# Symmetry classification of the one-dimensional second order equation of hydrodynamical type

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The paper contains a symmetry classification of the one-dimensional second order equation of hydrodynamical type $L(Lu) + \lambda Lu = F(u)$, where $L \equiv \partial_t + u\partial_x$. Some classes of exact solutions of this equation are pointed out.

In [1, 2] the following generalized Navier–Stokes equation

$$\lambda_1 L\vec{v} + \lambda_2 L(L\vec{v}) = F\left(\vec{v}^{\,2}\right)\vec{v} + \lambda_4 \nabla p, \tag{1}$$

was proposed, where

$$L \equiv \frac{\partial}{\partial t} + v^l \frac{\partial}{\partial x_l} + \lambda_3 \triangle, \quad l = 1, 2, 3,$$

$\vec{v} = \left(v^1, v^2, v^3\right)$, $v^l = v^l(t, \vec{x})$, $p = p(t, \vec{x})$, $\nabla$ is the gradient, $\triangle$ is the Laplace operator, $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$ are arbitrary real parameters, $F\left(\vec{v}^{\,2}\right)$ is an arbitrary differentiable function.

In the one-dimensional scalar case, when $\lambda_3 = 0$, $\lambda_4 = 0$, equation (1) has the form

$$\lambda_1 Lu + \lambda_2 L(Lu) = F(u), \tag{2}$$

where $u = u(t, x)$, $L \equiv \partial_t + u\partial_x$.

In the case when $\lambda_2 = 0$ and $F(u) = 0$, equation (2) is known to describe the simple wave

$$u = \varphi(x - tu), \tag{3}$$

where $\varphi$ is an arbitrary function. Formula (3) gives the general solution of the equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

If $\lambda_2 \neq 0$, then equation (2) can be rewritten in the form

$$L(Lu) + \lambda Lu = F(u), \quad \lambda = \mathrm{const}. \tag{4}$$

Equation (4), in expanded form, is written as follows

$$\frac{\partial^2 u}{\partial t^2} + 2u\frac{\partial^2 u}{\partial t \partial x} + \frac{\partial u}{\partial t}\frac{\partial u}{\partial x} + u\left(\frac{\partial u}{\partial x}\right)^2 + u^2\frac{\partial^2 u}{\partial x^2} + \lambda\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = F(u).$$

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This equation with arbitrary $F(u)$ is evidently invariant under the two-dimensional algebra of translations that is determined by the operators

$$P_0 = \partial_t, \quad P_1 = \partial_x. \tag{5}$$

In the present paper we carry out a symmetry classification of the equation (4), i.e., we describe functions $F(u)$, with which the equation (4) admits more extensive Lie algebras than the two-dimensional algebra of translations (5).

## Symmetry classification

Symmetry classification of (4) is performed on the base of the Lie algorithm [3, 4, 5] in the class of first-order differential operators

$$X = \xi^0(t,x,u)\partial_t + \xi^1(t,x,u)\partial_x + \eta(t,x,u)\partial_u. \tag{6}$$

**Remark.** In cases **1.4, 2.3, 2.4** we assume that

$$\frac{\partial \xi^0}{\partial u} = 0, \quad \frac{\partial \xi^1}{\partial u} = 0.$$

It is obvious, that the cases $\lambda = 0$ and $\lambda \neq 0$ will be essentially different for the investigation of symmetries of the equation (4). If $\lambda \neq 0$, then one can always set $\lambda \equiv 1$ (there exists a change of variables). For this reason we consider the cases $\lambda = 0$ and $\lambda = 1$ separately.

**I.** Let us consider equation (4), when $\lambda = 0$, i.e., the equation

$$L(Lu) = F(u). \tag{7}$$

Symmetry classification of (7) leads to five distinct cases.

**Case 1.1.** $F(u)$ is an arbitrary continuously differentiable function. The maximal invariance algebra in this case is the two-dimensional algebra (5).

