On new exact solutions of the multidimensional nonlinear d'Alembert equation

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On the present paper new classes of exact solutions of the nonlinear d'Alembert equation in the space $R_{1,n}$, $n \geq 2$,

$$\Box u + \lambda u^k = 0 \tag{1}$$

are built. Here $\Box u = u_{00} - u_{11} - \cdots - u_{nn}$, $u_{\mu\nu} = \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}$, $u = u(x)$, $x = (x_0, x_1, \dots, x_n)$; $\mu, \nu = 0, 1, \dots, n$. Symmetry properties of equation (1) have been studied in papers [1, 2] in which it was established that equation (1) is invariant under the extended Poincaré algebra $A\tilde{P}(1,n)$:

$$J_{0a} = x_0\partial_a + x_a\partial_0, \quad J_{ab} = x_b\partial_a - x_a\partial_b, \quad P_\mu = \partial_\mu,$$
$$S = -x^\mu\partial_\mu + \frac{2u}{k-1}\partial_u \quad (a, b = 1, \dots, n;\ \mu = 0, 1, \dots, n).$$

Using the subgroup structure of the group $\tilde{P}(1,2)$ in papers [1, 2] some classes of exact solutions of equation (1) in the space $R_{1,2}$ were built. The analogous results in the space $R_{1,3}$ were obtained in [3, 4]. The generalization of results for the n-dimensional case was considered in [5, 6]. In order to find exact solutions, symmetry ansatzes reducing equation (1) to ordinary differential equations were applied in above mentioned papers.

In the present paper in order to build exact solutions of equation (1), symmetry ansatzes reducing equation (1) to equations of two invariant variables are used. We are interested in these ansatzes because a reduced equation often has additional symmetries. This fact permits to apply these ansatzes for finding new solutions of the present equation. Let us cite as an example the ansatz $u = u(x_0 - x_n, x_1, \dots, x_{n-1})$ which was considered in [6]. The corresponding reduced equation has the infinite group of invariance. Note that this ansatz is built by one-dimensional subalgebra $\langle P_0 + P_n\rangle$.

In the present paper the series of ansatzes of such a kind as $u = u(\omega_1, \omega_2)$, where $\omega_1 = x_0 - x_m$, $\omega_2 = x_0^2 - x_1^2 - \cdots - x_m^2$, $2 \leq m \leq n$, is considered. These ansatzes are built by the subalgebras $AE_1[1, m-1] \oplus AE[m+1, n]$, where $AE_1[1, m-1] = \langle G_1, \dots, G_{m-1}, J_{12}, \dots, J_{m-2,m-1}\rangle$, $AE[m+1, n] = \langle P_{m+1}, \dots, P_n, J_{m+1,m+2}, \dots, J_{n-1,n}\rangle$, $G_a = J_{0a} - J_{am}$, $a = 1, \dots, m-1$, and if $m = n$ we think $AE[m+1, n] = 0$. The ansatz $u = u(\omega_1, \omega_2)$ reduces equation (1) to the equation

$$4\omega_1 u_{12} + 4\omega_2 u_{22} + 2(m+1)u_2 + \lambda u^k = 0. \tag{2}$$

Let us investigate symmetry of the equation (2).

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Theorem 1. *The maximal algebra of invariance of equation (2) in the case of $k \neq 0$, $\frac{m+1}{m-1}$ and $m > 1$ in the Lie sense is the 4-dimensional Lie algebra $A(4)$ which is generated by such operators:*

$$X_1 = \omega_1 \frac{\partial}{\partial w_1} + \omega_2 \frac{\partial}{\partial \omega_2} - \frac{1}{k-1} u \frac{\partial}{\partial u}, \quad X_2 = \omega_2 \frac{\partial}{\partial w_2} - \frac{1}{k-1} u \frac{\partial}{\partial u},$$
$$M = \omega_1^l \left(\omega_1 \frac{\partial}{\partial w_1} + \omega_2 \frac{\partial}{\partial \omega_2} - \frac{m-1}{2} u \frac{\partial}{\partial u} \right), \quad l = \frac{(m-1)(k-1)}{2} - 1.$$

