two equations determine the functional F of equation (1). The integration of the 60 remaining equations yields the following coefficients of the vector field X:

$$\begin{aligned}
&\xi_j = (2c_1t + c_2)x_j + w_{jl}x_l + v_jt + a_j,\\
&\tau = 2c_1t^2 + 2c_2t + 2c_3,\\
&\phi = \alpha(t)R,\\
&\sigma = \frac{m}{\hbar}(c_1\vec{x}^2 + v_kx_k) + \beta(t),
\end{aligned} \tag{5}$$

where c_i, v_j and a_j are real constants, w_{jl} is an antisymmetric matrix with real constant coefficients, and α and β are real functions of time. The two remaining determining equations which contain the functions u and v thus read

$$\alpha(t)R^2u'(R^2) + (2c_1t + c_2)u(R^2) + \frac{1}{2}\beta'(t) = 0, \tag{6}$$

$$\alpha(t)R^2v'(R^2) + (2c_1t + c_2)v(R^2) - \frac{1}{2}(\alpha'(t) + 2nc_1) = 0. \tag{7}$$

For the cases $n = 1, 2$ the resulting equations are exactly the same, with the understanding that in equation (7) the dimension n has to be inserted. In order to calculate the maximal symmetry, we solve the ordinary differential equation (7) for α and then (6) for β, requiring that the resulting functions do not depend on R. Neglecting the case of constant functions $u = C$ — which can be transformed to zero by the map $\psi \longmapsto e^{iCt}\psi$ — this leads to the following six possible cases.

1. For arbitrary functions u and v one has to require that their coefficients and the inhomogeneous terms in equations (6) and (7) vanish, which leaves only the centrally extended Galilei algebra $\mathfrak{g}(n = 3) = \langle H, P_j, J_{jk}, G_j, Q\rangle$ with ten generators:

$$\begin{aligned} &H = \partial_t, \quad P_j = \partial_{x_j}, \quad J_{jk} = x_j\partial_{x_k} - x_k\partial_{x_j}, \\ &G = t\partial_{x_j} + \frac{m}{\hbar}x_j\partial_S, \quad Q = \partial_S. \end{aligned} \tag{8}$$

2. A larger algebra is obtained if u and v are of the from

$$u(R^2) = \lambda_1 R^{2k}, \quad v(R^2) = \lambda_2 R^{2k},$$

in which case equations (6) and (7) reduce to linear inhomogeneous equations in u and v, respectively. Requiring the coefficients and the inhomogeneous term to vanish allows the maximal Lie symmetry to contain an additional generator

$$D = 2t\partial_t + x_k\partial_{x_k} - \frac{1}{k}R\partial_R, \tag{9}$$

and this algebra $\langle H, P_j, J_{jk}, G_j, Q, D\rangle$ has been named the Galilei similitude algebra [16]. D generates the dilations.

3. Calculations of the previous case show that the Lie symmetry has an extra generator if $k = \frac{1}{n} = \frac{1}{3}$:

$$C = t^2\partial_t + tx_k\partial_{x_k} + \frac{m}{2\hbar}\vec{x}^2\partial_S - ntR\partial_R, \tag{10}$$

yielding the maximal Lie symmetry algebra of the free linear Schrödinger equation [17] $\langle H, P_j, J_{jk}, G_j, Q, D, C\rangle$ (Schrödinger algebra). The transformations generated by C are called projective or conformal transformations.

4. If $u(R^2) = \lambda_1 \ln(R^2)$ and $v = \lambda_3$ is a constant, we obtain the maximal Lie symmetry algebra $\langle H, P_j, J_{jk}, G_j, Q, D, B\rangle$, where

$$B = R\partial_R - 2\lambda_1 t\partial_S. \tag{11}$$

Note that for nonvanishing λ_1 the constant λ_3 can be transformed to zero by the map $\psi \longmapsto e^{-\lambda_3 t + i\lambda_1\lambda_3 t^2}\psi$.