**Case 1.2.** $F(u) = a\exp(bu)$, $a, b = \mathrm{const}$, $a \neq 0$, $b \neq 0$. Without loss of generality we can put $b \equiv 1$ (there exists a change of variables). The maximal invariance algebra of the equation

$$L(Lu) = a\exp(u) \tag{8}$$

is a three-dimensional algebra, whose basis elements are given by the operators

$$P_0 = \partial_t, \quad P_1 = \partial_x, \quad Y = t\partial_t + (x - 2t)\partial_x - 2\partial_u. \tag{9}$$

The finite transformations which are generated by the operator $Y$ in (9) have the form:

$$\begin{aligned} &t \to \tilde{t} = t\exp(\theta), \\ &x \to \tilde{x} = (x - 2\theta t)\exp(\theta), \\ &u \to \tilde{u} = u - 2\theta. \end{aligned}$$

Hereafter $\theta$ is a real group parameter of the corresponding Lie group.

We note that $Y$ in (9) can be represented as the linear combination of the dilatation and Galilei operators

$$Y = (t\partial_t + x\partial_x) - 2(t\partial_x + \partial_u) = D - 2G.$$

The operators $D$ and $G$ commute, thus the transformations corresponding to $Y$ can be interpreted as a composition of dilatation and Galilei transformations, i.e., as a composition of dilatation on $t$ and $x$ with a change of inertial system. On the other hand, the operators (9) form a subalgebra of extended Galilei algebra, although the extended Galilei algebra is not the invariance algebra of the equation (8). The same results are valid for other cases of equation (4).

**Case 1.3.** $F(u) = a(u+b)^p$, $a, b, p = \text{const}$, $a \neq 0$, $p \neq 0$, $p \neq 1$. The maximal invariance algebra of the equation

$$L(Lu) = a(u+b)^p \tag{10}$$

is a three-dimensional algebra, whose basis elements are given by the operators

$$\begin{gathered} P_0 = \partial_t, \quad P_1 = \partial_x, \\ R = t\partial_t + \left(\frac{p-3}{p-1}x - \frac{2b}{p-1}t\right)\partial_x - \frac{2}{p-1}(u+b)\partial_u. \end{gathered} \tag{11}$$

The operator $R$ generates the following finite transformations:

$$\begin{aligned} & t \to \tilde{t} = t\exp(\theta), \\ & x \to \tilde{x} = x\exp\left(\frac{p-3}{p-1}\theta\right) - bt\exp(\theta), \\ & u \to \tilde{u} = (u+b)\exp\left(-\tfrac{2}{p-1}\theta\right) - b. \end{aligned}$$

If $b \neq 0$, then $R$ can be again represented as a linear combination of dilatation and Galilei operators.

**Case 1.4.** $F(u) = au + b$, $a, b = \text{const}$, $a \neq 0$. In consequence of a change of variables one can always set $a \equiv 1$ or $a \equiv -1$. Let us consider these cases.

**a)** The invariance algebra of the equation

$$L(Lu) = u + b \tag{12}$$

is a seven-dimensional algebra, whose basis elements are given by the operators

$$\begin{aligned} & P_0 = \partial_t, \quad P_1 = \partial_x, \\ & Y_1 = (x+bt)\partial_x + (u+b)\partial_u, \\ & Y_2 = \cosh t\partial_x + \sinh t\partial_u, \\ & Y_3 = \sinh t\partial_x + \cosh t\partial_u, \\ & Y_4 = \cosh t\partial_t + (x+bt)\sinh t\partial_x + ((x+bt)\cosh t + b\sinh t)\partial_u, \\ & Y_5 = \sinh t\partial_t + (x+bt)\cosh t\partial_x + ((x+bt)\sinh t + b\cosh t)\partial_u. \end{aligned} \tag{13}$$