Theorem 2. *The maximal algebra of invariance of equation (2) in the case of $k = \frac{m+1}{m-1}$ and $m > 1$ in the sense of Lie is the 4-dimensional Lie algebra $B(4)$ which is generated by such operators:*

$$S = \omega_1 \ln \omega_1 \frac{\partial}{\partial \omega_1} + \omega_2 \ln \omega_1 \frac{\partial}{\partial \omega_2} - \frac{m-1}{2} (\ln \omega_1 + 1) u \frac{\partial}{\partial u},$$
$$Z_1 = \omega_1 \frac{\partial}{\partial \omega_1} + \omega_2 \frac{\partial}{\partial \omega_2} - \frac{m-1}{2} u \frac{\partial}{\partial u},$$
$$Z_2 = \omega_2 \frac{\partial}{\partial \omega_2} - \frac{m-1}{2} u \frac{\partial}{\partial u}, \quad Z_3 = \omega_1 \frac{\partial}{\partial \omega_2}.$$

Let us consider two cases.

1. The case $k \neq \frac{m+1}{m-1}$. Classify one-dimensional subalgebras of the algebra $A(4)$ with respect to G-conjugation, where $G = \exp A(4)$. Ansatzes, built by these subalgebras, reduce the equation (2) to ordinary differential equations. Note that the operators of the algebra $A(4)$ satisfy the following commutation relations: $[X_1, X_2] = 0$, $[X_1, X_3] = 0$, $[X_1, M] = lM$, $[X_2, X_3] = -X_3$, $[X_2, M] = 0$, $[X_3, M] = 0$.

Theorem 3. *Let K be one-dimensional subalgebra of the algebra $A(4)$. Then K is conjugated with one of the following algebras:* 1) $K_1 = \langle X_1 + \alpha X_2 \rangle$; 2) $K_2 = \langle X_2 \rangle$; 3) $K_3 = \langle X_1 + \alpha X_3 \rangle$ $(\alpha = \pm 1)$; 4) $K_4 = \langle X_3 \rangle$; 5) $K_5 = \langle M + \alpha X_2 \rangle$ $(\alpha = 0, \pm 1)$; 6) $K_6 = \langle M + \alpha X_3 \rangle$ $(\alpha = \pm 1)$.

The following ansatzes correspond to the subalgebras K_1–K_6 of the theorem 3:

$$K_1: \quad u = \omega^{\frac{\alpha+1}{1-k}} \varphi(\omega), \quad \omega = \omega_2 \omega_1^{-\alpha-1};$$
$$K_2: \quad u = \omega_2^{\frac{1}{1-k}} \varphi(\omega), \quad \omega = \omega_1;$$
$$K_3: \quad u = \omega_1^{\frac{1}{1-k}} \varphi(\omega), \quad \omega = \frac{\omega_2}{\omega_1} - \alpha \ln \omega_1;$$
$$K_4: \quad u = \varphi(\omega), \quad \omega = \omega_1;$$
$$K_5: \quad u = (\omega_1^1 \omega_2)^{\frac{1}{1-k}} \varphi(\omega), \quad \omega = \frac{\alpha}{l} \omega_1^{-l} + \ln \frac{\omega_2}{\omega_1};$$
$$K_6: \quad u = \omega_1^{\frac{1-m}{2}} \varphi(\omega), \quad \omega = \frac{\omega_2}{\omega_1} + \frac{\alpha}{l} \omega_1^{-l}.$$

These ansatzes reduce equation (2) to ordinary differential equations with an unknown function $\varphi(\omega)$:

$$K_1: \quad -4\alpha\omega\ddot{\varphi} + \frac{4(l-\alpha k)}{k-1} \dot{\varphi} + \lambda \varphi^k = 0;$$
$$K_2: \quad -\frac{4\omega}{k-1} \dot{\varphi} - \frac{4l}{(k-1)^2} \varphi + \lambda \varphi^k = 0;$$
$$K_3: \quad -4\alpha\ddot{\varphi} + \frac{4l}{k-1} \dot{\varphi} + \lambda \varphi^k = 0;$$

$$K_4: \quad \lambda\varphi^k = 0;$$

$$K_5: \quad -4\alpha\ddot{\varphi} + \frac{4\alpha}{k-1}\dot{\varphi} + \lambda\varphi^k = 0;$$

$$K_6: \quad -4\alpha\ddot{\varphi} + \lambda\varphi^k = 0.$$

The equation corresponding to the subalgebra K_1, in case $\alpha = 0$ has the solution

$$\varphi^{1-k} = \frac{\lambda(k-1)^2}{4l}(\omega + C).$$

In consequence we obtain the following solution of equation (1)

$$u^{1-k} = \frac{\lambda(k-1)^2}{4l}(\omega_2 + C\omega_1). \tag{3}$$

If $\alpha = \frac{2l}{k+1}$, then the equation corresponding to the subalgebra K_1 assumes

$$-\frac{8l\omega}{k+1}\ddot{\varphi} - \frac{4l}{k+1}\dot{\varphi} + \lambda\varphi^k = 0.$$