5. If $u(R^2) = \lambda_1 \ln(R^2)$ and $v(R^2) = \lambda_2 \ln(R^2) + \lambda_3$ with $\lambda_2 \neq 0$, equation (7) leads to a simple differential equation for $\alpha(t)$ and equation (6) determines $\beta(t)$ up to a constant. Hence, the maximal symmetry algebra is $\langle H, P_j, J_{jk}, G_j, Q, D, A\rangle$, where

$$A = e^{2\lambda_2 t}\left(R\partial_R - \frac{\lambda_1}{\lambda_2}\partial_S\right). \tag{12}$$

6. Finally, if u and v vanish identically, the maximal Lie symmetry algebra is $\langle H, P_j, J_{jk}, G_j, Q, D', I\rangle$, the direct sum of the Schrödinger algebra (though with a different representative D' of the generator of dilations) with a one-dimensional algebra generated by I, where

$$D' = 2t\partial_t + x_k\partial_{x_k}, \tag{13}$$

$$I = R\partial_R. \tag{14}$$

The invariance under I reflects real homogeneity of the equation (1); together with Q it generates complex rescalings of ψ.

2.2. $n > 3$. In all cases the algebras remain symmetry algebras for arbitrary dimension n. We believe that they are still maximal, but we have no proof of maximality for arbitrary n. The algebras of the cases 1–3 and 6 have been studied in [18], and the finite transformations they generate are well known. The structure of the algebra of case 4 was investigated in [12, 19, 20].

As for the generators B and A, they generate the following finite transformations:

$$\psi \mapsto g_\epsilon^B\psi, \qquad g_\epsilon^B\psi(\vec{x},t) = \exp(\epsilon(1 - i2\lambda_1 t))\psi(\vec{x},t),$$
$$\psi \mapsto g_\epsilon^A\psi, \qquad g_\epsilon^A\psi(\vec{x},t) = \exp\left(\epsilon\left(1 - i\frac{\lambda_1}{\lambda_2}\right)\right)e^{2\lambda_2 t}\psi(\vec{x},t),$$

3. Reduction and exact solutions

Using the operators of symmetry we will construct ansätze reducing equation (1) to a system of ordinary differential equations (ODEs). The algebras of the cases 1–3 and 6 are subalgebras of the maximal symmetry algebra of the linear Schrödinger equation; their structure was studied in detail and corresponding ansätze are well known. Thus we concentrate on the cases 4 and 5, and particularly on the reduction by those subalgebras containing the "new" generators A and B. The solutions obtained in this way might reflect the nonlinear structure of equation (1) with $f(\rho) := (\lambda_1 + i\lambda_2)\ln\rho + i\lambda_3$. We consider mainly the case of three spatial variables, $n = 3$.

3.1. Case 4: $f(\rho) := \lambda_1\ln\rho + i\lambda_3$; or $u(R^2) = \lambda_1\ln(R^2)$, $v(R^2) = \lambda_3$

1. $\langle B + G_1, G_2, G_3\rangle$. The ansatz

$$\psi(\vec{x},t) = \exp\left\{\frac{x_1}{t} + g(t) + i\left[-2\lambda_1 x_1 + \frac{m}{2\hbar}\frac{\vec{x}^2}{t} + h(t)\right]\right\} \tag{15}$$

reduces equation (1) to the system

$$\frac{dg}{dt} = \left(\frac{2\hbar\lambda_1}{m} - \frac{3}{2}\right)\frac{1}{t} + 2d\frac{1}{t^2} + \lambda_3,$$
$$\frac{dh}{dt} = -\frac{2\hbar\lambda_1^2}{m} + \frac{\hbar}{2m}\frac{1}{t^2} - 2\lambda_1 g(t).$$

Having solved this system we find the solution

$$\psi(\vec{x},t) = t^k \exp\left\{\lambda_3 t + (x_1 - 2d)\frac{1}{t} + c_1 + i\left[-2\lambda_1 x_1 + \frac{m}{2\hbar}\frac{\vec{x}^2}{t} - \lambda_1\lambda_3 t^2 - \right.\right.$$
$$\left.\left. - 2\lambda_1 k t\ln t + 2\lambda_1\left(k - c_1 - \frac{\hbar\lambda_1}{m}\right)t + 4d\lambda_1\ln t - \frac{\hbar}{2m}\frac{1}{t} + c_2\right]\right\},$$

where $k := \frac{2\hbar\lambda_1}{m} - \frac{3}{2}$ and c_1, c_2 are real constants.