The operators $Y_1$–$Y_3$ generate the following finite transformations (because the transformations for $Y_4$ and $Y_5$ are cumbersome we omit their explicit form):

$$\begin{aligned} Y_1: \quad & t \to \tilde{t} = t, \\ & x \to \tilde{x} = (x+bt)\exp(\theta) - bt, \\ & u \to \tilde{u} = (u+b)\exp(\theta) - b. \end{aligned}$$

$$\begin{aligned} Y_2: \quad & t \to \tilde{t} = t, \\ & x \to \tilde{x} = x + \theta\cosh t, \\ & u \to \tilde{u} = u + \theta\sinh t. \end{aligned}$$

$$\begin{aligned} Y_3:\quad & t\to\tilde t=t,\\ & x\to\tilde x=x+\theta\sinh t,\\ & u\to\tilde u=u+\theta\cosh t.\end{aligned}$$

The operator $Y_1$ in (13) can be again represented as a linear combination of the dilatation and Galilei operators.

**b)** The invariance algebra of the equation

$$L(Lu)=-u+b \tag{14}$$

is a seven-dimensional algebra, whose basis elements are given by the operators

$$\begin{aligned} & P_0=\partial_t,\quad P_1=\partial_x,\\ & R_1=(x-bt)\partial_x+(u-b)\partial_u,\\ & R_2=\cos t\partial_x-\sin t\partial_u,\\ & R_3=\sin t\partial_x+\cos t\partial_u,\\ & R_4=-\cos t\partial_t+(x-bt)\sin t\partial_x+((x-bt)\cos t-b\sin t)\partial_u,\\ & R_5=\sin t\partial_t+(x-bt)\cos t\partial_x-((x-bt)\sin t+b\cos t)\partial_u.\end{aligned} \tag{15}$$

The operators $R_1$–$R_3$ generate the following finite transformations (because the transformations for $R_4$ and $R_5$ are cumbersome we omit their explicit form):

$$\begin{aligned} R_1:\quad & t\to\tilde t=t,\\ & x\to\tilde x=(x-bt)\exp(\theta)+bt,\\ & u\to\tilde u=(u-b)\exp(\theta)+b.\end{aligned}$$

$$\begin{aligned} R_2:\quad & t\to\tilde t=t,\\ & x\to\tilde x=x+\theta\cos t,\\ & u\to\tilde u=u-\theta\sin t.\end{aligned}$$

$$\begin{aligned} R_3:\quad & t\to\tilde t=t,\\ & x\to\tilde x=x+\theta\sin t,\\ & u\to\tilde u=u+\theta\cos t.\end{aligned}$$

The operator $R_1$ in (15) can be again represented as a linear combination of dilatation and Galilei operators.

**Case 1.5.** $F(u)=a$, $a=\mathrm{const}$. In the case $a\neq 0$ (there exists a change of variables) without loss of generality we can admit that $a\equiv 1$. Thus we consider the cases $a=0$ and $a=1$ separately.

**a)** The maximal invariance algebra of the equation

$$L(Lu)=0 \tag{16}$$

is a ten-dimensional algebra, whose basis elements are given by the operators

$$\begin{aligned} & P_0=\partial_t,\quad P_1=\partial_x,\\ & G=t\partial_x+\partial_u,\quad D=t\partial_t+x\partial_x,\quad D_1=x\partial_x+u\partial_u,\\ & A_1=\frac12 t^2\partial_t+tx\partial_x+x\partial_u,\quad A_2=\frac12 t^2\partial_x+t\partial_u,\quad A_3=u\partial_t+\frac12 u^2\partial_x,\\ & A_4=(tu-x)\partial_t+\frac12 tu^2\partial_x+\frac12 u^2\partial_u,\\ & A_5=\left(t^2u-2tx\right)\partial_t+\left(\frac12 t^2u^2-2x^2\right)\partial_x+\left(tu^2-2xu\right)\partial_u.\end{aligned} \tag{17}$$

We note, that subalgebras $\langle P_0, P_1, G\rangle$ and $\langle A_1, -A_2, G\rangle$ in the representation (17) define two different representations of the Galilei algebra $AG(1,1)$ [3].