The particular solution of this equation is

$$\varphi^{1-k} = \frac{\lambda(k-1)^2}{4l}(\omega^{\frac{1}{2}} + C)^2.$$

Therefore, equation (1) has the following solution:

$$u^{1-k} = \frac{\lambda(k-1)^2}{4l}\left(\omega_2^{\frac{1}{2}} + C\omega_1^{\frac{k-1}{2(k+1)}}\right)^2. \tag{4}$$

If $\alpha = \frac{l(k+1)}{2}$, then the equation corresponding to the subalgebra K_1 assumes

$$-2l(k+1)\varphi\ddot{\varphi} - 2(k+2)\dot{\varphi} + \lambda\varphi^k = 0.$$

This equation has the solution

$$\varphi^{1-k} = \frac{\lambda(k-1)^2}{4l}\left(\omega^{\frac{1}{2}} + Cw^{\frac{k-1}{2(k+1)}}\right)^2.$$

Therefore, equation (1) has the following solution

$$u^{1-k} = \frac{\lambda(k-1)^2}{4l}\left\{\omega_2^{\frac{1}{2}} + C\omega_1^{\frac{l(k+1)+2}{2(k+1)}}\omega_2^{\frac{k-1}{2(k+1)}}\right\}^2. \tag{5}$$

The equations corresponding to the subalgebras K_5 and K_6 have such solutions:

$$\varphi^{1-k} = \frac{\lambda(k-1)^2}{4l}(1 + C\omega^l), \quad \varphi^{1-k} = \frac{\lambda(k-1)^2}{8\alpha(k+1)}(\omega + C)^2.$$

Therefore, equation (1) has the following solutions:

$$u^{1-k} = \frac{\lambda(k-1)^2}{4l}\omega_2(1 + C\omega_1^l), \tag{6}$$

$$u^{1-k} = \frac{\lambda(k-1)^2}{8\alpha(k+1)}\omega_1^{l-1}\left(\omega_2 + \frac{\alpha}{l}\omega_1^{1-l} + C\omega_1\right)^2. \tag{7}$$

2. The case $k = \frac{m+1}{m-1}$. The basis elements of the algebra $B(4)$ satisfy the following commutation relations: $[S, Z_1] = -Z_1$, $[S, Z_2] = 0$, $[S, Z_3] = 0$, $[Z_1, Z_2] = 0$, $[Z_1, Z_3] = 0$, $[Z_2, Z_3] = -Z_3$.

Theorem 4. *Let L be one-dimensional subalgebra of the algebra $B(4)$. Then L is conjugated with one of the following algebras:* 1) $L_1 = \langle Z_1 + \alpha Z_2\rangle$ $(\alpha = 0, \pm 1)$*;* 2) $L_2 = \langle Z_2\rangle$*;* 3) $L_3 = \langle Z_1 + \alpha Z_3\rangle$ $(\alpha = \pm 1)$*;* 4) $L_4 = \langle Z_3\rangle$*;* 5) $L_5 = \langle S + \alpha Z_2\rangle$*;* 6) $L_6 = \langle S + \alpha Z_3\rangle$ $(\alpha = \pm 1)$.

The following ansatzes correspond to the subalgebras L_1–L_6 of theorem 4:

$$
\begin{aligned}
&L_1: \quad u = \omega_2^{\frac{1-m}{2}}\varphi(\omega), \quad \omega = \omega_2\omega_1^{-d-1};\\
&L_2: \quad u = \omega_2^{\frac{1-m}{2}}\varphi(\omega), \quad \omega = \omega_1;\\
&L_3: \quad u = \omega_1^{\frac{1-m}{2}}\varphi(\omega), \quad \omega = \frac{\omega_2}{\omega_1} - \alpha\ln\omega_1;\\
&L_4: \quad u = \varphi(\omega), \quad \omega = \omega_1;\\
&L_5: \quad u = (\omega_1\ln^{\alpha+1}\omega_1)^{\frac{1-m}{2}}\varphi(\omega), \quad \omega = \frac{\omega_1\ln^{\alpha}\omega_1}{\omega_2};\\
&L_6: \quad u = (\omega_1\ln\omega_1)^{\frac{1-m}{2}}\varphi(\omega), \quad \omega = \frac{\omega_2}{\omega_1} - \alpha\ln(\ln\omega_1).
\end{aligned}
$$