2. $\langle B + \alpha H, J_{12} + \beta P_3\rangle$, $\alpha \in \mathbb{R}_{\neq 0}$, $\beta \in \mathbb{R}$. For $\lambda_3 = 0$, the ansatz

$$\psi(\vec{x},t) = \exp\left\{\frac{t}{\alpha} + g(\omega_1,\omega_2) + i\left[-\frac{\lambda_1}{\alpha}t^2 + h(\omega_1\omega_2)\right]\right\}, \tag{16}$$

with $\omega_1 := (x_1^2 + x_2^2)^{\frac{1}{2}}$ and $\omega_2 := \arctan\left(\frac{x_2}{x_1}\right) - \beta x_3$, reduces equation (1) to the system

$$h_{11} + h_{22}\left(1 + \frac{\beta^2}{\omega_1^2}\right) + \frac{h_1}{\omega_1} + 2g_1h_1 + 2g_2h_2\left(1 + \frac{\beta^2}{\omega_1^2}\right) -$$
$$- \frac{2md}{\hbar}\left(g_{11} + g_{22}\left(1 + \frac{\beta^2}{\omega_1^2}\right) + \frac{g_1}{\omega_1} + 2g_1^2 + 2g_1^2 + 2g_2^2\left(1 + \frac{\beta^2}{\omega_1^2}\right)\right) =$$
$$= \frac{2m}{\hbar}\left(\lambda_3 - \frac{1}{\alpha}\right),$$
$$g_{11} + g_{22}\left(1 + \frac{\beta^2}{\omega_1^2}\right) + \frac{g_1}{\omega_1} + g_1^2 + g_2^2\left(1 + \frac{\beta^2}{\omega_1^2}\right) -$$
$$- \frac{4m\lambda_1}{\hbar}g - h_1^2 - h_2^2\left(1 + \frac{\beta^2}{\omega_1^2}\right) = 0,$$

where subscripts denote derivatives, i.e. $g_1 := \partial g/\partial\omega_1$, etc.

3. $\langle B + \alpha H + \beta G_1, J_{23}\rangle$, $\alpha \in \mathbb{R}_{\neq 0}$, $\beta \in \mathbb{R}$. The ansatz

$$\psi(\vec{x},t) = \exp\left\{\frac{t}{\alpha} + g(\omega_1,\omega_2) + i\left[\frac{m\beta}{\hbar\alpha}x_1 t - \frac{\lambda_1}{\alpha}t^2 - \frac{m\beta^2}{3\hbar\alpha^2}t^3 + h(\omega_1\omega_2)\right]\right\}, \tag{17}$$

with $\omega_1 := \frac{\beta t^2}{2\alpha} - x_1$ and $\omega_2 := (x_2^2 + x_3^2)^{\frac{1}{2}}$ reduces equation (1) to the system

$$2g_1h_1 + 2g_2h_2 + h_{11} + h_{22} + \frac{h_2}{\omega_2} - \frac{2md}{\hbar}\left(g_{11} + g_{22} + \frac{g_2}{\omega_2} - 2g_1^2 - 2g_2^2\right) =$$
$$= \frac{2m}{\hbar}\left(\lambda_3 - \frac{1}{\alpha}\right),$$
$$h_1^2 + h_2^2 - \frac{2\beta m^2\omega_1}{\hbar^2\alpha} - g_{11} - g_{22} - \frac{g_2}{\omega_2} - g_1^2 - g_2^2 + \frac{4\lambda_1 m}{\hbar}g = 0.$$

For $\alpha = 1/\lambda_3$, $\lambda_3 \neq 0$ and $d \neq \hbar/2m$ we have found the following partial solution of this system:

$$g(\omega_1,\omega_2) = \frac{m\beta}{2\hbar\alpha\lambda_1}\omega_1 + \frac{m\hbar\lambda_1}{\hbar^2 - 4m^2d^2}\omega_2^2 + \frac{\hbar^2}{\hbar^2 - 4m^2d^2} + \frac{(\hbar^2 - 4m^2d^2)m\beta^2}{16\hbar^3\alpha^2\lambda_1^3},$$
$$h(\omega_1,\omega_2) = \frac{2md}{\hbar}f(\omega_1,\omega_2).$$