The finite transformations which are generated by the operators (17) have the form (because the transformations for $A_4$ and $A_5$ are cumbersome we omit their explicit form):

$$\begin{aligned} G:\quad & t \to \tilde t = t,\\ & x \to \tilde x = x + \theta t,\\ & u \to \tilde u = u + \theta. \end{aligned}$$

$$\begin{aligned} D:\quad & t \to \tilde t = t\exp(\theta),\\ & x \to \tilde x = x\exp(\theta),\\ & u \to \tilde u = u. \end{aligned}$$

$$\begin{aligned} D_1:\quad & t \to \tilde t = t,\\ & x \to \tilde x = x\exp(\theta),\\ & u \to \tilde u = u\exp(\theta). \end{aligned}$$

$$\begin{aligned} A_1:\quad & t \to \tilde t = \frac{2t}{2-\theta t},\\ & x \to \tilde x = \frac{4x}{(2-\theta t)^2},\\ & u \to \tilde u = u + \frac{2x\theta}{2-\theta t}. \end{aligned}$$

$$\begin{aligned} A_2:\quad & t \to \tilde t = t,\\ & x \to \tilde x = x + \frac{1}{2}\theta t^2,\\ & u \to \tilde u = u + \theta t. \end{aligned}$$

$$\begin{aligned} A_3:\quad & t \to \tilde t = t + \theta u,\\ & x \to \tilde x = x + \frac{1}{2}\theta u^2,\\ & u \to \tilde u = u. \end{aligned}$$

**b)** The maximal invariance algebra of the equation

$$L(Lu) = 1 \tag{18}$$

is a ten-dimensional algebra, whose basis elements are given by the operators

$$\begin{aligned} & P_0 = \partial_t, \quad P_1 = \partial_x, \quad G = t\partial_x + \partial_u,\\ & B_1 = t\partial_t + 3x\partial_x + 2u\partial_u, \quad B_2 = \left(x - \frac{1}{6}t^3\right)\partial_x + \left(u - \frac{1}{2}t^2\right)\partial_u,\\ & B_3 = \frac{1}{2}t^2\partial_t + \left(tx + \frac{1}{12}t^4\right)\partial_x + \left(x + \frac{1}{3}t^3\right)\partial_u, \quad A_2 = \frac{1}{2}t^2\partial_x + t\partial_u,\\ & B_4 = \left(u - \frac{1}{2}t^2\right)\partial_t + \left(\frac{1}{2}u^2 - \frac{1}{8}t^4\right)\partial_x + \left(tu - \frac{1}{2}t^3\right)\partial_u, \end{aligned} \tag{19}$$

$$B_5=\left(tu-x-\frac{1}{3}t^3\right)\partial_t+\left(\frac{1}{2}tu^2-\frac{1}{2}t^2x-\frac{1}{24}t^5\right)\partial_x+$$
$$+\left(\frac{1}{2}u^2+\frac{1}{2}t^2u-tx-\frac{5}{24}t^4\right)\partial_u,$$
$$B_6=\left(t^2u-2tx-\frac{1}{6}t^4\right)\partial_t+\left(\frac{1}{2}t^2u^2-2x^2-\frac{1}{3}t^3x-\frac{1}{72}t^6\right)\partial_x+$$
$$+\left(tu^2-2xu+\frac{1}{3}t^3u-t^2x-\frac{1}{12}t^5\right)\partial_u.$$

The algebra, generated by the operators (19), includes again two different Galilei algebras $\langle P_0, P_1, G\rangle$ and $\langle B_3, -A_2, G\rangle$ as subalgebras.