These ansatzes reduce equation (2) to ordinary differential equations with an unknown function $\varphi(\omega)$:

$$
\begin{aligned}
&L_1: \quad -4\alpha\omega^2\ddot{\varphi} + 2\alpha(m-3)\omega\dot{\varphi} + \lambda\varphi^{\frac{m+1}{m-1}} = 0;\\
&L_2: \quad -2(m-1)\omega\dot{\varphi} + \lambda\varphi^{\frac{m+1}{m-1}} = 0;\\
&L_3: \quad -4\alpha\ddot{\varphi} + \lambda\varphi^{\frac{m+1}{m-1}} = 0;\\
&L_4: \quad \lambda\varphi^{\frac{m+1}{m-1}} = 0;\\
&L_5: \quad -4\alpha\omega^3\ddot{\varphi} + 2((m-3)\alpha + m - 1)\omega^2\dot{\varphi} + \lambda\varphi^{\frac{m+1}{m-1}} = 0;\\
&L_6: \quad -4\alpha\ddot{\varphi} - 2(m-1)\dot{\varphi} + \lambda\varphi^{\frac{m+1}{m-1}} = 0.
\end{aligned}
$$

The equation corresponding to the subalgebra L_2 has the solution

$$
\varphi^{\frac{2}{1-m}} = -\frac{\lambda}{(m-1)^2}(\ln\omega + C).
$$

Therefore, the equation (2) has the following solution

$$
u^{\frac{2}{1-m}} = -\frac{\lambda}{(m-1)^2}(\omega_2\ln\omega_1 + C\omega_2). \tag{8}
$$

The equation corresponding to the subalgebra L_3 has the particular solution

$$
\varphi^{\frac{2}{1-m}} = -\frac{\lambda}{4\alpha m(m-1)}(\omega + C)^2.
$$

Therefore, equation (2) has the following solution

$$
u^{\frac{2}{1-m}} = -\frac{\lambda}{4\alpha m(m-1)\omega_1}(\omega_2 - \alpha\omega_1\ln\omega_1 + C\omega_1)^2. \tag{9}
$$

In the case of an equation corresponding to the subalgebra L_5 $\alpha = 0$ or $\alpha = \frac{1-m}{m-3}$ $(m \neq 3)$ we obtain the equations:

$$2(m-1)\omega^2\dot{\varphi} + \lambda\varphi^{\frac{m+1}{m-1}} = 0, \quad -\frac{4(1-m)}{m-3}\omega^3\ddot{\varphi} + \lambda\varphi^{\frac{m+1}{m-1}} = 0.$$

The solutions of these equation are:

$$\varphi^{\frac{2}{1-m}} = -\frac{\lambda}{(m-1)^2}(\omega^{-1} + C), \quad \varphi^{\frac{2}{1-m}} = -\frac{\lambda}{(m-1)^2}\omega^{-1}.$$

Hence equation (2) has the following solutions:

$$u^{\frac{2}{1-m}} = -\frac{\lambda}{(m-1)^2}\ln\omega_1(\omega_2 + C\omega_1), \tag{10}$$

$$u^{\frac{2}{1-m}} = -\frac{\lambda}{(m-1)^2}\omega_2\ln\omega_1. \tag{11}$$

Using the groups of invariance of equations (1) and (2) we can duplicate the solutions (3)–(11). In consequence we obtain multiparametric exact solutions of equation (1). Write out these solutions for equation (1) in the space $R_{1,3}$ using the following notations: $a = (a_0, a_1, a_2, a_3)$, $b = (b_0, b_1, b_2, b_3)$, $c = (c_0, c_1, c_2, c_3)$, $y_\mu = x_\mu + \alpha_\mu$ $(\mu = 0, 1, 2, 3)$, $a \cdot b = a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3$, $\varepsilon = \pm 1$.