The corresponding solution of equation (1) has then the form

$$\psi(\vec{x},t) = \exp\left\{\frac{t}{\alpha} + \frac{m\beta^2}{4\hbar\alpha^2\lambda_1}t^2 - \frac{m\beta}{2\hbar\alpha\lambda_1}x_1 + \frac{m\hbar\lambda_1}{\hbar^2 - 4m^2d^2}(x_1^2 + x_2^2) + \right.$$
$$+ \frac{\hbar^2}{\hbar^2 - 4m^2d^2} + \frac{(\hbar^2 - 4m^2d^2)m\beta^2}{16\hbar^3\alpha^2\lambda_1^3} +$$
$$+ i\left[\frac{m\beta}{\hbar\alpha}tx_1 + \left(\frac{m^2\beta^2}{2\hbar^2\alpha^2\lambda_1} - \frac{\lambda_1}{\alpha}\right)t^2 - \frac{m\beta^2}{3\hbar\alpha^2}t^3 - \frac{m^2d\beta}{\hbar^2\alpha\lambda - 1}x_1 + \right.$$
$$\left.\left. + \frac{2m^2d\lambda_1}{\hbar^2 - 4m^2d^2}(x_2^2 + x_3^2) + c\right]\right\}.$$

4. $\langle B + \alpha H, J_{jk}\rangle$, $\alpha \in \mathbb{R}_{\neq 0}$. The ansatz

$$\psi(\vec{x},t) = \exp\left\{\frac{t}{\alpha} + g(\omega) + i\left[-\frac{\lambda_1}{\alpha}t^2 + h(\omega)\right]\right\}, \tag{18}$$

where $\omega := \sqrt{x_1^2 + x_2^2 + x_3^2}$, reduces equation (1) to the system

$$\frac{d^2h}{d\omega^2} + \frac{2}{\omega}\frac{dh}{d\omega} + 2\frac{dg}{d\omega}\frac{dh}{d\omega} - \frac{2md}{\hbar}\left(\frac{d^2g}{d\omega^2} + \frac{2}{\omega}\frac{dg}{d\omega} + 2\left(\frac{dg}{d\omega}\right)^2\right) = \frac{2m}{\hbar}\left(\lambda_3 - \frac{1}{\alpha}\right),$$
$$\frac{d^2g}{d\omega^2} + \frac{2}{\omega}\frac{dg}{d\omega} + \left(\frac{dg}{d\omega}\right)^2 - \left(\frac{dh}{d\omega}\right)^2 - \frac{4m\lambda_1}{\hbar}g = 0.$$

Its partial solution for the case $\alpha = 1/\lambda_3$ and $d \neq \hbar/2m$ is

$$g(\omega) = \frac{\hbar}{\hbar^2 - 4m^2d^2}\left(m\lambda_1\vec{x}^2 + \frac{3}{2}\hbar\right),$$
$$h(\omega) = \frac{2m^2d\lambda_1}{\hbar^2 - 4m^2d^2}\vec{x}^2 + c,$$

where c is an arbitrary real constant. The corresponding solution of equation (1) has then the form

$$\psi(\vec{x},t) = \exp\left\{\frac{t}{\alpha} + \frac{\hbar}{\hbar^2 - 4m^2d^2}\left(m\lambda_1\vec{x}^2 + \frac{3}{2}\hbar\right) + \right.$$
$$\left. + i\left[\frac{2m^2d\lambda_1}{\hbar^2 - 4m^2d^2}\vec{x}^2 - \frac{\lambda_1}{\alpha}t^2 + c\right]\right\}.$$

3.2. Case 5: $f(\rho) := (\lambda_1 + i\lambda_2)\ln\rho$; or $u(R^2) = \lambda_1\ln(R^2)$, $v(R^2) = \lambda_2\ln(R^2)$; $\lambda_2 \neq 0$.