The finite transformations which are generated by the operators (19) have the form (because the transformations for $B_4$, $B_5$ and $B_6$ are cumbersome we omit their explicit form):

$$B_1:\quad \begin{aligned} &t\to\tilde t=t\exp(\theta),\\ &x\to\tilde x=x\exp(3\theta),\\ &u\to\tilde u=u\exp(2\theta).\end{aligned}$$

$$B_2:\quad \begin{aligned} &t\to\tilde t=t,\\ &x\to\tilde x=\left(x-\frac{1}{6}t^3\right)\exp(\theta)+\frac{1}{6}t^3,\\ &u\to\tilde u=\left(u-\frac{1}{2}t^2\right)\exp(\theta)+\frac{1}{2}t^2.\end{aligned}$$

$$B_3:\quad \begin{aligned} &t\to\tilde t=\frac{2t}{2-\theta t},\\ &x\to\tilde x=\frac{12x-2t^3}{3(2-\theta t)^2}+\frac{4t^3}{3(2-\theta t)^3},\\ &u\to\tilde u=u+\frac{2t^2}{(2-\theta t)^2}+\frac{12x-2t^3}{3t(2-\theta t)}-\frac{12x+t^3}{6t}.\end{aligned}$$

**II.** Let us consider equation (4) for $\lambda\neq 0$. As it was noticed above, we can set $\lambda\equiv 1$. Symmetry classification gives in this case four principally distinct cases.

**Case 2.1.** $F(u)$ is an arbitrary continuously differentiable function. The maximal invariance algebra of the equation

$$L(Lu)+Lu=F(u), \tag{20}$$

is the two-dimensional algebra (5).

**Case 2.2.** $F(u)=au^3-\frac{2}{9}u$, $a=\mathrm{const}$, $a\neq 0$. The maximal invariance algebra of the equation

$$L(Lu)+Lu=au^3-\frac{2}{9}u \tag{21}$$

is a three-dimensional algebra, whose basis elements are given by the operators

$$P_0=\partial_t,\quad P_1=\partial_x,\quad Z=\exp\left(\frac{1}{3}t\right)\left(\partial_t-\frac{1}{3}u\partial_u\right). \tag{22}$$

The operator $Z$ generates the following finite transformations:

$$t \to \tilde{t} = -3\ln\left(\exp\left(-\frac{1}{3}t\right) - \frac{\theta}{3}\right),$$
$$x \to \tilde{x} = x,$$
$$u \to \tilde{u} = u\left(1 - \frac{1}{3}\theta\exp\left(\frac{1}{3}t\right)\right).$$

**Case 2.3.** $F(u) = au + b$, $a, b = \text{const}$, $a \neq 0$. The invariance algebra of the equation

$$L(Lu) + Lu = au + b \tag{23}$$

is a five-dimensional algebra, whose basis elements are given by the operators

$$P_0 = \partial_t, \quad P_1 = \partial_x, \quad Z_1 = \left(x + \frac{b}{a}t\right)\partial_x + \left(u + \frac{b}{a}\right)\partial_u,$$

and two other operators depending on constant $a$ have the form

**a)** $a = -\frac{1}{4}$

$$Z_2 = \exp\left(-\frac{1}{2}t\right)\left(\partial_x - \frac{1}{2}\partial_u\right), \quad Z_3 = \exp\left(-\frac{1}{2}t\right)\left(t\partial_x + \left(1 - \frac{1}{2}t\right)\partial_u\right),$$

**b)** $a > -\frac{1}{4}$, $a \neq 0$

$$Z_4 = \exp(\alpha t)(\partial_x + \alpha\partial_u), \quad Z_5 = \exp(\beta t)(\partial_x + \beta\partial_u),$$

where

$$\alpha = \frac{-1 - \sqrt{4a+1}}{2}, \quad \beta = \frac{-1 + \sqrt{4a+1}}{2},$$