1) $u^{1-k} = \sigma(y \cdot y)(1 + \varepsilon(b \cdot y)^{k-2}), \quad \sigma = \dfrac{\lambda(k-1)^2}{4(k-2)}, \quad b \cdot b = 0;$

2) $u^{1-k} = \sigma\left\{[(y \cdot y)(1 + \varepsilon(b \cdot y)^{k-2})]^{\frac{1}{2}} + \alpha(b \cdot y)^{\frac{3(k-1)}{2(k+1)}}[1 + \varepsilon(b \cdot y)^{k-2}]^{\frac{k-1}{2(k+1)}}\right\}^2,$

$\sigma = \dfrac{\lambda(k-1)^2}{4(k-2)}, \quad b \cdot b = 0, \quad \alpha \in R;$

3) $u^{1-k} = \sigma\left\{[(y \cdot y)(1 + \varepsilon(b \cdot y)^{k-2})]^{\frac{1}{2}} + \alpha(b \cdot y)^{\frac{k(k-1)}{2(k+1)}}(y \cdot y)^{\frac{k-1}{2(k+1)}}\right\}^2,$

$\sigma = \dfrac{\lambda(k-1)^2}{4(k-2)}, \quad b \cdot b = 0, \quad \alpha \in R;$

4) $u^{1-k} = \sigma\varepsilon(b \cdot y)^{k-3}[(y \cdot y) + \varepsilon(b \cdot y)^{3-k}]^2, \quad \sigma = \dfrac{\lambda(k-1)^2}{8(k-2)(k+1)}, \ b \cdot b = 0;$

5) $u^{1-k} = \sigma[(y \cdot y) + (a \cdot y)^2][1 + \varepsilon(b \cdot y)^{\frac{k-3}{2}}], \quad \sigma = \dfrac{\lambda(k-1)^2}{2(k-3)},$

$a \cdot a = -1, \quad a \cdot b = 0, \quad b \cdot b = 0;$

6) $u^{1-k} = \sigma\Big\{[((y \cdot y) + (a \cdot y)^2)(1 + \varepsilon(b \cdot y)^{\frac{k-3}{2}})]^{\frac{1}{2}} +$

$+ \alpha(b \cdot y)^{\frac{k-1}{k+1}}(1 + \varepsilon(b \cdot y)^{\frac{k-3}{2}})^{\frac{k-1}{2(k+1)}}\Big\}^2,$

$\sigma = \dfrac{\lambda(k-1)^2}{2(k-3)}, \quad a \cdot a = -1, \quad a \cdot b = 0, \quad b \cdot b = 0, \quad \alpha \in R;$

7) $u^{1-k} = \sigma\Big\{[((y \cdot y) + (a \cdot y)^2)(1 + \varepsilon(b \cdot y)^{\frac{k-3}{2}})]^{\frac{1}{2}} +$

$+ \alpha(b \cdot y)^{\frac{(k-1)^2}{4(k+1)}}((y \cdot y) + (b \cdot y)^2)^{\frac{k-1}{2(k+1)}}\Big\}^2,$

$$\sigma = \frac{\lambda(k-1)^2}{2(k-3)}, \quad a\cdot a = -1, \quad a\cdot b = 0, \quad b\cdot b = 0, \quad \alpha \in R;$$

8) $$u^{1-k} = \sigma\varepsilon(b\cdot y)^{\frac{k-5}{2}}[(y\cdot y) + (a\cdot y)^2 + \varepsilon(b\cdot y)^{\frac{5-k}{2}}]^2;$$
$$\sigma = \frac{\lambda(k-1)^2}{4(k-3)(k+1)}, \quad a\cdot a = -1, \quad a\cdot b = 0, \quad b\cdot b = 0;$$

9) $$u^{-1} = -\frac{\lambda}{4}(y\cdot y)\ln(b\cdot y), \quad b\cdot b = 0, \quad k = 2;$$

10) $$u^{-1} = -\frac{\lambda\varepsilon}{24}(b\cdot y)^{-1}[(y\cdot y) - \varepsilon(b\cdot y)\ln(b\cdot y)]^2, \quad b\cdot b = 0, \quad k = 2;$$

11) $$u^{-2} = -\lambda\ln(b\cdot y)[(y\cdot y) + (a\cdot y)^2],$$
$$a\cdot a = -1, \quad a\cdot b = 0, \quad b\cdot b = 0, \quad k = 3;$$

12) $$u^{-2} = \frac{\lambda\varepsilon}{8}(b\cdot y)^{-1}[(y\cdot y) + (a\cdot y)^2 - \varepsilon(b\cdot y)]^2,$$
$$a\cdot a = -1, \quad a\cdot b = 0, \quad b\cdot b = 0, \quad k = 3.$$

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