1. $\langle A + \alpha P_1, G_2, G_3\rangle$, $\alpha \in \mathbb{R}_{\neq 0}$. The ansatz

$$\psi(\vec{x},t) = \exp\left\{\frac{1}{\alpha}e^{2\lambda_2 t}x_1 + g(t) + i\left[-\frac{\lambda_1}{\alpha\lambda_2}e^{2\lambda_2 t}x_1 + \frac{m}{2\hbar}\frac{x_2^2 + x_3^2}{t} + h(t)\right]\right\}$$

reduces equation (1) to the system of ODEs

$$\frac{dg}{dt} - 2\lambda_2 g = -\frac{1}{t} + \frac{1}{\alpha^2}\left(\frac{\hbar\lambda_1}{m\lambda_2} + 2d\right)e^{4\lambda_2 t},$$
$$\frac{dh}{dt} = -2\lambda_1 g + \frac{\hbar}{2m\alpha^2}\left(1 - \frac{\lambda_1^2}{\lambda_2^2}\right)e^{4\lambda_2 t}.$$

Having solved this system we obtain the following exact solution of equation (1):

$$\psi(\vec{x},t) = \exp\left\{\frac{1}{\alpha}e^{2\lambda_2 t}x_1 + ce^{2\lambda_2 t} + \frac{1}{2\lambda_2\alpha^2}\left(\frac{\hbar\lambda_1}{m\lambda_2} + 2d\right)e^{4\lambda_2 t} - \right.$$
$$-Ei(-2\lambda_2 t)e^{2\lambda_2 t} + i\left[-\frac{\lambda_1}{\alpha\lambda_2}e^{2\lambda_2 t}x_1 + \frac{m}{2\hbar}\frac{x_2^2+x_3^2}{t} - \frac{\lambda_1 c}{\lambda_2}e^{2\lambda_2 t} - \frac{\lambda_1}{\lambda_2}\ln(2\lambda_2 t) + \right.$$
$$\left.\left. + \frac{\hbar}{8m\alpha^2\lambda_2}\left(1 - 3\frac{\lambda_1^2}{\lambda_2^2} - \frac{4md}{\hbar}\right)e^{4\lambda_2 t} + \frac{\lambda_1}{\lambda_2}Ei(-2\lambda_2 t)e^{2\lambda_2 t}\right]\right\},$$

where c is a real constant and $Ei(ax) = \int \frac{\exp(ax)}{x}dx = \ln x + \sum\limits_{k=1} \frac{a^k x^k}{k!k}$. This solution is non-analytical in λ_2, and for $n=1$ can be written in explicit form.

2. $\langle A + \alpha J_{12}, G_3\rangle$, $\alpha \in \mathbb{R}_{\neq 0}$. The ansatz

$$\psi(\vec{x},t) = \exp\left\{\frac{1}{\alpha}e^{2\lambda_2 t}\arctan\left(\frac{x_2}{x_1}\right) + g(t,\omega) + \right.$$
$$\left. + i\left[-\frac{\lambda_2}{\alpha\lambda_1}e^{2\lambda_2 t}\arctan\left(\frac{x_2}{x_1}\right) + \frac{mx_3^2}{2\hbar t} + g(t,\omega)\right]\right\},$$

where $\omega = \sqrt{x_1^2+x_2^2}$, reduces equations (1) to the system

$$g_1 + \frac{\hbar}{2m}\left(h_{22} + \frac{h_2}{\omega} + 2g_2h_2 - \frac{2\lambda_1}{\alpha^2\lambda_2}e^{4\lambda_2 t}\frac{1}{r^2}\right) -$$
$$- d\left(g_{22} + \frac{g_2}{\omega} + 2g_2^2 + \frac{2}{\alpha^2}e^{4\lambda_2 t}\frac{1}{r^2}\right) - 2\lambda_2 g = 0,$$
$$h_1 + \frac{\hbar}{2m}\left(h_2^2 - g_{22} - \frac{g_2}{\omega} - g_2^2 - \frac{1}{\alpha^2}\left(1 - \frac{\lambda_1^2}{\lambda_2^2}\right)e^{4\lambda_2 t}\frac{1}{r^2}\right) + 2\lambda_1 g = 0,$$