**c)** $a < -\frac{1}{4}$

$$Z_6 = \exp(\gamma t)(\sin\delta t\partial_x + (\gamma\sin\delta t + \delta\cos\delta t)\partial_u),$$
$$Z_7 = \exp(\gamma t)(\cos\delta t\partial_x + (\gamma\cos\delta t - \delta\sin\delta t)\partial_u),$$

where

$$\gamma = -\frac{1}{2}, \quad \delta = \frac{\sqrt{-(4a+1)}}{2}.$$

The corresponding finite transformations have the form:

$$Z_1: \quad t \to \tilde{t} = t,$$
$$x \to \tilde{x} = \left(x + \frac{b}{a}t\right)\exp(\theta) - \frac{b}{a}t,$$
$$u \to \tilde{u} = \left(u + \frac{b}{a}\right)\exp(\theta) - \frac{b}{a}.$$

$$Z_4: \quad t \to \tilde{t} = t,$$
$$x \to \tilde{x} = x + \theta\exp(\alpha t),$$
$$u \to \tilde{u} = u + \alpha\theta\exp(\alpha t).$$

$$Z_3: \quad t \to \tilde{t} = t, \\ x \to \tilde{x} = x + \theta t \exp\left(-\frac{1}{2}t\right), \\ u \to \tilde{u} = u + \theta\left(1 - \frac{1}{2}t\right)\exp\left(-\frac{1}{2}t\right).$$

$$Z_6: \quad t \to \tilde{t} = t, \\ x \to \tilde{x} = x + \theta \sin\delta t \exp(\gamma t), \\ u \to \tilde{u} = u + \theta(\gamma\sin\delta t + \delta\cos\delta t)\exp(\gamma t).$$

$$Z_7: \quad t \to \tilde{t} = t, \\ x \to \tilde{x} = x + \theta \cos\delta t \exp(\gamma t), \\ u \to \tilde{u} = u + \theta(\gamma\cos\delta t - \delta\sin\delta t)\exp(\gamma t).$$

**Case 2.4.** $F(u) = a$, $a = \text{const}$. The invariance algebra of the equation

$$L(Lu) + Lu = a \tag{24}$$

is a five-dimensional algebra, whose basis elements are given by the operators

$$P_0 = \partial_t, \quad P_1 = \partial_x, \quad G = t\partial_x + \partial_u, \\ Q_1 = \left(x - \frac{a}{2}t^2\right)\partial_x + (u - at)\partial_u, \quad Q_2 = \exp(-t)(\partial_x - \partial_u). \tag{25}$$

The finite transformations for $Q_1$, $Q_2$ have the form:

$$Q_1: \quad t \to \tilde{t} = t, \\ x \to \tilde{x} = \left(x - \frac{a}{2}t^2\right)\exp(\theta) + \frac{a}{2}t^2, \\ u \to \tilde{u} = (u - at)\exp(\theta) + at.$$

$$Q_2: \quad t \to \tilde{t} = t, \\ x \to \tilde{x} = x + \theta\exp(-t), \\ u \to \tilde{u} = u - \theta\exp(-t).$$

## Construction of solutions

In the case when the equation (4) has the form

$$L(Lu) + \lambda Lu = a, \quad a, \lambda = \text{const} \tag{26}$$

the change of variables

$$t = \tau, \quad x = \omega + u\tau, \quad u = u \tag{27}$$

enable us to construct the general solution of (26). In consequence of the change of variables (27) we obtain:

$$L = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} \quad \to \quad \partial_\tau, \\ Lu = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} \quad \to \quad \frac{u_\tau}{1 + \tau u_\omega}.$$

After the change of variables the equation (26) has the form

$$\partial_\tau \left( \frac{u_\tau}{1+\tau u_\omega} \right) + \lambda \left( \frac{u_\tau}{1+\tau u_\omega} \right) = a. \tag{28}$$

Integrating (28) one time, we get the linear nonhomogeneous partial differential equation. Finding first integrals of the corresponding system of characteristic equations and doing the inverse change of variables we find the solutions of (26).