3. $\langle A + \alpha H, J_{12} + \beta P_3\rangle$, $\alpha \in \mathbb{R}_{\neq 0}$, $\beta \in \mathbb{R}$. The ansatz

$$\psi(\vec{x},t) = \exp\left\{\frac{1}{2\alpha\lambda_2}e^{2\lambda_2 t} + g(\omega_1,\omega_2) + i\left[-\frac{\lambda_1}{2\alpha\lambda_2^2}e^{2\lambda_2 t} + h(\omega_1,\omega_2)\right]\right\},$$

where $\omega_1 = \sqrt{x_1^2+x_2^2}$, $\omega_2 = \beta\arctan\left(\frac{x_1}{x_2}\right) - x_3$, reduces equations (1) to the system

$$h_{11} + \frac{h_1}{\omega_1} + 2g_1h_1 + h_{22}\left(1 + \frac{\beta^2}{\omega_1^2}\right) + h_2^2\left(1 + \frac{\beta^2}{\omega_1^2}\right) -$$
$$- \frac{2md}{\hbar}\left(g_{11} + \frac{g_1}{\omega_1} + 2g_1^2 + g_{22}\left(1 + \frac{\beta^2}{\omega_1^2}\right) + g_2^2\left(1 + \frac{\beta^2}{\omega_1^2}\right)\right) - \frac{4m\lambda_2}{m}g = 0,$$
$$g_{11} + \frac{g_1}{\omega_1} + g_1^2 - h_1^2 + g_{22}\left(1 + \frac{\beta^2}{\omega_1^2}\right) + g_2^2\left(1 + \frac{\beta^2}{\omega_1^2}\right) -$$
$$- h_2^2\left(1 + \frac{\beta^2}{\omega_1^2}\right) - \frac{4m\lambda_1}{m}g = 0.$$

4. $\langle A + \alpha P_3, J_{12}\rangle$, $\alpha \in \mathbb{R}_{\neq 0}$. The ansatz

$$\psi(\vec{x},t) = \exp\left\{\frac{1}{\alpha}e^{2\lambda_2 t}x_3 + g(t,\omega) + i\left[-\frac{\lambda_1}{\alpha\lambda_2}e^{2\lambda_2 t}x_3 + h(t,\omega)\right]\right\},$$

where $\omega = \sqrt{x_1^2 + x_2^2}$, reduces equations (1) to the system

$$g_1 + \frac{\hbar}{2m}\left(h_{22} + \frac{h_2}{\omega} + 2g_2h_2 - \frac{\lambda_1}{\alpha^2\lambda_2}e^{4\lambda_2 t}\right) - \\ - d\left(g_{22} + \frac{g_2}{\omega} + 2g_2^2 + \frac{2}{\alpha^2}e^{4\lambda_2 t}\right) - 2\lambda_2 g = 0,$$

$$h_1 + \frac{\hbar}{2m}\left(h_2^2 - g_{22} - \frac{g_2}{\omega} - g_2^2 - \frac{1}{\alpha^2}\left(1 - \frac{\lambda_1^2}{\lambda_2^2}\right)e^{4\lambda_2 t}\right) + 2\lambda_1 g = 0.$$

4. Conclusions

We have determined the maximal Lie symmetries of equation (1) with an F of the form $F[\psi,\bar{\psi}] := \hbar f(\rho)$, and have found six different algebras containing among others the centrally extended Galilei algebra, the Galilei similitude algebra, and the Schrödinger algebra. Reduction and ansätze for these algebras have been studied previously.

New maximal symmetry algebras, due to the nonlinear character of the equation, appear in the case $f(\rho) = (\lambda_1 + i\lambda_2)\ln(\rho)$ (see cases 5 and 6 in Section 2.1). For these cases we have obtained reduced equations for various subalgebras. The ansätze resulting from these reductions lead to differential equations which we have solved explicitly in some cases and thus we have obtained explicit solutions of (1). Those reduced equations, which we have not been able to solve explicitly, are still much more suitable to numerical treatments than the original equation (1). The list of subalgebras which we have used for reduction in the case of the new algebras is by no means complete. In view of the successes of the reduction technique it seems warranted to obtain a classification of their subalgebras. The non-Lie anzätze for the nonlinear Schrödinger equation were constructed by Fushchych and Chopyk [21].

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