**Remark.** We notice that the solution of equation $1+\tau u_\omega = 0$ in variables $(t,x,u)$ is $x = f(t)$, where $f(t)$ is an arbitrary function. Thus (26) is equivalent to an ordinary differential equation in this singular case.

Let us illustrate it on the example of equations (16). After the change of variables (27), equation (16) is rewritten in the form:

$$\partial_\tau \left( \frac{u_\tau}{1+\tau u_\omega} \right) = 0. \tag{29}$$

Integrating (29) we obtain

$$\frac{u_\tau}{1+\tau u_\omega} = g(\omega), \tag{30}$$

where $g(\omega)$ is an arbitrary function.

If $g(\omega) \equiv 0$, then $u_\tau = 0$ and we get the solution of type (3) (because, it is obvious that the solution of equation $Lu = 0$ is a solution of (16)). When $g(\omega) \neq 0$, in accordance with arbitrary choice of $g(\omega)$ we can set $g(\omega) = -2(dh(\omega)/d\omega)^{-1}$. Therefore (30) has the form

$$u_\tau + \frac{2\tau}{h'(\omega)} u_\omega = -\frac{2}{h'(\omega)}. \tag{31}$$

The system of characteristic equation for (31) is

$$\frac{d\tau}{1} = \frac{h'(\omega)d\omega}{2\tau} = \frac{h'(\omega)du}{-2}. \tag{32}$$

Hence, we obtain two first integrals:

$$\tau^2 - h(\omega) = C_1, \quad u \pm \int \frac{d\omega}{\sqrt{h(\omega)+C_1}} = C_2. \tag{33}$$

Integrating (33) and expressing $C_1$ and $C_2$ by $(\tau,\omega,u)$ we find a solution of (30) in the form

$$\Phi\left(C_1, C_2\right) = 0, \tag{34}$$

where $\Phi$ is an arbitrary function. Performing in (34) the inverse change of variables we get a solution of (16). For instance, we set $h(\omega) = \omega$. Then the expression

$$x - ut - t^2 = \varphi(u+2t), \tag{35}$$

defines the class of implicit solutions of equation (16), where $\varphi$ is an arbitrary function.

The same results we can obtain for other cases of (26). If $F(u) \neq \mathrm{const}$ in (4) then this method does not lead to solutions. Below we give some classes of solutions of equations (26):

**1.** $L(Lu) = 0$

1.1. $x - ut + \frac{C}{2}t^2 = \varphi(u - Ct)$;

1.2. $u \pm \ln(x - ut \mp t) = \varphi\left(t^2 - (x-ut)^2\right)$;

1.3. $u + \frac{t(x-ut)^3}{t^2(x-ut)^2 - 1} = \varphi\left(t^2 - \frac{1}{(x-ut)^2}\right)$;

1.4. $u = \varphi\left(\frac{x-ut}{\exp\left(t^2\right)}\right) - \frac{x-ut}{\exp\left(t^2\right)}\int \exp\left(t^2\right) dt$;

**2.** $L(Lu) = a$

$$x - ut + \frac{a}{3}t^3 + \frac{C}{2}t^2 = \varphi\left(u - \frac{a}{2}t^2 - Ct\right);$$

**3.** $L(Lu) + Lu = a$

$$x - ut - C(t+1)\exp(-t) + \frac{a}{2}t^2 = \varphi\left(u + C\exp(-t) - at\right),$$

$C = \mathrm{const}$, $\varphi$ is an arbitrary function.

Thus, we have investigated the symmetry classifications of (4) and pointed out all functions $F(u)$ under which the invariance algebra of (4) admits the extension. The new representations which may have an interesting physical interpretation are obtained. In the case $F(u) = \mathrm{const}$ we described the algorithm of construction of the general solution of (4) and pointed out some solutions. The symmetry properties of (4) can be used for a symmetry reduction and construction of the solutions and for their generation by finite group transformations [3, 4, 5].

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